ON THE DIOPHANTINE EQUATION $(X - Y)^m - XY = 0$

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Abstract. It is shown that the equation $(X - Y)^m - XY = 0$ has no integer solutions for $m \geq 4$.

The integer solutions to the equation

$$(X - Y)^m - XY = 0 \tag{1}$$

have been outlined for $m = 3$ by Bouhamida in sequences A045991 and A011379 of the Encylopedia of Integer Sequences [3]. The approach to other values of m is to introduce the auxiliary integer

$$k \equiv X - Y \tag{2}$$

and to consider the equation transformed from (1),

$$k^m - X(X - k) = 0, \tag{3}$$

which—considered a quadratic equation for X—

$$X^2 - kX - k^m = 0 \tag{4}$$

is solved as

$$X = \frac{k}{2}\left(1 \pm \sqrt{1 + 4k^{m-2}}\right). \tag{5}$$

This replaces the original problem by the question, whether perfect squares s^2 exist in the discriminant,

$$1 + 4k^{m-2} = s^2, \tag{6}$$

equivalent to

$$4k^{m-2} = (s + 1)(s - 1). \tag{7}$$

The left hand side is even, so this equation can only be solved if $s+1$ and $s-1$ are even, which we parametrize as

$$s = 2\sigma + 1. \tag{8}$$

This turns (7) into

$$k^{m-2} = \sigma(\sigma + 1). \tag{9}$$

According to Saradha [2] quoting Erdös and Selfridge [1], this does not have a solution in integers $m \geq 4$, $\sigma \geq 1$ and $k \geq 1$. (For even m, that is for squares on the right hand side, this was shown by Erdös and Rigge in 1939.) Since we are only interested in positive s, only these cases of $\sigma \geq 1$ are relevant, and this concludes the proof.

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References

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