Levine's sequence

6/4/97

A11784

Define an array

a_{1*} 1 1 — S_1

a_{2*} 1 2 — S_2

a_{3*} 1 1 2 — S_3

a_{4*} 1 1 2 3 — S_4

a_{5*} 1 1 1 2 2 3 4 — S_5

a_{6*} 1 1 1 1 2 2 2 3 3 4 4 5 6 7 — S_6

a_{7*} 1111111 22222 3333 4444 5555 666 777 88 99 10 10 11 12 13 14 — S_7

etc

in which if row a_{6*} is $a_{61} \cdots a_{6l}$ then row$_{7*}$ is

$\{1^{a_{6l}}\ 2^{a_{6,l-1}}\ 3^{a_{6,l-2}}, \ldots\}$.

The sequence $s(n)$ is the last term of a_{n*}. and runs

1	2	3	4	5	6	7	8
1	2	2	3	4	7	14	42

9	10	11	12
213	2387	175450	139759600

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and I have 3 more terms beyond that. Note that it has the following properties:

(0) last term of a_{n*} = $s(n)$ (defn)

(1) $\ell(n)$ = length of n^{th} row = $s(n+1)$

(2) $\sum_{j=1}^{\ell(n)} a_{nj} = s(n+2)$

(3) $\sum_{j=1}^{\ell(n)} \left(\ell(n)+1-j\right) a_{nj} = s(n+3)$

(4) No of 1's in ~~the~~ a_{n*} = $s(n-1)$ (for $n \geq 2$)

(5) No of 2's in a_{n*} (for $n \geq 5$) = $s(n-2)$

and by following down three rows we get

(6) Let b_{n*} = row a_{n*} read backward $(1 \leq j \leq \ell(n))$

Then

(6a) $$s(n+4) = \sum_{j=1}^{\ell(n)} j \left\{ \left(s_{n+2} - \sum_{r=1}^{j-1} b_{nr} \right) b_{nj} - \binom{b_{nj}}{2} \right\}$$

The file seqs95 in bin does all this.

To use it:

```
a1:=[1,1];
a2:=st(a1);
a3:=st(a2);
sum1(a3);
sum2(a3);
sum3(a3);
```

Here's how to do it in S (from CLM)

(Colin Mallows)

\# Splus

> st <— function(v)

reverse

```
function(v)
{
        revv <- rev(v)
        a <- NULL
        for(i in 1:length(v))
                a <- c(a, rep(i, revv[i]))
        a
}
```

Concatenate   repeat

Then

v <— c(1,1)   (means string 1 1)

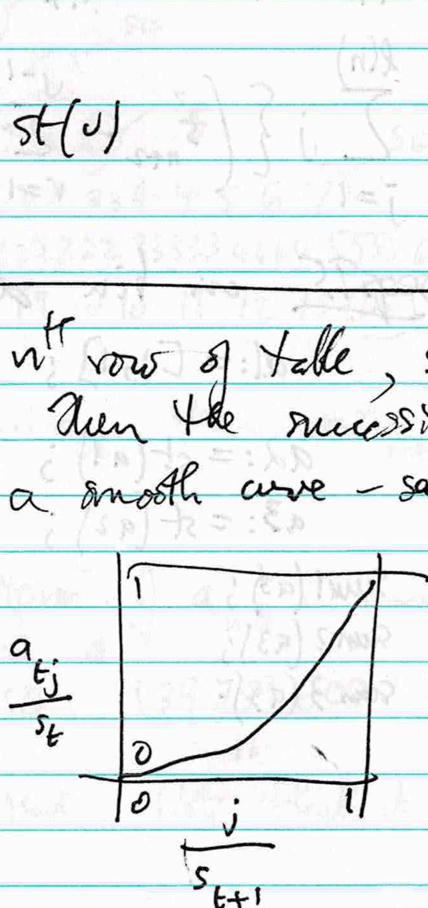
for (j in 1:6)

