DCL-Chemy II

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A Synopsis of the Basics as Covered in DCL-Chemy

The idea of the *dynamic coefficient list* (*DCL* for short) is that coefficients may assume diverse values over the course of an iterative procedure. In the first article in this series (called 'DCL-Chemy Transforms Fibonacci-type Sequences to Arrays', or 'DCL-Chemy'), the generalized Fibonacci-sequence formula $(c)F_n + (b)F_{n+1} = F_{n+2}$; $F_0 = 0,\ F_1 = 1$ is further generalized to

$$(\gamma)F_n + (\beta)F_{n+1} = F_{n+2}\ ; \quad F_0 = 0,\ F_1 = 1$$

Where β and γ are the lists $\beta = [b_1, b_2 \ldots b_i]$ and $\gamma = [c_1, c_2 \ldots c_j]$.

A sequence φ_λ (where λ = the *order* of φ_λ = LCM(i,j)) is generated by applying terms in β and γ in order, according to the iteration being performed. I.e., at the 1st iteration, the initial F_0 and F_1 are multiplied by c_1 and b_1 respectively; at the 2nd iteration, F_1 and F_2 are multiplied by c_2 and b_2; on the 3rd iteration c_3 and b_3 apply, and so on. After λ iterations, the cycle repeats.

This process generates one sequence: to then start with $\beta = [b_2, b_3 \ldots b_\lambda, b_1]$ and $\gamma = [c_2, c_3 \ldots c_\lambda, c_1]$ generates another. Permuting β and γ cyclically generates λ distinct sequences which, in the context of an array, are aligned vertically and designated as $S_1, S_2 \ldots S_\lambda$. Arrays are typically represented by $\Phi_\lambda\,[\beta][\gamma]$, with β, γ and λ in numerical form.

Define F_{n+1}/F_n, for $n \to \infty$, as a *limit ratio*. As a rule, each sequence in an array converges, simultaneously and in two directions, to λ positive and λ negative limit ratios. Formulas derived in DCL-Chemy use elements of Φ_λ to find coefficients for λ different quadratic equations (Q_j) that have roots corresponding to specific limit ratios.

The symbol Q^k was defined as the equation with Q's roots taken to the kth power. In contexts where, as in $Q^{\underline{k}}$, the exponent is underlined, it is taken to mean the exponent operates over a *polytonic* base. E.g., in Φ_λ [1,2,3][3,2,1], $c_2^{\underline{5}} = 2\cdot 1\cdot 3\cdot 2\cdot 1 = 12$. If $\beta = [\mathbb{N}] = [1,2,3\ldots\infty]$, then, e.g., $b_1^{\underline{n}} = n!$ and $b_3^{\underline{7}} = 7!/2!$. (For what it's worth, the exponential and factorial functions are thus, to some extent, unified.)

These and other formulas, operations and symbols will appear in what follows as tools for exploring the effects of reversing (inverting) the order of the terms in β and γ.

DCL-Chemy II: Reflections and Other Symmetries

Coefficient List Inversions Create a Zigzagged Hall of Mirrors

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Enter the Labyrinth

Whereas cyclical permutation of the terms in the *DCL*s (coefficient lists) $\beta[b_1, b_2\dots b_\lambda]$ and $\gamma[c_1, c_2\dots c_\lambda]$ generates the sequences S_j in an array Φ_λ, certain other permutations create arrays that are closely related in varied and surprising ways. The other permutations to be investigated here reverse the order of most or all of the terms in β and γ. In general, such reversals are referred to as *inversions*.

More precise definitions will follow; for now, the simple $\gamma = [1]$ case is considered. Define the mapping $\beta \rightarrow \cancel{\beta}$ as $[b_1, b_2\dots b_\lambda] \rightarrow [b_\lambda, b_{\lambda-1}\dots b_1]$. This turns Φ_λ into $\cancel{\Phi}_\lambda$. For a numerical example, $\beta = [1,2,3]$, gives $\cancel{\beta} = [3,2,1]$. These arrays Φ_3 and $\cancel{\Phi}_3$ are, as seen below, closely related, but much more intimately than one might at first suspect.

Φ_3 [1,2,3][1]	S_1	S_2	S_3	$\cancel{\Phi}_3$ [3,2,1][1]	S_1	S_2	S_3
	25	37	13		25	13	37
	–11	–10	–9		–9	–10	–11
$F_{-\lambda}$	3	7	4		7	3	4
	–2	–3	–1		–2	–1	–3
	1	1	1		1	1	1
F_0	0	0	0		0	0	0
	1	1	1		1	1	1
	1	2	3		3	2	1
F_λ	3	7	4		7	3	4
	10	9	11		10	11	9
	13	25	37		37	25	13
$F_{2\lambda}$	36	84	48		84	36	48
	121	109	133		121	133	109
	157	302	447		447	302	157

Table 1: The symmetries of $\beta \rightarrow \cancel{\beta}$

A cursory comparison of Φ_3 and $\cancel{\Phi}_3$ reveals several relationships and symmetries. E.g., it could be said that the $\beta \rightarrow \cancel{\beta}$ mapping straightens out the right-to-left descending diagonals of Φ_3 to create the columns of $\cancel{\Phi}_3$. Then vice versa for $\cancel{\beta} \rightarrow \beta$.

The left-to-right descending diagonals are the same in each array, and, in one of the three cases, in phase.

Moreover, a series S_j in Φ_3 taken from F_0 through $F_{n<0}$ has the same terms (disregarding signs) as found in a column S_i in $\cancel{\Phi}_3$ from F_0 through $F_{n>0}$. Hence, signage aside, $\beta \rightarrow \cancel{\beta}$ is akin to either flipping Φ_3 on both a horizontal and vertical axis or rotating it through 180º.

We now restate, from among those formulas derived in DCL-Chemy, a generalized quadratic equation:

$$Q_j^k = F_{\lambda,\,j} \cdot x_j^2 - (F_{\lambda+k,\,j} + F_{\lambda-k,\,j+k} \cdot c_j^k \cdot (-1)^k)\, x_j + F_{\lambda,\,j+k} \cdot c_j^k \cdot (-1)^k \tag{1.1}$$

Let $\underline{k} = k = 1$, and apply this formula to the two arrays above for these six equations:

$$
\begin{array}{ll}
Q_1 = 3x_1^2 - 8x_1 - 7 & \qquad \cancel{Q}_1 = 7x_1^2 - 8x_1 - 3 \\
Q_2 = 7x_2^2 - 6x_2 - 4 & \qquad \cancel{Q}_2 = 3x_2^2 - 10x_2 - 4 \\
Q_3 = 4x_3^2 - 10x_3 - 3 & \qquad \cancel{Q}_3 = 4x_3^2 - 6x_3 - 7
\end{array}
$$

Table 2: Quadratics with coefficients derived from table 1

A property of $Q = ax^2 + bx + c$ is that $Q^{-1} = cx^2 + bx + a$; but for $-c$, then $Q^{-1} = cx^2 - bx - a$. That is, for a always positive, if c in Q is negative, then the sign of b in Q^{-1} is reversed. Note that for every Q_j in table 2 there is a $\cancel{Q}_i$ with the same b while a and c are exchanged; yet, while the c coefficient of each equation is negative, the b coefficients in each Q_j and $\cancel{Q}_i$ are all of negative sign. Hence, in these examples, for every root r_j in Φ_3 there is an $\cancel{r}_i$ in $\cancel{\Phi}_3$ such that $r_j \cdot \cancel{r}_i = -1$; i.e., a root in Φ_3 has a negative inverse in $\cancel{\Phi}_3$.

