New series for old functions

Peter Bala

pbala@talktalk.net

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1 Introduction

One way to derive Mercator's series for the natural logarithm function starts from the integral expression

(1.1) $$\log(1+x)=\int_0^1 \frac{x}{1+xt}\,dt.$$

Expanding the integrand as a Taylor series in t and integrating term by term yields Mercator's expansion

$$\log(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\cdots.$$

We can get more rapidly converging series for $\log(1+x)$ by a simple modification of the above approach; the idea is to rewrite the integrand in (1.1) before carrying out the expansion and term by term integration.

Example

The quadratic polynomial $1+\frac{x^2}{1+x}t(1-t)$ in t vanishes when $t=-\frac{1}{x}$, and hence is divisble by the linear polynomial $1+xt$ in t. It follows that the quotient

$$\frac{1+\frac{x^2}{1+x}t(1-t)}{1+xt}=\frac{x}{1+x}(1+x-xt),$$

a linear polynomial in t. We can thus write the integrand of (1.1) in the form

$$\frac{x}{1+xt}=\frac{x^2}{1+x}\left\{\frac{1+x-xt}{1+\frac{x^2}{1+x}t(1-t)}\right\}.$$

Integrating both sides between 0 and 1 gives

$$\begin{aligned}\log(1+x)&=\int_0^1\frac{x}{1+xt}dt\\ &=\frac{x^2}{1+x}\int_0^1\left\{\frac{1+x-xt}{1+\frac{x^2}{1+x}t(1-t)}\right\}dt\\ &=\frac{x^2}{1+x}\sum_{k=0}^{\infty}\left(\frac{-x^2}{1+x}\right)^k\int_0^1(1+x-xt)t^k(1-t)^k dt.\end{aligned}$$

The integrals in the series can be evaluated making use of the beta function result

(1.2) $$\mathrm{B}(p+1,q+1) \;=\; \int_0^1 t^p(1-t)^q dt \;=\; \frac{p!q!}{(p+q+1)!},$$

where p and q are nonnegative integers. After a short calculation we obtain the expansion

$$\log(1+x) = \frac{x+2}{x}\sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{x^2}{1+x}\right)^n \frac{1}{n\binom{2n}{n}},$$

which converges provided $\left|\frac{x^2}{1+x}\right| < 4.$

More generally, if m and n are nonnegative integers we can write the integrand of (1.1) in the form

(1.3) $$\frac{x}{1+xt} = \left\{\frac{P_{m,n}(t)}{1+R_{m,n}(x)t^m(1-t)^n}\right\},$$

where $P_{m,n}(t)$ is a polynomial in t (with coefficients rational functions in x) and

$$R_{m,n}(x) = (-1)^{m+1}\frac{x^{m+n}}{(1+x)^n}.$$

Integrating both sides of (1.3) between 0 and 1 gives

$$\begin{aligned}\log(1+x) &= \int_0^1 \frac{x}{1+xt}dt \\ &= \int_0^1 \left\{\frac{P_{m,n}(t)}{1+R_{m,n}(x)t^m(1-t)^n}\right\}dt \\ &= \sum_{k=0}^{\infty}\left(-R_{m,n}(x)\right)^k \int_0^1 P_{m,n}(t)t^{mk}(1-t)^{nk}dt\end{aligned}$$

The integrals can be evaluated using (1.2) to produce a series expansion for $\log(1+x)$. Some examples of these expansions for small values of m and n are listed in the next section.

Once we have these new expansions for the logarithmic function $\log(1+x)$ we can obtain new series expansions for the inverse tangent function $\tan^{-1}(x)$ and the functions $\frac{\sin^{-1}(x)}{\sqrt{1-x^2}}$ and $\left(\sin^{-1}(x)\right)^2$ by means of the relations

$$\tan^{-1}(x) = \frac{i}{2}\left(\log(1-ix) - \log(1+ix)\right), \tag{1.4}$$

$$\frac{\sin^{-1}(x)}{\sqrt{1-x^2}} = \frac{1}{\sqrt{1-x^2}}\tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right), \tag{1.5}$$

and

$$\left(\sin^{-1}(x)\right)^2 = \int_0^x \frac{\sin^{-1}(t)}{\sqrt{1-t^2}}\,dt. \tag{1.6}$$

Some examples are listed in Section 4.

