Asymptotics of sequence A034691

(*Václav Kotěšovec, published Sep 09 2014*)

The generating function for the sequence A034691 in the OEIS is

$$U(x) = \prod_{k=1}^{\infty} \frac{1}{(1-x^k)^{2^{k-1}}}$$

Main result:

$$a_n \sim e^{\sum_{k=2}^{\infty} \frac{1}{k\,(2^k-2)}} * \frac{e^{\sqrt{2n}}\ 2^n}{\sqrt{2\pi}\ e^{1/4}\ 2^{3/4}\ n^{3/4}}$$

Proof:
We have Maclaurin series

$$\log(1-x) = -\sum_{k=1}^{\infty} \frac{x^k}{k}$$

$$\log(U(x)) = \log\left(\prod_{j=1}^{\infty} \frac{1}{(1-x^j)^{2^{j-1}}}\right) = -\sum_{j=1}^{\infty} 2^{j-1} * \log(1-x^j) = \sum_{j=1}^{\infty} 2^{j-1} \sum_{k=1}^{\infty} \frac{x^{jk}}{k} = \sum_{k=1}^{\infty} \frac{x^k}{k\,(1-2x^k)}$$

The saddle-point method is used, see [2], equation (12.9).

$$a_n \sim \frac{U(r_n)}{\sqrt{2\pi * b(r_n) * r_n^n}}$$

The saddle-point equation is

$$r_n * \frac{U'(r_n)}{U(r_n)} = n$$

$$x * \frac{U'(x)}{U(x)} = x * \frac{d}{dx} \log\,(U(x)) = x * \frac{d}{dx} \sum_{k=1}^{\infty} \frac{x^k}{k\,(1-2x^k)} = \sum_{k=1}^{\infty} \frac{x^k}{(1-2x^k)^2}$$

$$\sum_{k=1}^{\infty} \frac{r_n^k}{\left(1-2r_n^k\right)^2} = n$$

An asymptotic solution is (set $k = 1$)

```
Solve[r / (1 - 2 r)^2 == n, r]
{{r → -(-1 - 4 n + Sqrt[1 + 8 n])/(8 n)}, {r → (1 + 4 n + Sqrt[1 + 8 n])/(8 n)}}
```

The dominant root is

$$r_n \sim \frac{1}{2} - \frac{\sqrt{8n+1}}{8n} + \frac{1}{8n}$$

Now we compute

$$\frac{1}{r_n^n} \sim \frac{1}{\left(\frac{1}{2} - \frac{\sqrt{8n+1}}{8n} + \frac{1}{8n}\right)^n} \sim 2^n\, e^{\sqrt{n/2}}$$

It is important to note that taking only two terms the asymptotic expansion $\frac{1}{2} - \frac{1}{2\sqrt{n}}$ is insufficient, three terms are needed. An eventual term $n^{-3/2}$ can be ignored. We obtain:

```
Limit[1 / (1 / 2 - Sqrt[8 n + 1] / 8 / n + 1 / 8 / n) ^ n / (2 ^ n * E ^ (Sqrt[n / 2])), n → Infinity]
Limit[1 / (1 / 2 - Sqrt[8 n + 1] / 8 / n) ^ n / (2 ^ n * E ^ (Sqrt[n / 2])), n → Infinity]
Limit[1 / (1 / 2 - Sqrt[8 n + 1] / 8 / n + 1 / 8 / n + c / n ^ (3 / 2)) ^ n / (2 ^ n * E ^ (Sqrt[n / 2])), n → Infinity]

1

e^(1/4)

1
```

$$b(x) = \frac{x\,U'(x)}{U(x)} + \frac{x^2\,U''(x)}{U(x)} - \frac{x^2\,U'(x)^2}{U(x)^2} = \frac{x\,U'(x)}{U(x)} + x^2\left(\frac{d}{dx}\right)^2 \log\left(U(x)\right)$$

$$b(x) = \frac{x\,U'(x)}{U(x)} + x^2\left(\frac{d}{dx}\right)^2 \sum_{k=1}^{\infty} \frac{x^k}{k\,(1-2x^k)} = \frac{x\,U'(x)}{U(x)} + \sum_{k=1}^{\infty} \frac{x^k\,(2\,(k+1)\,x^k + k - 1)}{(1-2x^k)^3}$$

$$b(r_n) \sim n + \sum_{k=1}^{\infty} \frac{r_n^k\,(2\,(k+1)\,r_n^k + k - 1)}{(1-2\,r_n^k)^3}$$

$$b(r_n) \sim n + \frac{4r_n^2}{(1-2\,r_n)^3} + \sum_{k=2}^{\infty} \frac{r_n^k\,(2\,(k+1)\,r_n^k + k - 1)}{(1-2\,r_n^k)^3}$$

For $k > 1$ the sum tends to a constant as *n* tends to infinity

```
FullSimplify[
 Limit[r^k * (2 (k + 1) r^k + k - 1) / (1 - 2 r^k) ^ 3 /.
   r → (1 / 2 - Sqrt[8 n + 1] / 8 / n + 1 / 8 / n), n → Infinity]]

2^k (2^k (-1 + k) + 2 (1 + k))
------------------------------
         (-2 + 2^k)^3
```

```
N[Sum[2^k (2^k (-1 + k) + 2 (1 + k)) / (-2 + 2^k)^3, {k, 2, Infinity}], 14]

6.5966596802914
```

$$\sum_{k=2}^{\infty} \frac{r_n^k\,(2\,(k+1)\,r_n^k + k - 1)}{(1-2\,r_n^k)^3} \sim \sum_{k=2}^{\infty} \frac{2^k(2^k(k-1) + 2(k+1))}{(2^k-2)^3} = c = \ 6.596659680291\ldots$$