Asymmetric Inversions

For $\lambda > 1$, it seems that certain relationships inherent to the roots and coefficients of $\lambda = 1$ quadratics are now expressed in more expansive ways. Interactions that are typical of roots r_+ and r_- in a single equation will also be found among roots of equations taken from related arrays. To further elaborate, consider arrays with terms in γ not all identical; i.e., for $\gamma[c_1, c_2 \ldots c_\lambda]$, some $c_j \neq c_i$. In this polytonic γ case, two types of inversions will be investigated: *symmetrical* and *asymmetrical*. The latter kind are considered first.

Definition: To effect the desired alignment of $\cancel{\beta}$ and $\cancel{\gamma}$, an *asymmetric inversion* ($\Phi_\lambda \to \cancel{\Phi}_\lambda$) leaves the initial term of γ fixed. I.e., this technique reverses the order of all terms in β, but for γ, only the final $\lambda - 1$ terms of the arrangement are inverted in $\cancel{\gamma}$: i.e.,

$$\beta \to \cancel{\beta} = [b_1, b_2 \ldots b_\lambda] \to [\cancel{b}_\lambda, \cancel{b}_{\lambda-1} \ldots \cancel{b}_1] \qquad \gamma \to \cancel{\gamma} = [c_1, c_2 \ldots c_\lambda] \to [\cancel{c}_1, \cancel{c}_\lambda, \cancel{c}_{\lambda-1} \ldots \cancel{c}_2]$$

Arrays formed on $\beta[1]$, $\gamma[1,2,3]$ and $\cancel{\beta}[1]$, $\cancel{\gamma}[1,3,2]$ are seen below.

Φ_3 [1][1,2,3]	S_1	S_2	S_3	$\cancel{\Phi}_3$ [1][1,3,2]	S_1	S_2	S_3
	$^{-4}/_{18}$	$^{-6}/_{6}$	$^{-5}/_{12}$		$^{-5}/_{12}$	$^{-6}/_{6}$	$^{-4}/_{18}$
$F_{-\lambda}$	$^{3}/_{6}$	$^{4}/_{6}$	$^{2}/_{6}$		$^{4}/_{6}$	$^{3}/_{6}$	$^{2}/_{6}$
	$^{-1}/_{6}$	$^{-1}/_{3}$	$^{-1}/_{2}$		$^{-1}/_{6}$	$^{-1}/_{2}$	$^{-1}/_{3}$
	$^{1}/_{3}$	$^{1}/_{1}$	$^{1}/_{2}$		$^{1}/_{2}$	$^{1}/_{1}$	$^{1}/_{3}$
F_0	0	0	0		0	0	0
	1	1	1		1	1	1
	1	1	1		1	1	1
F_λ	3	4	2		4	3	2
	6	5	4		6	4	5
	9	13	10		10	13	9
$F_{2\lambda}$	21	28	14		28	21	14
	48	41	34		48	34	41

Table 3: The effect of $\gamma \to \cancel{\gamma}$ for $\gamma[1,2,3]$

The formula in (1.1) now applies to arrays in table 3

$$Q_1 = 3x_1^2 - 5x_1 - 4 \qquad \cancel{Q}_1 = 4x_1^2 - 5x_1 - 3$$
$$Q_2 = 4x_2^2 - 3x_2 - 4 \qquad \cancel{Q}_2 = 3x_2^2 - x_2 - 6$$
$$Q_3 = 2x_3^2 - x_3 - 9 \qquad \cancel{Q}_3 = 2x_3^2 - 3x_3 - 8$$

Table 4: Quadratics with coefficients derived from table 3

In the patterns that appear below, the reason for the $\cancel{\gamma}[\epsilon_1, \epsilon_\lambda, \epsilon_{\lambda-1} \ldots \epsilon_2]$ offset becomes apparent.

Φ_3 [1,2,3][1,2,3]	S_1	S_2	S_3	$\cancel{\Phi}_3$ [3,2,1][1,3,2]	S_1	S_2	S_3
	$-7/9$	$-5/2$	$-11/12$		$-11/12$	$-5/2$	$-7/9$
$F_{-\lambda}$	$2/3$	$3/2$	$2/3$		$3/2$	$2/3$	$2/3$
	$-1/3$	$-1/1$	$-1/2$		$-1/3$	$-1/2$	$-1/1$
	$1/3$	$1/1$	$1/2$		$1/2$	$1/1$	$1/3$
F_0	0	0	0		0	0	0
	1	1	1		1	1	1
	1	2	3		3	2	1
F_λ	4	9	4		9	4	4
	15	11	14		15	14	11
	19	40	54		54	40	19
$F_{2\lambda}$	68	153	68		153	68	68
	261	193	224		261	224	193

Table 5: The offset in $\cancel{\gamma}$ preserves the $\Phi_3/\cancel{\Phi}_3$ relationships found in tables 1 and 3

$$Q_1 = 4x_1^2 - 13x_1 - 9 \qquad \cancel{Q}_1 = 9x_1^2 - 13x_1 - 4$$
$$Q_2 = 9x_2^2 - 5x_2 - 8 \qquad \cancel{Q}_2 = 4x_2^2 - 11x_2 - 12$$
$$Q_3 = 4x_3^2 - 11x_3 - 12 \qquad \cancel{Q}_3 = 4x_3^2 - 5x_3 - 18$$

Table 6: Quadratics with coefficients derived from table 5

Below, roots from tables 2, 4 and 6 are paired off in the top tiers so that their products are integers, equal (in absolute value) to a term in γ. In the lower tiers, the roots are summed to equal a term in β,