It turns out that the expansions we obtain by this method for the functions $\left(\sin^{-1}(x)\right)^2$ and $\frac{\sin^{-1}(x)}{\sqrt{1-x^2}}$ are in fact power series in x, and hence must be just the usual Maclaurin expansions for these functions in a disguised form. Nevertheless, these equivalent expansions are useful for finding new series for the constants π, $\zeta(2)$, $\zeta(3)$, $\zeta(4)$ and Catalan's constant G. Some examples of these new representations for these constants may be found in Sections 5 through 9.

2 Series expansions for log(1+*x*)

(2.1) $$\log(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(2+x)}{n\binom{2n}{n}} \frac{x^{2n-1}}{(1+x)^n} \qquad (m=n=1)$$

(2.2) $$\log(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{n(9+15x+4x^2)-(3+6x+2x^2)}{2n(2n-1)\binom{3n}{n}} \frac{x^{3n-2}}{(1+x)^{2n}} \qquad (m=1, n=2)$$

(2.3) $$\log(1+x) = \sum_{n=1}^{\infty} \frac{n(9+3x-2x^2)-(3-x^2)}{2n(2n-1)\binom{3n}{n}} \frac{x^{3n-2}}{(1+x)^{n}} \qquad (m=2, n=1)$$

(2.4) $$\log(1+x) = \sum_{n=1}^{\infty} \frac{(x^3-6x-4)-2n(x^3-2x^2-12x-8)}{2n(2n-1)\binom{4n}{2n}} \frac{x^{4n-3}}{(1+x)^{2n}} \qquad (m=n=2)$$

(2.5) $$\log(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\mathrm{P}(n,x)}{3n(3n-1)(3n-2)\binom{4n}{n}} \frac{x^{4n-3}}{(1+x)^{3n}} \qquad (m=1, n=3)$$

where $\mathrm{P}(n,x) = n^2(64+176x+148x^2+27x^3) - n(48+140x+128x^2+27x^3)$
$+ 2(4+12x+12x^2+3x^3)$

(2.6) $$\log(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\mathrm{P}(n,x)}{3n(3n-1)(3n-2)\binom{4n}{n}} \frac{x^{4n-3}}{(1+x)^{n}} \qquad (m=3, n=1)$$

where $\mathrm{P}(n,x) = n^2(64+16x-12x^2+9x^3) - n(48+4x-8x^2+9x^3) + 2(4+x^3)$

3 Series for log(2)

We can obtain an endless supply of rapidly converging series for log(2) by specialising these generalised expansions for log(1+*x*). Here are some typical results:

(3.1) $$\log(2) = \frac{3}{4}\sum_{n=0}^{\infty}\frac{(-1)^n}{2^n(2n+1)\frac{(2n)!}{n!^2}}$$

(3.2) $$\log(2) = \frac{3}{16}\sum_{n=0}^{\infty}\frac{14n+11}{(4n+1)(4n+3)\binom{4n}{2n}}\frac{1}{4^n}$$

(3.3) $$\log(2) = \frac{9}{64}\sum_{n=0}^{\infty}(-1)^n\frac{171n^2+231n+74}{(6n+1)(6n+3)(6n+5)\binom{6n}{3n}}\frac{1}{8^n}$$

(3.4) $$\log(2) = \frac{3}{128}\sum_{n=0}^{\infty}\frac{14560n^3+27504n^2+16466n+3105}{(8n+1)(8n+3)(8n+5)(8n+7)\binom{8n}{4n}}\frac{1}{16^n}$$

(3.5) $$\log(2) = \frac{1}{3}\sum_{n=0}^{\infty}\frac{5n+4}{(3n+1)(3n+2)\binom{3n}{n}}\frac{1}{2^n}$$

(3.6) $$\log(2) = \frac{1}{12}\sum_{n=0}^{\infty}(-1)^n\frac{28n+17}{(3n+1)(3n+2)\binom{3n}{n}}\frac{1}{4^n}$$

(3.7) $$\log(2) = \frac{1}{8}\sum_{n=0}^{\infty}(-1)^n\frac{77n^2+101n+34}{(4n+1)(4n+2)(4n+3)\binom{4n}{n}}\frac{1}{2^n}$$