If $k = 1$ then we obtain

$$\frac{4r_n^2}{(1-2\,r_n)^3} \sim n\left(\sqrt{8n+1} - 1\right)$$

Together

$$b(r_n) \sim n + n\left(\sqrt{8n+1} - 1\right) + c \sim (2n)^{3/2}$$

We have

$$U(r_n) = e^{\log(U(r_n))} = e^{\sum_{k=1}^{\infty} \frac{r_n^k}{k\,(1-2r_n^k)}}$$

$$\sum_{k=1}^{\infty} \frac{r_n^k}{k\,(1-2r_n^k)} = \frac{r_n}{(1-2r_n)} + \sum_{k=2}^{\infty} \frac{r_n^k}{k\,(1-2r_n^k)}$$

Contribution of the first term is

```
FullSimplify[r^k / (1 - 2 r^k) /. {r → (1 / 2 - Sqrt[8 n + 1] / 8 / n + 1 / 8 / n), k → 1}]
1/4 (-1 + Sqrt[1 + 8 n])
```

$$\frac{r_n}{(1-2r_n)} \sim \sqrt{n/2} - 1/4$$

```
Simplify[
 Limit[r^k / (1 - 2 r^k) / k /.
   {r → (1 / 2 - Sqrt[8 n + 1] / 8 / n + 1 / 8 / n)}, n → Infinity]]
1/((-2 + 2^k) k)
```

$$\frac{r_n^k}{k\,(1-2r_n^k)} \sim \frac{1}{k\,(2^k-2)}$$

$$U(r_n) = e^{\sum_{k=1}^{\infty} \frac{r_n^k}{k\,(1-2r_n^k)}} \sim e^{\sqrt{n/2} - 1/4 + \sum_{k=2}^{\infty} \frac{1}{k\,(2^k-2)}}$$

Constant A247003

$$e^{\sum_{k=2}^{\infty} \frac{1}{k\,(2^k-2)}} = 1.39764900508365028506507459852679115900781142944 \ldots$$

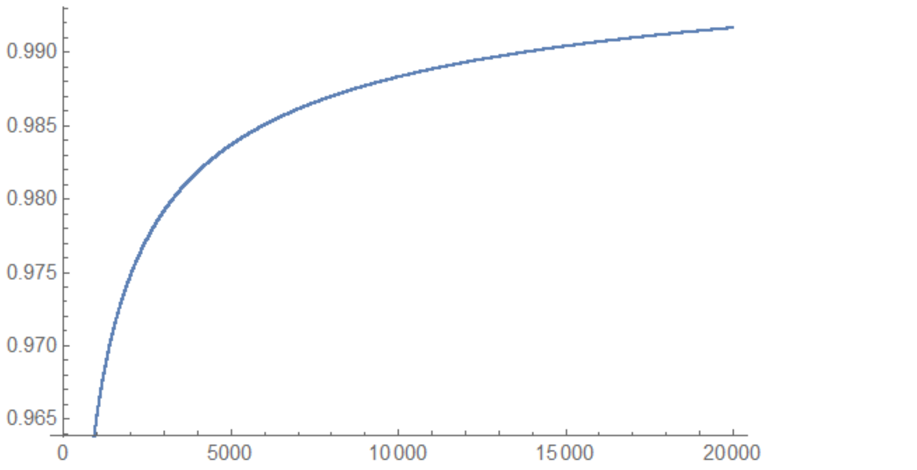
The final asymptotic is

$$a_n \sim \frac{U(r_n)}{\sqrt{2\pi * b(r_n) * r_n^n}} = \frac{e^{\sqrt{n/2} - 1/4 + \sum_{k=2}^{\infty} \frac{1}{k\,(2^k-2)}}}{\sqrt{2\pi * (2n)^{3/2}}} * 2^n\, e^{\sqrt{n/2}} = e^{\sum_{k=2}^{\infty} \frac{1}{k\,(2^k-2)}} * \frac{e^{\sqrt{2n}}\ 2^n}{\sqrt{2\pi}\ e^{1/4}\, 2^{3/4}\, n^{3/4}}$$

Note that the asymptotic formula in the article [1] (Theorem 3) is incorrect!

Numerical verification (for 20000 terms), ratio tends to 1**:**

```
ListPlot[
 Table[
  N[A034691[[n]] /
    (Exp[Sum[1 / (k * (2^k - 2)), {k, 2, Infinity}]] *
      2^n * E^(Sqrt[2 * n]) /
       (Sqrt[2 * Pi] * E^(1 / 4) * 2^(3 / 4) * n^(3 / 4)))],
  {n, 1, Length[A034691]}]]
```



References:

[1] N. J. A. Sloane and Thomas Wieder, The Number of Hierarchical Orderings, Order 21 (2004), 83-89

[2] A. M. Odlyzko, Asymptotic enumeration methods, pp. 1063-1229 of R. L. Graham et al., eds., Handbook of Combinatorics, 1995

Saddle point approximation

$$[z^n]f(z) \sim (2\pi b(r_0))^{-1/2} f(r_0) r_0^{-n} \quad \text{as} \quad n \to \infty\,, \tag{12.9}$$

where r_0 is the saddle point (where $r^{-n}f(r)$ is minimized, so that $r_0 f'(r_0)/f(r_0) = n$) and

$$b(r) = r\frac{f'(r)}{f(r)} + r^2\frac{f''(r)}{f(r)} - r^2\left(\frac{f'(r)}{f(r)}\right)^2 = r\left(r\frac{f'(r)}{f(r)}\right)'\,. \tag{12.10}$$

[3] OEIS - The On-Line Encyclopedia of Integer Sequences

This article was published on the website http://web.telecom.cz/vaclav.kotesovec/math.htm, 9.9.2014
and in the OEIS, Sep 09 2014