Roots from	table 2	table 4	table 6
	$r_{1+} \cdot \cancel{r}_{1-} = -1$	$r_{1+} \cdot \cancel{r}_{1-} = -1$	$r_{1+} \cdot \cancel{r}_{1-} = -1$
Products:	$r_{2+} \cdot \cancel{r}_{3-} = -1$	$r_{2+} \cdot \cancel{r}_{3-} = -2$	$r_{2+} \cdot \cancel{r}_{3-} = -2$
	$r_{3+} \cdot \cancel{r}_{2-} = -1$	$r_{3+} \cdot \cancel{r}_{2-} = -3$	$r_{3+} \cdot \cancel{r}_{2-} = -3$
	$r_{1+} + \cancel{r}_{2-} = 3$	$r_{1+} + \cancel{r}_{2-} = 1$	$r_{1+} + \cancel{r}_{2-} = 3$
Sums:	$r_{2+} + \cancel{r}_{1-} = 1$	$r_{2+} + \cancel{r}_{1-} = 1$	$r_{2+} + r_{1-} = 1$
	$r_{3+} + \cancel{r}_{3-} = 2$	$r_{3+} + \cancel{r}_{3-} = 1$	$r_{3+} + \cancel{r}_{3-} = 2$

Table 7: Roots of quadratics combined as products and sums equal terms in γ and β respectively

The signs of roots in each pairing can be reversed for the same result. Note that in table 6, Q_3 and $\cancel{Q}_2$ are the same equation. Their roots multiply to return γ and $\cancel{\gamma}$ terms, and also, multiplied by a, the c coefficient.

Given the proper matchings, $r_{j+} \cdot \cancel{r}_{i-}$ products are in γ and $\cancel{\gamma}$, and $r_{j+} + \cancel{r}_{i-}$ sums are in β and $\cancel{\beta}$. Without the adjustment in the $\cancel{\gamma}$ inversion, the root pairings in table 7 would not produce these same products and sums, and the relationships between the roots that are considered next would not exist.

To get somewhat formal about it: taking it as given that, for $\lambda > 1$, the patterns in tables 1 through 6 will hold for all Φ_λ and their $\cancel{\Phi}_\lambda$ inversions, we prove, for the $\gamma[1]$ case, a theorem about the roots found in table 2. In the following provisional proof, a strikethrough in any symbol identifies it with $\cancel{\Phi}_\lambda$.

Theorem 1: (Given $\gamma[1]$.) For each Q_j there is a $\cancel{Q}_i$ such that $r_{j+} \cdot \cancel{r}_{i-} = r_{j-} \cdot \cancel{r}_{i+} = -c_j = -\cancel{c}_i$.

Proof: For this demonstration, Q_j and $\cancel{Q}_i$ are chosen to have the same b coefficient; i.e., $b = \cancel{b}$. Then, using a simplified version of (1.1) and availing of the $a = \cancel{c}$, $c = \cancel{a}$ symmetry, the two equations are expressed as

$$Q_j = F_{\lambda,\, j} \cdot x^2 - (F_{\lambda+1,\, j} - F_{\lambda-1,\, j+1} \cdot c_j)x \ - F_{\lambda,\, j+1} \cdot c_j \tag{1.2}$$

$$\cancel{Q}_i = F_{\lambda,\, j+1} \cdot x^2 - (F_{\lambda+1,\, j} - F_{\lambda-1,\, j+1} \cdot c_j)x \ - F_{\lambda,\, j} \cdot c_j$$

Both equations, Q_j and $\cancel{Q}_i$ above, based on the $a_j = |\cancel{c}_i|$ and $|c_j| = \cancel{a}_i$ exchange, apply to Φ_λ. The roots are expressed below in quadratic formula form. (It will be shown in theorem 2 that all equations in Φ_λ share a common discriminant (D); here $D = b^2 - 4ac$. Moreover, as a consequence of the $\Phi_\lambda/\cancel{\Phi}_\lambda$ symmetries, this D is common to $\cancel{Q}_i$ as well.)

$$r_{j+} \cdot \cancel{r}_{i-} = \frac{-b+\sqrt{D}}{2F_{\lambda,\, j}} \cdot \frac{-b-\sqrt{D}}{2F_{\lambda,\, j+1}} = \frac{b^2 - b^2 + 4(F_{\lambda,\, j} \cdot -F_{\lambda,\, j+1} \cdot c_j)}{4F_{\lambda,\, j} \cdot F_{\lambda,\, j+1}} = -c_j$$

QED

If the equations in (1.2) had instead been designed to apply to $\cancel{\Phi}_\lambda$, the proof would have returned $\cancel{c}_i$. As a corollary, $r_+^{\underline{\lambda}} = \cancel{r}_+^{\underline{\lambda}}$ and $r_-^{\underline{\lambda}} = \cancel{r}_-^{\underline{\lambda}}$. That is, the product of all the positive (negative) roots in Φ_λ is equal to the product of all the positive (negative) roots in $\cancel{\Phi}_\lambda$. This is because $D = \cancel{D}$ and each a_j = some $\cancel{a}_i$. Thus the roots r_j and $\cancel{r}_i$ are the same combinations of $-b \pm D^{1/2}$ over the same $2a$ in a different order, so the products are identical. Note then:

$$Q^{\underline{\lambda}} = \cancel{Q}^{\underline{\lambda}} = x^2 - (r_+^{\underline{\lambda}} + \cancel{r}_-^{\underline{\lambda}})x + r_+^{\underline{\lambda}} \cdot \cancel{r}_-^{\underline{\lambda}} \text{ and } r_+^{\underline{\lambda}} \cdot \cancel{r}_-^{\underline{\lambda}} = \cancel{r}_+^{\underline{\lambda}} \cdot r_-^{\underline{\lambda}} = r_+^{\underline{\lambda}} \cdot r_-^{\underline{\lambda}} = \cancel{r}_+^{\underline{\lambda}} \cdot \cancel{r}_-^{\underline{\lambda}} = c_j^{\underline{\lambda}} \cdot (-1)^\lambda$$

An example of theorem 1 uses the equations $Q_1 = 3x^2 - 8x - 7$ and $\cancel{Q}_1 = 7x^2 - 8x - 3$ in table 2:

$$r_{1+} \cdot \cancel{r}_{1-} = \frac{8+\sqrt{64-4\cdot 3\cdot -7}}{2\cdot 3} \cdot \frac{8-\sqrt{64-4\cdot 7\cdot -3}}{2\cdot 7} = \frac{64-(64+84)}{84} = -1 = -c_1$$

Matching 'b' coefficients works for table 4 roots as well. E.g., take $Q_3 = 2x^2 - x - 9$ and $\cancel{Q}_2 = 3x^2 - x - 6$:

$$r_{3+} \cdot \cancel{r}_{2-} = \frac{1+\sqrt{1-4\cdot 2\cdot(-9)}}{2\cdot 2} \cdot \frac{1-\sqrt{1-4\cdot 3\cdot(-6)}}{2\cdot 3} = \frac{1-(1+72)}{24} = -3 = -c_3$$