(3.8) $$\log(2) = \frac{1}{32}\sum_{n=0}^{\infty}(-1)^n\frac{415n^2+487n+134}{(4n+1)(4n+2)(4n+3)\binom{4n}{n}}\frac{1}{8^n}$$

(3.9) $$\log(2) = \frac{1}{12}\sum_{n=0}^{\infty}(-1)^n \frac{34n+25}{(4n+1)(4n+3)\binom{4n}{2n}}\frac{1}{18^n}$$

(3.10) $$\log(2) = \frac{1}{108}\sum_{n=0}^{\infty}(-1)^n \frac{2800n^2+3680n+1123}{(6n+1)(6n+3)(6n+5)\frac{(6n)!(2n)!}{(4n)!(3n)!n!}}\frac{1}{162^n}$$

(3.11) $$\log(2) = \frac{1}{432}\sum_{n=0}^{\infty}\frac{156128n^3+291728n^2+171658n+31441}{(8n+1)(8n+3)(8n+5)(8n+7)\binom{8n}{4n}}\frac{1}{324^n}$$

(3.12) $$\log(2) = \frac{1}{972}\sum_{n=0}^{\infty}(-1)^n \frac{361944n^3+672036n^2+391770n+70743}{(8n+1)(8n+3)(8n+5)(8n+7)\frac{(8n)!(3n)!}{(6n)!(4n)!n!}}\frac{1}{1458^n}$$

The first 10 terms of this last series gives a value for log(2) which is correct in the first 46 decimal places.

4 Series for inverse trigonometric functions

(4.1) $$\tan^{-1} x = \frac{x}{(1+x^2)} \sum_{n=0}^{\infty} \frac{1}{(2n+1)\binom{2n}{n}} \left(\frac{4x^2}{1+x^2}\right)^n \qquad \text{(Euler)}$$

(4.2) $$\tan^{-1} x = \frac{x}{(1+x^2)^2} \sum_{n=0}^{\infty} \frac{4n(1+2x^2) + 3 + 5x^2}{(4n+1)(4n+3)\binom{4n}{2n}} \left(\frac{4x^2}{1+x^2}\right)^{2n}$$

(4.3) $$\tan^{-1} x = \frac{x}{(1+x^2)^3} \sum_{n=0}^{\infty} \frac{P(n,x)}{(6n+1)(6n+3)(6n+5)\binom{6n}{3n}} \left(\frac{4x^2}{1+x^2}\right)^{3n}$$

where $P(n,x) = 36n^2(1+3x^2+3x^4) + 6n(8+23x^2+21x^4) + 15 + 40x^2 + 33x^4$

(4.4) $$\tan^{-1} x = \frac{x}{(1+x^2)^4} \sum_{n=0}^{\infty} \frac{P(n,x)}{(8n+1)(8n+3)(8n+5)(8n+7)\binom{8n}{4n}} \left(\frac{4x^2}{1+x^2}\right)^{4n}$$

where $P(n,x) = 512n^3(1+4x^2+6x^4+4x^6) + 64n^2(15+59x^2+86x^4+54x^6)$
$+ 8n(71+272x^2+381x^4+224x^6) + 105 + 385x^2 + 511x^4 + 279x^6$

(4.5) $$\tan^{-1} x = \frac{x}{(1+x^2)} \sum_{n=0}^{\infty} (-1)^n \frac{n(4+2x^2) + 3 + 2x^2}{(4n+1)(4n+3)\binom{4n}{2n}} \left(\frac{4x^4}{1+x^2}\right)^n$$

(4.6) $$\tan^{-1} x = \frac{x}{(1+x^2)^2} \sum_{n=0}^{\infty} \frac{P(n,x)}{(8n+1)(8n+3)(8n+5)(8n+7)\binom{8n}{4n}} \left(\frac{4x^4}{1+x^2}\right)^{2n}$$

where $P(n,x) = 64n^3(8+12x^2+2x^4-x^6) + 16n^2(60+92x^2+19x^4-7x^6)$
$+ 4n(142+225x^2+58x^4-14x^6) + (105+175x^2+56x^4-8x^6)$

