NOTES ON STEPHAN'S CONJECTURES 72, 73, AND 74

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Recently Stephan [3] posted 117 conjectures based on an extensive analysis of the On-line Encyclopedia of Integer Sequences [1, 2]. Here we give entirely elementary proofs of (slightly corrected forms of) conjectures 72, 73, and 74.

All three of these conjectures concern the number of "non-palindromic reversible strings," although the restrictions on the characters included vary. How should we interpret this phrase? Note that the group $\mathbb{Z}/2\mathbb{Z}$ acts on any (reasonable) set of strings by mapping the non-trivial element to reversal of strings:

$$a_1a_2\dots a_k \mapsto a_ka_{k-1}\dots a_1.$$

A "non-palindromic reversible string" is an orbit of size two under this action; palindromes, of course, generate orbits of size one.

All three proofs follow an extremely simple outline: first, count both the strings and the palindromes in the set. Then subtract the one result from the other and divide by 2.

Proposition 1 (Conjecture 72). *The number of non-palindromic reversible strings with n beads of 4 possible colors is*

$$\begin{cases} \frac{1}{2}\left(4^n - 2^n\right) & n \text{ even,} \\ \frac{1}{2}\left(4^n - 2^{n+1}\right) & n \text{ odd.} \end{cases}$$

Remark. The original conjecture [3] asserts that there should be 4 such strings when $n = 1$, when in fact there are none; every one-letter string is a palindrome.

Proof. There are 4^n strings total.

When n is even, there are $4^{n/2} = 2^n$ palindromes (the first $n/2$ characters determine the rest of the string).

When n is odd, there are $4^{(n+1)/2} = 2^{n+1}$ palindromes (the center character may be freely chosen, while the first $(n-1)/2$ characters determine the last $(n-1)/2$ characters). □

Proposition 2 (Conjecture 73). *The number of non-palindromic reversible strings with $n-1$ beads, of which 4 are black and $n-5$ white, is*

$$\begin{cases} \frac{1}{48}\left(n^4 - 10n^3 + 32n^2 - 38n + 15\right) & n \text{ odd,} \\ \frac{1}{48}\left(n^4 - 10n^3 + 32n^2 - 32n\right) & n \text{ even.} \end{cases}$$

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Remark. The formulas for the odd and even cases have been switched from the original conjecture [3].

Proof. There are $\binom{n-1}{4}$ strings total.

When n is odd, $n-1$ is even, and the first $(n-1)/2$ beads of a palindrome determine the rest of the string. Exactly two of these beads must be black, so there are $\binom{(n-1)/2}{2}$ palindromes.

When n is even, $n-1$ is odd. The center bead of any such palindrome must be white. The first $(n-2)/2$ beads determine the rest of the palindrome; of these, exactly two must be black, so there are $\binom{(n-2)/2}{2}$ palindromes.

Fortunately,

$$\frac{1}{2}\left(\binom{n-1}{4}-\binom{(n-1)/2}{2}\right)=\frac{1}{48}\left(n^4-10n^3+32n^2-38n+15\right)$$

and

$$\frac{1}{2}\left(\binom{n-1}{4}-\binom{(n-2)/2}{2}\right)=\frac{1}{48}\left(n^4-10n^3+32n^2-32n\right).$$

□

Proposition 3 (Conjecture 74)**.** *The number of non-palindromic reversible strings with n black beads and $n-1$ white beads is*

$$\begin{cases}\frac{1}{4}\left(\binom{2n}{n}-\binom{n}{n/2}\right) & n \text{ even},\\ \frac{1}{2}\left(\binom{2n-1}{n-1}-\binom{n-1}{(n-1)/2}\right) & n \text{ odd}.\end{cases}$$

Remark. The original conjecture [3] asserts that there should be 1 such string when $n=1$, when in fact there are none; every one-letter string is a palindrome.

Remark. We have used $\binom{2n}{n}=2\binom{2n-1}{n-1}$ to slightly simplify the n odd case of the conjecture.

Proof. First, consider the case where $n=2k$ is even. Then there are $\binom{4k-1}{2k-1}$ strings, in total. Any such palindrome has a white center bead. The first $2k-1$ beads determine the rest of the string; of those, $k-1$ must be white, so there are $\binom{2k-1}{k-1}$ palindromes. Fortunately,

$$\begin{aligned}\frac{1}{2}\left(\binom{4k-1}{2k-1}-\binom{2k-1}{k-1}\right)&=\frac{1}{4}\left(\frac{4k}{2k}\binom{4k-1}{2k-1}-\frac{2k}{k}\binom{2k-1}{k-1}\right)\\&=\frac{1}{4}\left(\binom{2n}{n}-\binom{n}{n/2}\right).\end{aligned}$$

When $n = 2k+1$ is odd, there are $\binom{4k+1}{2k}$ strings in total. The center of each palindrome is now black, and there will be $\binom{2k}{k}$ palindromes. Fortunately,

$$\frac{1}{2}\left(\binom{4k+1}{2k} - \binom{2k}{k}\right) = \frac{1}{2}\left(\binom{2n-1}{n-1} - \binom{n-1}{(n-1)/2}\right).$$

□

References

[1] Sloane, N. J. A. The On-Line Encyclopedia of Integer Sequences, published electronically at http://www.research.att.com/ñjas/sequences/, 2004.

[2] Sloane, N. J. A. The on-line encyclopedia of integer sequences. *Notices Amer. Math. Soc.* **50** (2003), pp. 912–915.

[3] Stephan, Ralf. Prove or Disprove: 100 Conjectures from the OEIS. arXiv:math.CO/0409509

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