The technique also extends to table 6 roots. For instance, take $Q_2 = 9x^2 - 5x - 8$ and $\cancel{Q}_3 = 4x^2 - 5x - 18$:

$$r_{2+}\cdot \cancel{r}_{3-} = \frac{5+\sqrt{25-4\cdot 9\cdot(-8)}}{2\cdot 9}\cdot\frac{5-\sqrt{25-4\cdot 4\cdot(-18)}}{2\cdot 4} = \frac{25-(25+288)}{144} = -2 = -c_2$$

As for the corollary: from table 1, $r_+^{\lambda} = r_{1+}\cdot r_{2+}\cdot r_{3+} = 12.08276\ldots = \cancel{r}_{1+}\cdot \cancel{r}_{2+}\cdot \cancel{r}_{3+} = \cancel{r}_+^{\lambda}$

Then for $r_+^{\lambda}\cdot r_-^{\lambda} = c_j^{\lambda}\cdot(-1)^{\lambda}$: from table 2 we have $r_+^{\lambda} = 7.772,\;\; r_-^{\lambda} = -.772$ and $r_+^{\lambda}\cdot r_-^{\lambda} = -6$

As evidenced in table 7, another complementary relationship between the roots of Φ_λ and $\cancel{\Phi}_\lambda$ is that for each Q_j there is a $\cancel{Q}_i$ such that $r_{j+} + \cancel{r}_{i-} = r_{j-} + \cancel{r}_{i+} = b_j = \cancel{b}_i$. To effect this result, Q_j and $\cancel{Q}_i$ are chosen to share the *a* coefficient. Here, even the simple $\cancel{\gamma}[1]$ case is left for the reader to prove…

Some Curious Properties of Q_j Convergents

A remarkable relationship between Φ_λ and $\cancel{\Phi}_\lambda$ emerges when one seeks *convergents* for r_j. Convergents are, in this application of the concept, a series of ordinary fractions that approach ever more closely to the roots of Q_j. For $\lambda = 1$, the convergents of $r_{j\pm}$ are simply terms of the sequence from which the coefficients for Q were taken. E.g., let $Q_\phi = x^2 - x - 1$: then the convergents have successive terms of the ϕ sequence as numerators and denominators:

$$\ldots{}^{5}/_{-8},\ {}^{-3}/_{5},\ {}^{2}/_{-3},\ {}^{-1}/_{2},\ {}^{1}/_{-1},\ {}^{0}/_{1},\ {}^{1}/_{0},\ {}^{1}/_{1},\ {}^{2}/_{1},\ {}^{3}/_{2},\ {}^{5}/_{3},\ {}^{8}/_{5},\ {}^{13}/_{8},\ {}^{21}/_{13}\ldots$$

These fractions converge to $-1/\phi$ on the left and ϕ on the right; i.e., the zeros of $x^2 - x - 1$.

Another symbol will be helpful here. With the numerator sequence as a reference, let the displacement of the denominator sequence be represented by Δ. Let Δ_0 represent a series of fractions with numerators equal to denominators. Then Δ_1 signifies a shifting of the denominators one place to the right, as in the series of fractions above. Δ_k is a rightward shift of k places and, in general, convergence to the roots of Q^k. For example, Δ_3 gives the sequence

$$\ldots{}^{5}/_{-21},\ {}^{-3}/_{13},\ {}^{2}/_{-8},\ {}^{-1}/_{5},\ {}^{1}/_{-3},\ {}^{0}/_{2},\ {}^{1}/_{-1},\ {}^{1}/_{1},\ {}^{2}/_{0},\ {}^{3}/_{1},\ {}^{5}/_{1},\ {}^{8}/_{2},\ {}^{13}/_{3},\ {}^{21}/_{5}\ldots$$

These fractions are converging to $-1/\phi^3$ on the left and ϕ^3 on the right.

Now, what happens for $\lambda > 1$? Take for an example Φ_3 [1,2,3][1], S_1/S_1, Δ_1;

$$\ldots{}^{25}/_{-36},\ {}^{-11}/_{25},\ {}^{3}/_{-11},\ {}^{-2}/_{3},\ {}^{1}/_{-2},\ {}^{0}/_{1},\ {}^{1}/_{0},\ {}^{1}/_{1},\ {}^{3}/_{1},\ {}^{10}/_{3},\ {}^{13}/_{10},\ {}^{36}/_{13},\ {}^{121}/_{36}\ldots$$

We've seen that the ratio of adjacent terms of this series converges to six values simultaneously, i.e., the roots of Q_1, Q_2 and Q_3. But what if the objective is a series of consecutive fractions that will converge to the roots of only a single equation in this set? Here a shift Δ_2 doesn't help, for the fractions now converge to the 2nd power of the same six roots. A shift of Δ_3, however, fulfills the condition, for then $\Delta_k = \Delta_\lambda$, and the equation Q_j^{λ} has the same roots for all j;

$$\ldots{}^{25}/_{-302},\ {}^{-11}/_{133},\ {}^{3}/_{-36},\ {}^{-2}/_{25},\ {}^{1}/_{-11},\ {}^{0}/_{3},\ {}^{1}/_{-2},\ {}^{1}/_{1},\ {}^{3}/_{0},\ {}^{10}/_{1},\ {}^{13}/_{1},\ {}^{36}/_{3},\ {}^{121}/_{10}\ldots$$

This is a start: these fractions are indeed converging to the roots of one equation. Since a linear shift seems to work only for Δ_λ let's try another, sideways, course of action… such as, say, S_1/S_2, Δ_1;

$\ldots{}^{25}/_{-84}$, ${}^{-11}/_{37}$, ${}^{3}/_{-10}$, ${}^{-2}/_{7}$, ${}^{1}/_{-3}$, ${}^{0}/_{1}$, ${}^{1}/_{0}$, ${}^{1}/_{1}$, ${}^{3}/_{2}$, ${}^{10}/_{7}$, ${}^{13}/_{9}$, ${}^{36}/_{25}$, ${}^{121}/_{84}\ldots$

This works—the terms converge to two numbers; $r_{-} = -.29753\ldots$ and $r_{+} = 1.44039\ldots$ Or, more precisely, $\cancel{r}_{1-}$ and $\cancel{r}_{1+}$, because the surprise here is that *fractions composed of terms of sequences in* Φ_3 *converge to the roots of equations in* $\cancel{\Phi}_3$ (in this case, to $\cancel{Q}_1$). This works both ways and in general, sequences in Φ_λ form fractions that converge to roots in $\cancel{\Phi}_\lambda$ and vice versa. To confirm this Φ_3 [1,2,3][1] example, we set the 16th term of S_1 ($F_{16,1} = 213442$) over the 15th term of S_2 ($F_{15,2} = 148183$) to find the quotient $1.44039\ldots \approx \cancel{r}_{1+}$, a root of $\cancel{Q}_1 = 7x^2 - 8x - 3$ in $\cancel{\Phi}_3$ [3,2,1][1].