Inverse sine

(4.7) $$(\sin^{-1} x)^2 = \frac{1}{2}\sum_{n=1}^{\infty} \frac{(2x)^{2n}}{n^2 \binom{2n}{n}} \qquad \text{(Euler)}$$

(4.8) $$\frac{\sin^{-1} x}{\sqrt{1-x^2}} = \sum_{n=1}^{\infty} \frac{(2x)^{2n-1}}{n \binom{2n}{n}}$$

(4.9) $$(\sin^{-1} x)^2 = \sum_{n=1}^{\infty} \frac{8n^2(1+x^2) - 2n(1+4x^2) + 2x^2}{(2n(2n-1))^2 \binom{4n}{2n}} (2x)^{4n-2}$$

(4.10) $$\frac{\sin^{-1} x}{\sqrt{1-x^2}} = \sum_{n=1}^{\infty} \frac{8n(1+x^2) - 2(1+2x^2)}{2n(2n-1) \binom{4n}{2n}} (2x)^{4n-3}$$

(4.11) $$(\sin^{-1} x)^2 = \sum_{n=1}^{\infty} \frac{\mathrm{P}(n,x)}{(3n(3n-1)(3n-2))^2 \binom{6n}{3n}} (2x)^{6n-4},$$

where $\mathrm{P}(n,x) = 648n^4(1+x^2+x^4) - 324n^3(2+3x^2+4x^4)$
$+18n^2(11+24x^2+52x^4) - 6n(3+8x^2+48x^4) + 32x^4$

(4.12) $$\frac{\sin^{-1} x}{\sqrt{1-x^2}} = \sum_{n=1}^{\infty} \frac{\mathrm{P}(n,x)}{(3n(3n-1)(3n-2)) \binom{6n}{3n}} (2x)^{6n-5},$$

where $\mathrm{P}(n,x) = 144n^2(1+x^2+x^4) - 24n(4+5x^2+6x^4) + 4(3+4x^2+8x^4)$

5 A collection of results relating to π

$\boldsymbol{\pi \neq \frac{22}{7}}$

(5.1) $$\frac{22}{7} - \pi = 240 \sum_{n=1}^{\infty} \frac{1}{(4n+1)(4n+2)(4n+3)(4n+5)(4n+6)(4n+7)}$$

This is a series companion formula to the integral result of Dalzell,

(5.2) $$\frac{22}{7} - \pi = \int_0^1 \frac{x^4(1-x)^4}{1+x^2}\, dx.$$

Dalzell, D. P. "On 22/7." *J. London Math. Soc.* 19, 133-134, 1944.

If we expand the integrand in (5.2) into a series and integrate term by term we obtain the alternating series

(5.3) $$\frac{22}{7} - \pi = 24 \sum_{n=2}^{\infty} \frac{(-1)^n}{(2n+1)(2n+2)(2n+3)(2n+4)(2n+5)} .$$

It is also interesting to note that

(5.4) $$\frac{22}{7} + \pi = -240 \sum_{n=-2}^{\infty} \frac{1}{(4n+1)(4n+2)(4n+3)(4n+5)(4n+6)(4n+7)} .$$

A list of series for π derived from the new inverse tangent and inverse sine function expansions

(5.5) $$\pi = \sum_{n=0}^{\infty} \frac{10n+6}{(3n+1)(3n+2)\binom{3n}{2n}} \frac{1}{2^n}$$

(5.6) $$\pi = 2\sum_{n=0}^{\infty} \frac{187n^3+342n^2+201n+38}{(5n+1)(5n+2)(5n+3)(5n+4)\binom{5n}{2n}} \frac{(-1)^n}{2^n}$$

(5.7) $$\pi = 2\sum_{n=0}^{\infty} \frac{820n^3+1533n^2+902n+165}{(8n+1)(8n+3)(8n+5)(8n+7)\binom{8n}{4n}} \frac{(-1)^n}{4^n}$$

(5.8) $$\pi = 2\sum_{n=0}^{\infty} \frac{6n+5}{(4n+1)(4n+3)\binom{4n}{2n}} (-2)^n$$

