A note on A018248

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NAME: The 10-adic integer $x = ...1787109376$ satisfies $x^2 = x$.

DATA: $6, 7, 3, 9, 0, 1, 7, 8, 7, 1, 8, 0, 0, 4, 7, 3, ...$

The table below shows the trailing digits of the integers 2^{10^n}, 4^{10^n} and 6^{10^n}, with the final $n+1$ digits in bold.

n		2^{10^n}		4^{10^n}		6^{10^n}
2		..03205**376**		...35301**376**		...41477**376**
3		...6806**9376**		...4902**9376**		...1078**9376**
4		...967**09376**		...63**09376**		...239**09376**
5		...83**109376**		...79**109376**		...55**109376**
6		...4**7109376**		...0**7109376**		...6**7109376**
7		...**87109376**		...**87109376**		...**87109376**

Claim. For $n >= 2$, the final $n+1$ digits of either 2^{10^n}, 4^{10^n} or 6^{10^n}, when read in reverse order, give the first $n+1$ entries in A018248.
The proof is an easy consequence of the following result due to Euler: *the congruence*

$$a^{p^r} \equiv a^{p^{r-1}} \ (mod\ p^r) \tag{1}$$

holds for all integers $a \in \mathbb{Z}$, for all primes p and all positive integers r.
a) Let $n \geq 2$. First we show that the integers 2^{10^n} and $2^{10^{n+1}}$ have the same final $n+1$ decimal digits, that is,

$$2^{10^{n+1}} \equiv 2^{10^n} \left(\text{mod } 10^{n+1}\right) \tag{2}$$

or, equivalently,

$$2^{10^n}\left(2^{9(10^n)} - 1\right) \equiv 0 \left(\text{mod } 2^{n+1}5^{n+1}\right).$$

Clearly, 2^{n+1} divides 2^{10^n}. Thus to prove (2) it suffices to show that

$$2^{9(10^n)} - 1 \equiv 0 \left(\text{mod } 5^{n+1}\right). \tag{3}$$

Setting $a = 2^m$, m a nonnegative integer, $r = n+1$ and $p = 5$ in Euler's congruence (1) yields

$$2^{m5^{n+1}} \quad \equiv \quad 2^{m5^n} \left(\text{mod } 5^{n+1}\right)$$

leading to

$$2^{m5^n}\left(2^{4m(5^n)} - 1\right) \equiv 0 \left(\text{mod } 5^{n+1}\right)$$

and hence

$$2^{4m(5^n)} - 1 \equiv 0 \pmod{5^{n+1}}. \tag{4}$$

Setting $m = \dfrac{9\,(2^n)}{4}$ (an integer for $n \geq 2$) in (4) yields (3) and thus establishes (2).

An immediate consequence of this result is that

$$x := \text{the 10-adic limit}\{n \to \infty\}\, 2^{10^n} \text{mod } 10^n$$

is a well-defined 10-adic integer.

b) Still with $n >= 2$, we show next that the integers 2^{10^n} and 4^{10^n} have the same final $n+1$ decimal digits.

Put $m = \dfrac{2^n}{4}$ in (4) to find

$$2^{10^n} - 1 \equiv 0 \pmod{5^{n+1}}. \tag{5}$$

Multiplying the congruence (5) by 2^{10^n} we see that

$$4^{10^n} - 2^{10^n} \equiv 0 \pmod{10^{n+1}}. \tag{6}$$

Thus the integers 2^{10^n} and 4^{10^n} have the same final $n+1$ decimal digits.
It follows from (6) that

$$x^2 = \text{the 10-adic limit\_}\{n \to \infty\}\, 4^{10^n} = \text{the 10-adic limit\_}\{n \to \infty\}\, 2^{10^n} = x.$$

Therefore, x is an idempotent in the ring of 10-adic integers (with its rightmost digit equal to 6) and so must be A018248 (the other 3 idempotents being $0, 1$ and A018247 $= 1 - x = \ldots 18212890625$).
c) By an argument similar to that which proved (5) we can show that

$$3^{10^n} - 1 \equiv 0 \pmod{5^{n+1}}. \tag{7}$$

Multiplying (7) by 2^{10^n} leads to the congruence

$$6^{10^n} - 2^{10^n} \equiv 0 \pmod{10^{n+1}}, \tag{8}$$

showing that the integers 2^{10^n} and 6^{10^n} also have the same final $n+1$ decimal digits.