A cursory appraisal reveals a general pattern, where S_1/S_2 in Φ_λ converges to the roots of $\cancel{Q}_1$. Moving to the left, S_λ/S_1 gives convergence to the roots of $\cancel{Q}_2$, $S_{\lambda-1}/S_\lambda$ to $\cancel{Q}_3$, and etc. Moving the numerator sequence leftward while holding a constant denominator (i.e., S_j/S_{j+1}, $S_{j-1}/S_{j+1}\ldots$) while simultaneously incrementing Δ_k causes a corresponding increase in the power $\underline{k}$ of $\cancel{Q}_j^{k}$.

Beyond these patterns there are, for ever-larger λ, no limits to making arbitrary S_j/S_i pairings and Δ_k shifts, It seems that many of these combinations produce convergence to numbers that combine to form integral coefficients for some Q_j, but such other patterns as may exist here aren't so easy to discern.

However, some of the statements above must at present be qualified, because attempts to derive $\cancel{\Phi}_3$ roots by creating convergents from Φ_3 [1][1,2,3] sequences have not been successful thus far.

In any case, returning to the Φ_3 [1,2,3][1] example; to pursue this a bit further leads to the discovery that Φ_λ and $\cancel{\Phi}_\lambda$ are not really distinct. We take the array that appears in table 1, retain the vertical alignment and apply the shift S_{j-1}/S_j, Δ_1 repeatedly to create the configuration below.

25						
–11	37					
3	–10	13				
–2	7	–9	25			
1	–3	4	–11	37		
0	1	–1	3	–10	13	
1	0	1	–2	7	–9	25
1	1	0	1	–3	4	–11
3	2	1	0	1	–1	3
10	7	3	1	0	1	–2
13	9	4	1	1	0	1
36	25	11	3	2	1	0
121	84	37	10	7	3	1
157	109	48	13	9	4	1
	302	133	36	25	11	3
		447	121	84	37	10
			157	109	48	13
				302	133	36
					447	121
						157

Table 8: The array $\Phi_{3\Delta 1}$ [1,2,3][1], where the *n*th column is shifted down by $n - 1$ rows

Call an array that is thus ordered, say, Φ_Δ. Now perform these shifts on $\cancel{\Phi}_3$ [3,2,1][1] to create $\cancel{\Phi}_{3\Delta}$. Then define the symbol ♦ as an operation on an array; specifically, a 90º CCW rotation followed by a vertical axis flip. Then $♦\Phi_\Delta = \cancel{\Phi}_\Delta$, and so of course $♦\cancel{\Phi}_\Delta = \Phi_\Delta$. The illusion of $\Phi_\lambda/\cancel{\Phi}_\lambda$ duality is thus dispelled…

Q_j Derived from Φ_λ by (1.1) Share a Common Discriminant

Consider the square below, which represents four adjacent terms of Φ_Δ:

d	f
g	h

The arrays in table 8 may be used to verify that the relationship $dh - gf = \pm 1$ holds throughout. This is a Φ_Δ variant of the more general formula expressed below in (1.3). This formula applies to Φ_λ and is useful for proving a basic theorem, with a method that showcases a novel algebraic technique.

Theorem 2: The equations in Φ_λ share a common discriminant of the form

$$D = (F_{\lambda+1,\, j} + F_{\lambda-1,\, j+1} \cdot c_j)^2 - 4c_j^{\frac{\lambda}{}} \cdot (-1)^\lambda$$

Proof: The proof depends on the equation

$$F_{n,\, j} \cdot F_{n,\, j+1} = F_{n+1,\, j} \cdot F_{n-1,\, j+1} - c_{j+1}^{\underline{n-1}} \cdot (-1)^n \tag{1.3}$$

(This is a 2D version of the Fibonacci identity, $F_n^2 = F_{n+1} \cdot F_{n-1} - (-1)^n$)

(1.2) , the alternate version of (1.1) used in theorem 1, is generalized to be useful in this situation as well.

$$Q_j^{\underline{k}} = F_{\lambda,\, j} \cdot x^2 - (F_{\lambda+k,\, j} + F_{\lambda-k,\, j+k} \cdot c_j^{\underline{k}} \cdot (-1)^k)x \;+ F_{\lambda,\, j+k} \cdot c_j^{\underline{k}} \cdot (-1)^k \tag{1.4}$$

Then the discriminant of Q_j can be stated as $D = (F_{\lambda+1,\, j} - F_{\lambda-1,\, j+1} \cdot c_j)^2 - 4F_{\lambda,\, j} \cdot F_{\lambda,\, j+1} \cdot (-c_j)$

Clearing the parentheses; $D = F_{\lambda+1,\, j}^2 + F_{\lambda-1,\, j+1}^2 \cdot c_j^2 - 2c_j \cdot F_{\lambda+1,\, j} \cdot F_{\lambda-1,\, j+1} + 4c_j \cdot F_{\lambda,\, j} \cdot F_{\lambda,\, j+1}$

But, from (1.3): $4c_j \cdot F_{\lambda,\, j} \cdot F_{\lambda,\, j+1} = 4c_j \cdot F_{\lambda+1,\, j} \cdot F_{\lambda-1,\, j+1} - 4c_j^{\underline{\lambda}} \cdot (-1)^\lambda$

Substituting and collecting terms; $D = F_{\lambda+1,\, j}^2 + F_{\lambda-1,\, j+1}^2 \cdot c_j^2 + 2c_j \cdot F_{\lambda+1,\, j} \cdot F_{\lambda-1,\, j+1} - 4c_j^{\underline{\lambda}} \cdot (-1)^\lambda$

Factoring; $D = (F_{\lambda+1,\, j} + F_{\lambda-1,\, j+1} \cdot c_j)^2 - 4c_j^{\underline{\lambda}} \cdot (-1)^\lambda$

QED

An assertion here is that $c_j \cdot c_{j+1}^{\underline{\lambda-1}} = c_j^{\underline{\lambda}}$ is a valid algebraic operation, even if it isn't in the textbooks yet. If another proof of the theorem can be found, we can work backwards to prove (1.3).

Note that theorem 2 also shows that $F_{\lambda+1,j} + F_{\lambda-1,j+1} \cdot c_j$ has the same value for any/all choices of j in an array.