(5.9) $$\pi = \sum_{n=0}^{\infty} \frac{3n+2}{(4n+1)(4n+3)\binom{4n}{2n}} 4^{n+1}$$

(5.10) $$\pi = \frac{8}{3}\sum_{n=0}^{\infty} \frac{7n+6}{(6n+1)(6n+5)\frac{(6n)!n!}{(3n)!(2n)!(2n)!}} (-4)^n$$

(5.11) $$\frac{\pi}{\sqrt{3}} = \frac{1}{2}\sum_{n=0}^{\infty} \frac{14n+11}{(4n+1)(4n+3)\binom{4n}{2n}} \frac{(-1)^n}{3^n}$$

(5.12) $$\frac{\pi}{\sqrt{3}} = \frac{1}{6}\sum_{n=0}^{\infty} \frac{352n^2+488n+163}{(6n+1)(6n+3)(6n+5)\frac{(6n)!(2n)!}{(4n)!(3n)!n!}} \frac{1}{9^n}$$

(5.13) $$\frac{\pi}{\sqrt{3}} = \frac{1}{12}\sum_{n=0}^{\infty} \frac{10528n^3+19984n^2+12038n+2285}{(8n+1)(8n+3)(8n+5)(8n+7)\binom{8n}{4n}} \frac{1}{9^n}$$

6 Results for $\zeta(2)$

Recall

$$\zeta(2) := \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6}.$$

Here are a variety of formulas for $\zeta(2)$.

As a limit

(6.1) $$\zeta(2) = \lim_{n\to\infty} \sum_{k=1}^{n-1} \frac{\ln(n)-\ln(k)}{n-k}$$

(6.2) $$\zeta(2) = \lim_{n\to\infty} \frac{np(n)}{\sum_{k=1}^{n} kp(n-k)},$$

where p(n) counts the number of partitions of n - sequence A000041 in Sloane's Online Encyclopedia of Integer Sequences.

As an integral

(6.3) $$\zeta(2) = \int_0^1 (x\wedge -x)\wedge(x\wedge -x)\wedge(x\wedge -x)\ldots\, dx$$

where the notation $a\wedge b\wedge c\wedge\ldots$ denotes the tower of powers $a^{b^{c^{\cdot^{\cdot^{\cdot}}}}}$.

An interesting series

(6.4) $$\zeta(2) = 1 + 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2(1+n^2+n^4)}.$$

Maple can evaluate this sum but can't evaluate the companion result for Napier's constant

(6.5) $$e = 2\sum_{n=1}^{\infty} \frac{1}{n!(1+n^2+n^4)}.$$

Leonhard Euler (1707 - 1783)

In celebration of Euler's tercentenary we offer the amusing

(6.6) $$\zeta(2) = \frac{4}{3}\int_0^{\frac{\pi}{2}} \tan^{-1}(\tan^{1707}(\tan^{-1}(\tan^{1783}(x))))\, dx$$

Further series for ζ(2)

Putting $x = \frac{1}{2}$ in the expansion $(\sin^{-1} x)^2 = \frac{1}{2}\sum_{n=1}^{\infty} \frac{(2x)^{2n}}{n^2 \binom{2n}{n}}$ results in the well-known series

$$\zeta(2) = \frac{\pi^2}{6} = 3\sum_{n=1}^{\infty} \frac{1}{n^2 \binom{2n}{n}} .$$

Using the new representations for $(\sin^{-1} x)^2$ given in this website produces an infinite sequence of faster and faster converging series for $\zeta(2)$, which continues with

(6.7) $$\frac{\pi^2}{6} = 3\sum_{n=1}^{\infty} \frac{20n^2 - 8n + 1}{(2n(2n-1))^2 \binom{4n}{2n}} ,$$ (see Mohammed [(1)], Example 4)

(6.8) $$\frac{\pi^2}{6} = 3\sum_{n=1}^{\infty} \frac{1701n^4 - 1944n^3 + 729n^2 - 96n + 4}{(3n(3n-1)(3n-2))^2 \binom{6n}{3n}} ,$$

(6.9) $$\frac{\pi^2}{6} = 12\sum_{n=1}^{\infty} \frac{87040n^6 - 173568n^5 + 131968n^4 - 47456n^3 + 8084n^2 - 552n + 9}{(4n(4n-1)(4n-2)(4n-3))^2 \binom{8n}{4n}} ,$$

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(1) Mohamud Mohammed *Infinite families of accelerated series for some classical constants by the Markov–WZ Method* Discrete Mathematics and Theoretical Computer Science **7**, 2005, 11-24

7 Series for $\zeta(3)$

Making use of the Taylor series expansion for $(\sin^{-1}(x))^2$ given in (4.7), the integral representation

$$\zeta(3) = 10\int_0^{1/2} \frac{(\sinh^{-1}(t))^2}{t}\,dt \qquad (7.1)$$

[L.Lewin *Polylogarithms and associated functions*, North-Holland, New York, 1981, Sec. 6.3]

is easily seen to be equivalent to the series

$$\zeta(3) = \frac{5}{2}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^3\binom{2n}{n}} \qquad (7.2)$$

(Hjortnaes 1954).

Using the new series expansions for $(\sin^{-1}(x))^2$ provides further results of this type. Examples include

$$\zeta(3) = \frac{5}{2}\sum_{n=1}^{\infty} \frac{24n^3 + 4n^2 - 6n + 1}{(2n(2n-1))^3\binom{4n}{2n}}, \qquad (7.3)$$

$$\zeta(3) = \frac{5}{2}\sum_{n=1}^{\infty} (-1)^{n+1}\frac{9477n^6 - 11421n^5 + 5265n^4 - 1701n^3 + 558n^2 - 108n + 8}{(3n(3n-1)(3n-2))^3\binom{6n}{3n}}, \qquad (7.4)$$

and

$$\zeta(3) = 20\sum_{n=1}^{\infty} \frac{P(n)}{(4n(4n-1)(4n-2)(4n-3))^3\binom{8n}{4n}}, \qquad (7.5)$$

where

$$\mathrm{P}(n) = 1671168n^9 - 4161536n^8 + 4278272n^7 - 2340864n^6 + 712064n^5 - 98496n^4 - 6360n^3 + 4476n^2 - 594n + 27.$$

8 Series for $\zeta(4)$

Comtet's result

(8.1) $$\zeta(4) = \frac{36}{17}\sum_{n=1}^{\infty}\frac{1}{n^4\binom{2n}{n}}$$

follows from the identity

(8.2) $$\zeta(4) = \frac{144}{17}\int_0^1 \frac{\sin^{-1}(x/2)\log^2(x)}{\sqrt{1-(x/2)^2}}\,dx$$

by replacing $\frac{\sin^{-1}(x/2)}{\sqrt{1-(x/2)^2}}$ with its Maclaurin series and integrating term by term. If we use the equivalent expansions for $\frac{\sin^{-1}(x)}{\sqrt{1-x^2}}$, some of which are listed in section (4), we can extend Comtet's result to

(8.3) $$\zeta(4) = \frac{36}{17}\sum_{n=1}^{\infty}\frac{80n^4-48n^3+24n^2-8n+1}{(2n(2n-1))^4\binom{4n}{2n}},$$

(8.4) $$\zeta(4) = \frac{36}{17}\sum_{n=1}^{\infty}\frac{\mathrm{P}(n)}{(3n(3n-1)(3n-2))^4\binom{6n}{3n}}$$

where $\mathrm{P}(n) = 137781n^8 - 275562n^7 + 240570n^6 - 122472n^5 + 41877n^4 - 10908n^3 + 2232n^2 - 288n + 16,$

and so on.

9 Series for Catalan's constant

The Dirichlet β function is defined as

(9.1) $$\beta(s) = \frac{1}{1^s} - \frac{1}{3^s} + \frac{1}{5^s} - \frac{1}{7^s} + \cdots \qquad \text{where } \operatorname{Re} s \geq 1.$$

It is an example of an L-series. The values $\beta(1)$, $\beta(3)$, $\beta(5)$, $\ldots$ of the Dirichlet β function at the positive odd integers are rational multiples of powers of π. Explicitly

(9.2) $$\beta(2n+1) = \frac{E_n^*}{2(2n)!}\left(\frac{\pi}{2}\right)^{2n+1}$$

where E_n^* is an Euler number (secant number). The first few values are

n	= 0	1	2	3	4	5	
E_n^*	= 1	1	5	61	1385	50521	(Sloane's A000364).