Symmetrical Inversions

Returning to the subject of coefficient list inversions, we'll now look at the symmetrical type. In this case, the symbol for the inverted array will be underlined.

Definition: A *symmetric* inversion $\Phi_\lambda \rightarrow \underline{\Phi}_\lambda$ is symmetrical with respect to the order of terms in β and γ. That is, it directly inverts (reverses) the sequential order of the terms in each list:

$$\beta \rightarrow \underline{\beta} = [b_1, b_2 \ldots b_\lambda] \rightarrow [b_\lambda, b_{\lambda-1} \ldots b_1] \qquad \gamma \rightarrow \underline{\gamma} = [c_1, c_2 \ldots c_\lambda] \rightarrow [c_\lambda, c_{\lambda-1} \ldots c_1]$$

For example:

Φ_3 [1,2,3][1,2,3]	S_1	S_2	S_3
F_{-4}	$^{-14}/_{18}$	$^{-15}/_{6}$	$^{-11}/_{12}$
$F_{-\lambda}$	$^{4}/_{6}$	$^{9}/_{6}$	$^{4}/_{6}$
F_{-2}	$^{-2}/_{6}$	$^{-3}/_{3}$	$^{-1}/_{2}$
F_{-1}	$^{1}/_{3}$	$^{1}/_{1}$	$^{1}/_{2}$
F_0	0	0	0
F_1	1	1	1
F_2	1	2	3
F_λ	4	9	4
F_4	15	11	14
F_5	19	40	54
$F_{2\lambda}$	68	153	68

$\underline{\Phi}_3$ [3,2,1][3,2,1]	S_1	S_2	S_3
F_{-4}	$^{-14}/_{6}$	$^{-11}/_{18}$	$^{-15}/_{12}$
$F_{-\lambda}$	$^{8}/_{6}$	$^{3}/_{6}$	$^{6}/_{6}$
F_{-2}	$^{-2}/_{2}$	$^{-1}/_{3}$	$^{-3}/_{6}$
F_{-1}	$^{1}/_{1}$	$^{1}/_{3}$	$^{1}/_{2}$
F_0	0	0	0
F_1	1	1	1
F_2	3	2	1
F_λ	8	3	6
F_4	11	15	14
F_5	57	36	20
$F_{2\lambda}$	136	51	102

Table 9: Symmetric inversion of Φ_3 [1,2,3][1,2,3]

The formula in (1.1) gives the equations:

$$Q_1 = 4x_1^2 - 13x_1 - 9 \qquad \underline{Q_1} = 8x_1^2 - 5x_1 - 9$$

$$Q_2 = 9x_2^2 - 5x_2 - 8 \qquad \underline{Q_2} = 3x_2^2 - 13x_2 - 12$$

$$Q_3 = 4x_3^2 - 11x_3 - 12 \qquad \underline{Q_3} = 6x_3^2 - 11x_3 - 8$$

Table 10: Quadratic coefficients as extracted by (1.1) from symmetrically inverted arrays

In this special case where $\beta = \gamma$ (i.e., $b_1 = c_1$, $b_2 = c_2 \ldots b_\lambda = c_\lambda$) all of the equations in the two sets share a common discriminant. There are many $\beta \neq \gamma$ cases where this relationship holds, such as say β[1,5,9], γ[2,4,6]. Exactly what conditions on β and γ are required to ensure that $D = \underline{D}$ is an interesting question, apparently related to properties of hyperbolic curves such as the one that appears later in (1.5). A similar question is, when do row sums and products in Φ_λ equal, row for row, the row sums and products in $\underline{\Phi}_\lambda$.

When $D = \underline{D}$, the DCL-Chemy formula $Q_j^\lambda = x_j^2 - (F_{\lambda+1,j} + c_j \cdot F_{\lambda-1,j+1})x_j + c_j^\lambda \cdot (-1)^\lambda$ says that $Q^\lambda = \underline{Q}^\lambda$. Then the theorem below shows that solving this, or any single equation in the sets above, gives the roots of all of the others.

Theorem 3: (*i*); $r_{j\pm} = \dfrac{r_\pm^\lambda - c_j \cdot F_{\lambda-1,j+1}}{F_{\lambda,j}}$ and (*ii*); $r_\pm^\lambda = F_{\lambda,j} \cdot r_{j\pm} + c_j \cdot F_{\lambda-1,j+1}$

Proof: First the root $r_{j\pm}$ on the right side of (*ii*) is expressed in quadratic formula form, and the term $2F_{\lambda,j}$ is cleared from the denominator;

$$2F_{\lambda,j} \cdot r_\pm^\lambda = F_{\lambda,j} \cdot F_{\lambda+1,j} - F_{\lambda,j} \cdot c_j \cdot F_{\lambda-1,j+1} \pm F_{\lambda,j} \cdot D^{1/2} + 2F_{\lambda,j} \cdot c_j \cdot F_{\lambda-1,j+1}$$

$F_{\lambda,j}$ is divided out; $2r_{\pm}^{\lambda} = F_{\lambda+1,\,j} - c_j \cdot F_{\lambda-1,\,j+1} \pm D^{1/2} + 2c_j \cdot F_{\lambda-1,\,j+1}$

Collecting terms leaves a quadratic formula version of $r_{\pm}^{\lambda}$: $2r_{\pm}^{\lambda} = F_{\lambda+1,\,j} + c_j \cdot F_{\lambda-1,\,j+1} \pm D^{1/2}$

QED

For an example, r_{1+}^{λ} = 3.8365 · 1.26066 · 3.5865 = 17.3459 is the product of the positive roots of the Q_j in table 10. Using (*ii*) in theorem 3, take r_{1+} and $F_{j,k}$ terms from table 9 for 4 · 3.8365 + 2 = 17.3459.

Recalling that D = Đ, this theorem applies there as well.

Roots of Q_j and $\underline{Q}_i$ as Points on a Hyperbolic Curve

Regarding $\Phi_\lambda \to \underline{\Phi}_\lambda$, the special $\beta = \gamma$ case closely relates to a certain well-known curve. To look first at an elementary example is instructive: take $Q_\phi = x^2 - x - 1$. Two formulas that equate the roots of this equation to its coefficients can be combined to create a third formula:

$$i)\ r_+ + r_- = -b \qquad ii)\ r_+ \cdot r_- = c \qquad \text{Since } b = c \text{ in } Q_\phi\text{, then:} \quad iii)\ r_+ \cdot r_- + r_+ + r_- = 0$$

A more general rendering of *iii* is

$$xy + x + y = 0 \tag{1.5}$$

Thus when $b = c$, the roots of Q in $\lambda = 1$will combine to identify a point on (1.5). Evidently, when $\beta = \gamma$, this holds true for all λ. The (by now expected) twist is that *a point on* (1.5) *takes a root from each* Φ_3 *and* $\underline{\Phi}_3$. For an example, let $x = r_{1+} = 3.8365\ldots$; this is a root of $Q_1 = 4x^2 - 13x - 9$ in Φ_3 [1,2,3][1,2,3]. Then let $y = \underline{r}_{1-} = -.7932\ldots$, a root from $\underline{Q}_1 = 8x^2 - 5x - 9$ in $\underline{\Phi}_3$ [3,2,1][3,2,1]. Combine the two in (1.5) for 3.8365(–.7932) + 3.8365 –.7932 = 0. These roots combine as the red points in the graph below.