Little is known about the values of $\beta(2n)$ at the positive even integers. $\beta(2)$ is known as Catalan's constant and denoted by G (sometimes K).

(9.3) $$G = 1 - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \cdots = 0.91596\ 55941\ 77219\ 01505\ \ldots.$$

Unlike $\zeta(3)$, it is not known if G is irrational.

D. M Bradley in "Representations of Catalan's Constant" catalogues and proves a large number of infinite series and integral representations for G. In particular we have the integral formula (entry (34) in Bradley)

(9.4) $$G = \frac{1}{8}\pi\log(2+\sqrt{3}) + \frac{3}{4}\int_0^{\frac{\pi}{6}} \frac{x}{\sin(x)}\,dx\ .$$

Make the change of variable $x = \sin^{-1}(y)$ in the integral to give

(9.5) $$G = \frac{1}{8}\pi\log(2+\sqrt{3}) + \frac{3}{4}\int_0^{\frac{1}{2}} \frac{\sin^{-1}(y)}{y\sqrt{1-y^2}}\,dy\ .$$

Now replace $\sin^{-1}(y)/\sqrt{1-y^2}$ by its Taylor series expansion

$$\frac{\sin^{-1} y}{\sqrt{1-y^2}} = \sum_{n=1}^{\infty} \frac{(2y)^{2n-1}}{n\binom{2n}{n}}$$

and integrate term by term to obtain

$$(9.6) \qquad G = \frac{1}{8}\pi \log(2+\sqrt{3}) + \frac{3}{8}\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2\binom{2n}{n}} \text{; (entry (62) in Bradley).}$$

Because of the fast convergence of the series, this representation for G has been used to calculate Catalan's contant to a large number of decimal places.

If in (9.5) we use the new series expansions for $\sin^{-1}(y)/\sqrt{1-y^2}$ given in this website we find new representations for Catalan's constant:

$$(9.7) \qquad G = \frac{1}{8}\pi \log(2+\sqrt{3}) + \frac{3}{16}\sum_{n=0}^{\infty} \frac{40n^2+54n+19}{(4n+1)^2(4n+3)^2\binom{4n}{2n}}$$

$$(9.8) \qquad G = \frac{1}{8}\pi \log(2+\sqrt{3}) + \frac{3}{32}\sum_{n=0}^{\infty} \frac{6804n^4+17172n^3+15903n^2+6405n+956}{(6n+1)^2(6n+3)^2(6n+5)^2\binom{6n}{3n}}$$

$$(9.9) \qquad G = \frac{1}{8}\pi \log(2+\sqrt{3}) + \frac{3}{32}\sum_{n=0}^{\infty} \frac{\mathrm{p}(n)}{(8n+1)^2(8n+3)^2(8n+5)^2(8n+7)^2\binom{8n}{4n}},$$

where $\mathrm{p}(n) = 1392640n^6 + 5056512n^5 + 7466752n^4 + 5731040n^3 + 2409488n^2 + 526414n + 46889,$

and so on.

Bradley (entry (4) in the above reference) also gives the integral representation

$$(9.10) \qquad G = \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{x}{\sin(x)}\,dx$$

$$= \frac{1}{2}\int_0^{1} \frac{\sin^{-1}(x)}{x\sqrt{1-x^2}}\,dx$$

which produces the poorly converging series

$$(9.11) \qquad 2G = \sum_{n=0}^{\infty} \frac{4^n}{(2n+1)^2\binom{2n}{n}} \text{ ; entry (61) in Bradley.}$$

Using the expansion (4.10)

$$\frac{\sin^{-1} x}{\sqrt{1-x^2}} = \sum_{n=1}^{\infty} \frac{8n(1+x^2)-2(1+2x^2)}{2n(2n-1)\binom{4n}{2n}}(2x)^{4n-3}$$

in (9.10) leads to the representation

$$(9.12) \qquad 2G = \sum_{n=0}^{\infty} \frac{(32n^2+36n+11)\ 16^n}{(4n+1)^2(4n+3)^2\binom{4n}{2n}} .$$

Again the convergence is slow.