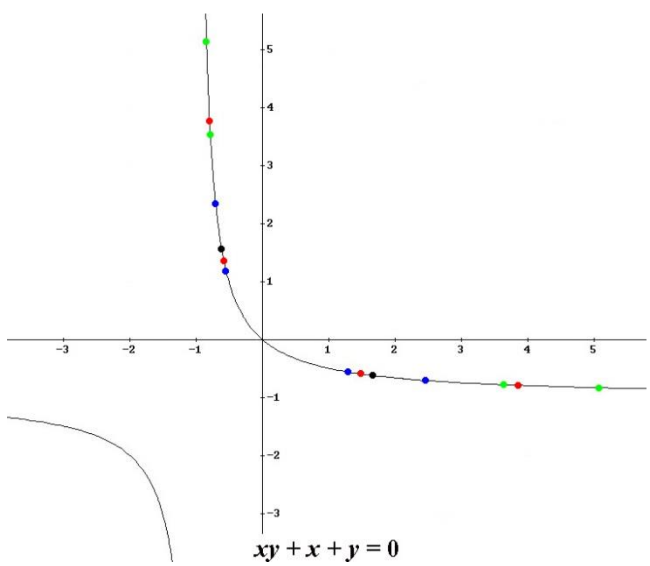

Figure 1: Quadratic roots combine as points on a hyperbola

The graph of ((1.5)) consists of curves that are mirror images, symmetrical with respect to the diagonals that pass through the point at (–1,–1). Call the section passing through the origin l_1 and the other l_2. Note that certain points are marked on l_1: the black points are the roots of Q_ϕ; the colored dots each take a root from a table 10 Q_j, and a root from a corresponding $\underline{Q}_i$ in Φ_3. These table 10 equations are restated below.

$Q_1 = 4x_1^2 - 13x_1 - 9$	$\underline{Q}_1 = 8x_1^2 - 5x_1 - 9$
$Q_2 = 9x_2^2 - 5x_2 - 8$	$\underline{Q}_2 = 3x_2^2 - 13x_2 - 12$
$Q_3 = 4x_3^2 - 11x_3 - 12$	$\underline{Q}_3 = 6x_3^2 - 11x_3 - 8$

Table 11: Equations in table 10 are color-coded to correlate with points on the graph in figure 1

The symmetries we've identified make it very simple to map a point from l_1 to l_2. Since $x = y$ at both [0,0] and [–2,–2], then we need only add 2 to each root and reverse its sign. Thus to map the black dots to l_2, take $(1.618 + 2)(–1) = –3.618$ and $(–.618 + 2)(–1) = –1.382$. Now $(x + 3.618)(x + 1.382)$ gives the coefficients of a new equation, $Q_\phi' = x^2 + 5x + 5$. The same procedure will map the colored dots to l_2 (not shown in the graphic), and (after multiplying by the original a coefficient) gives the equations below.

$Q_1' = 4x_1^2 + 29x_1 + 33$	$\underline{Q}_1' = 8x_1^2 + 37x_1 + 33$
$Q_2' = 9x_2^2 + 41x_2 + 38$	$\underline{Q}_2' = 3x_2^2 + 25x_2 + 26$
$Q_3' = 4x_3^2 + 27x_3 + 26$	$\underline{Q}_3' = 6x_3^2 + 35x_3 + 38$

Table 12: Equations derived by mapping l_1 points to l_2

We now have six more equations with integer coefficients, and it's natural to wonder at this point if there are analogs to Φ_3 and $\underline{\Phi}_3$, that is, arrays from whence these new coefficients too may be taken. Although at first glance it seems plausible, a closer look shows that it likely is not. For example, the formula (1.1) ensures that the a coefficient of Q_{j+1} always divides c of Q_j, a relationship that is seldom fulfilled in table 12. The discriminant and certain other qualities are unchanged by the $l_1 \rightarrow l_2$ mapping though; this property is explained in the third article in this series, DCL-Chemy III: Hyper-Quadratics.

Given arbitrary β and γ, an interesting pursuit could be to find a general method for determining when $D = \underline{D}$. If and when that is accomplished, there is the matter of finding a curve as in (1.5) that combines roots of the two Q-sets, or to determine if/when such a form exists. In many cases such as, say, $\beta[1,3,4]$, $\gamma[4,3,1]$ then $D \neq \underline{D}$. But in general, if β and γ are chosen as arithmetical sequences in $\mathbb{N}$, then apparently $D = \underline{D}$.

Take, for a numerical example, Φ_3 [2,4,6][1,2,3]; here $D = \underline{D}$ and the roots of Φ_3 and $\underline{\Phi}_3$ combine to form points on the graph of $2xy + x + y = 0$. Symmetric inversion of, say, Φ_3 [1,2,3][3,2,1] and Φ_3 [2,4,6][7,6,5] gives $D = \underline{D}$, but try finding a curve on which these roots reside. To restate a problem outlined above: Given Φ_λ and $\underline{\Phi}_\lambda$ with $D = \underline{D}$, is there always a version of (1.5), or a curve of any kind, that contains the points r_{j+}, $\underline{r}_{i-}$ and r_{j-}, $\underline{r}_{i+}$ on its line(s)? If so, given such a pair, how is the specific curve identified?

ADDENDA

Define the symmetric group of order n (S_n) as the set of the n! possible permutations (p) on n objects. Now map $\beta \rightarrow S_\lambda$, so as to generate $(\lambda - 1)!$ different arrays ($(\lambda - 1)!$ because each p gets cycled λ times). What patterns/relationships will emerge when *all* of these arrays are then considered as a set?

An infinite, aperiodic array is $\Phi_\infty[\mathbb{Z}][1]$, where $\mathbb{Z}$ = the ring of integers. This array is generated for $F_{n\geq 0}$ by a now familiar process, using as usual the initial terms $F_0 = 0$, $F_1 = 1$. For the column S_1 let $\beta = [1,2,3\ldots\infty]$, for S_2, $\beta = [2,3,4\ldots\infty]$ and etc. Then, for generation of columns leftwards of S_1: $S_0 = \beta[0,1,2,3\ldots\infty]$; $S_{-1} = \beta[-1,0,1,2,3\ldots\infty]$ and so forth. The result is

	S_{-3}	S_{-2}	S_{-1}	S_0	S_1	S_2	S_3	S_4	S_5
F_0	0	0	0	0	0	0	0	0	0
	1	1	1	1	**1**	1	1	1	1
	–3	–2	–1	0	1	2	3	4	5
	7	3	1	1	3	7	13	21	31
	–10	–2	0	2	10	30	68	130	222
F_5	7	1	1	7	43	157	421	931	1807
	–3	0	3	30	225	972	3015	7578	16485
	1	1	13	157	1393	6961	24541	69133	166657
	0	4	68	972	9976	56660	223884	698908	1849712
	1	21	421	6961	81201	516901	2263381	7757121	22363201

Table 13: $\Phi_\infty[\mathbb{Z}][1]$

This array is now reconfigured, using the numeral **1** as a pivot point.

1				7	–2	1	0	**1**	0	1	–2	7			
2			13	–3	1	0	**1**	**1**	**1**	0	1	–3	13		
3		21	–4	1	0	**1**	**2**	**3**	**2**	**1**	0	1	–4	21	
4	31	–5	1	0	**1**	**3**	**7**	**10**	**7**	**3**	**1**	0	1	–5	31
5	–6	1	0	**1**	**4**	**13**	**30**	**43**	**30**	**13**	**4**	**1**	0	1	–6
6	1	0	**1**	**5**	**21**	**68**	**157**	**225**	**157**	**68**	**21**	**5**	**1**	0	1
7	0	**1**	**6**	**31**	**130**	**421**	**972**	**1393**	**972**	**421**	**130**	**31**	**6**	**1**	0
8	**1**	**7**	**43**	**222**	**931**	**3015**	**6961**	**9976**	**6961**	**3015**	**931**	**222**	**43**	**7**	**1**
	7	6	5	4	3	2	1	0	–1	–2	–3	–4	–5	–6	–7

Table 14: Shifting the columns reveals the symmetry hidden in table 13

One result, distinguished by bold numbers, is a symmetrical triangle. Note that the columns in the triangle are, aside from the offset, the same as those in the table 13 array. Let a number in this field be identified (according to the numbers at the margin) as $N_{j,k}$, where j designates a row and k a column. Then we have $N_{j-2,k} + (j-1)N_{j-1,k} = N_{j,k} = N_{j,k+2} + (k+1)N_{j,k+1}$. So, a slightly variant form of the formula that produced Φ_∞ in table 13 now applies in the horizontal direction as well. This is reminiscent of table 8, which was created by a similar kind of shifting process,

And, indeed, a multiplicative relationship seen earlier in reference to table 8 also applies here:

d	*f*
g	*h*

Then $dh = fg \pm 1$ is true in this case as well. In closing, we extend the array in figure 14 from 8 to –8 in the vertical direction and highlight the zeros in bold.

-8	1	7	43	222	931	3015	6961	9976	6961	3015	931	222	43	7	1
-7	**0**	1	6	31	130	421	972	1393	972	421	130	31	6	1	**0**
-6	1	**0**	1	5	21	68	157	225	157	68	21	5	1	**0**	1
-5	-6	1	**0**	1	4	13	30	43	30	13	4	1	**0**	1	-6
-4	31	-5	1	**0**	1	3	7	10	7	3	1	**0**	1	-5	31
-3	-130	21	-4	1	**0**	1	2	3	2	1	**0**	1	-4	21	-130
-2	421	-68	13	-3	1	**0**	1	1	1	**0**	1	-3	13	-68	421
-1	-972	157	-30	7	-2	1	**0**	1	**0**	1	-2	7	-30	157	-972
0	1393	-225	43	-10	3	-1	1	**0**	1	-1	3	-10	43	-225	1393
1	-972	157	-30	7	-2	1	**0**	1	**0**	1	-2	7	-30	157	-972
2	421	-68	13	-3	1	**0**	1	1	1	**0**	1	-3	13	-68	421
3	-130	21	-4	1	**0**	1	2	3	2	1	**0**	1	-4	21	-130
4	31	-5	1	**0**	1	3	7	10	7	3	1	**0**	1	-5	31
5	-6	1	**0**	1	4	13	30	43	30	13	4	1	**0**	1	-6
6	1	**0**	1	5	21	68	157	225	157	68	21	5	1	**0**	1
7	**0**	1	6	31	130	421	972	1393	972	421	130	31	6	1	**0**
8	1	7	43	222	931	3015	6961	9976	6961	3015	931	222	43	7	1
	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7

Table 15: Extending the range of table 14

An obvious name for this configuration is an *X-array*. What sort of relationships or patterns can be teased out here? For another example, take $\beta = [\ldots -1,-1,-1,0,1,1,1\ldots]$. This gives the array below:

1	1	2	3	5	8	13	21	13	8	5	3	2	1	1
0	1	1	2	3	5	8	13	8	5	3	2	1	1	**0**
1	**0**	1	1	2	3	5	8	5	3	2	1	1	**0**	1
-1	1	**0**	1	1	2	3	5	3	2	1	1	**0**	1	-1
2	-1	1	**0**	1	1	2	3	2	1	1	**0**	1	-1	2
-3	2	-1	1	**0**	1	1	2	1	1	**0**	1	-1	2	-3
5	-3	2	-1	1	**0**	1	1	1	**0**	1	-1	2	-3	5
-8	5	-3	2	-1	1	**0**	1	**0**	1	-1	2	-3	5	-8
13	-8	5	-3	2	-1	1	**0**	1	-1	2	-3	5	-8	13
-8	5	-3	2	-1	1	**0**	1	**0**	1	-1	2	-3	5	-8
5	-3	2	-1	1	**0**	1	1	1	**0**	1	-1	2	-3	5
-3	2	-1	1	**0**	1	1	2	1	1	**0**	1	-1	2	-3
2	-1	1	**0**	1	1	2	3	2	1	1	**0**	1	-1	2
-1	1	**0**	1	1	2	3	5	3	2	1	1	**0**	1	-1
1	**0**	1	1	2	3	5	8	5	3	2	1	1	**0**	1
0	1	1	2	3	5	8	13	8	5	3	2	1	1	**0**
1	1	2	3	5	8	13	21	13	8	5	3	2	1	1

Table 16: Table 15 with a binary β

This is an interesting outcome. In a typical horizontal representation, the Fibonacci sequence has a center at zero and a left and right side. In Table 16, the sequence is 'bent' 90° at each zero, and the former 'sides' are now orthogonal to one another. The multiplication $(-1)\beta$ rotates it 180°; what does it do to the array?