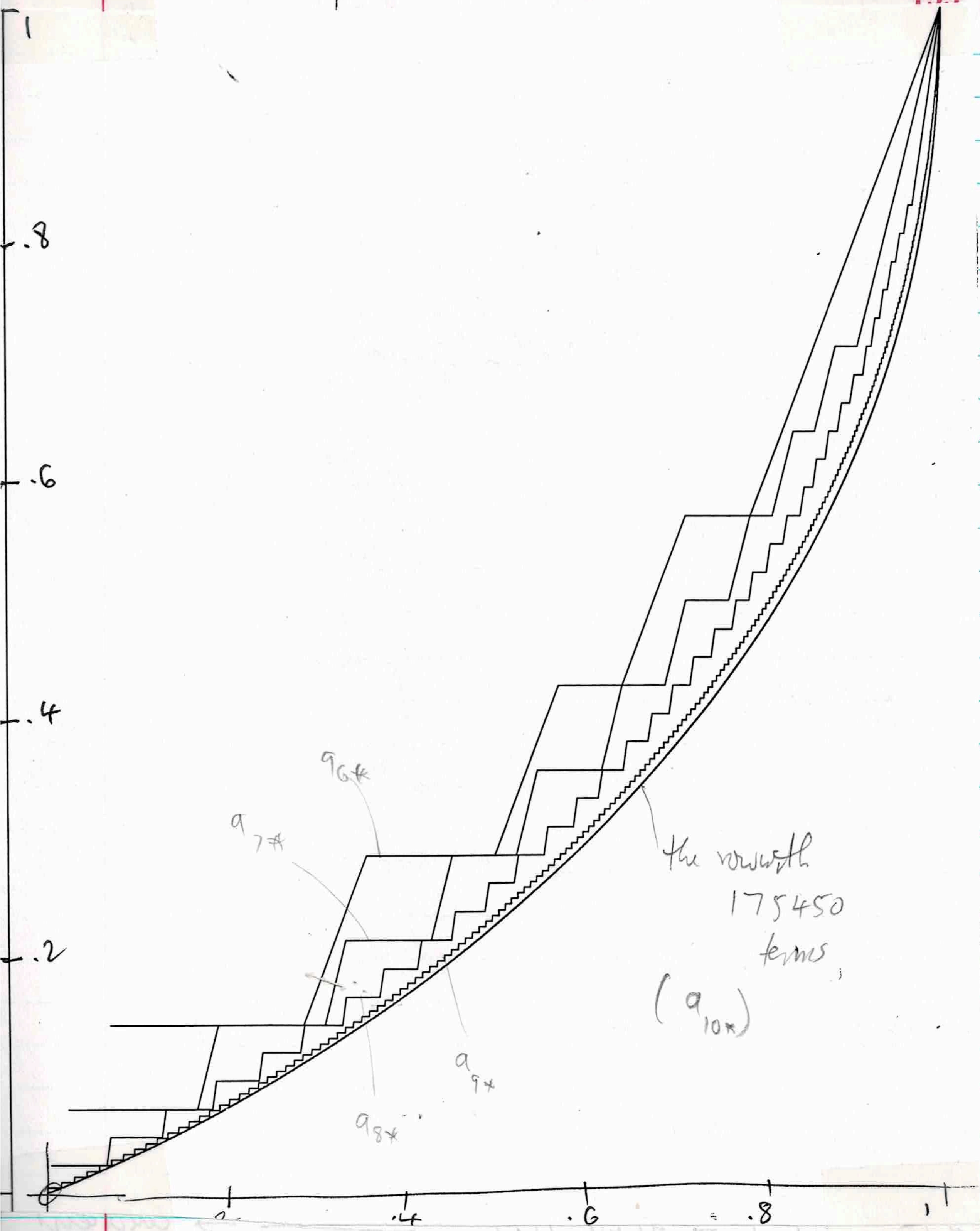
{ v <— ~~step(v)~~ st(v)

print(v)

}

Colin's analysis Look at n^{th} row of table, scale it to have length 1 & height 1. Then the successive rows appear to converge to a smooth curve – see opposite. Call this $g(x)$. Definition




1
.8
.6
.4
.2
.2
.4
.6
.8
1
a_{6*}
a_{7*}
a_{8*}
a_{9*}
the row with
175450
terms
(a_{10*})

$$1\ 1\ 1\ 1\quad 2\ 2\ 2\quad 3\ 3\ \ldots\ k\ k\ k\ \ldots$$

look at last k in row t

Then

(1) $$k = a(t, s_{t-1}) + \ldots + a(t, s_{t-1} - k + 1)$$

Now $$g\left(\frac{j}{s_{t+1}}\right) \cong \frac{a(t,j)}{s_t}$$

Then from (1),

$$k = \sum_{j=s_{t-1}-k+1}^{s_{t-1}} a(t,j)$$

and this becomes

$$\frac{k}{s_{t+1}s_t} \cong \sum_{j=s_{t-1}-k+1}^{s_{t-1}} g\left(\frac{j}{s_{t+1}}\right) \frac{\Delta j}{s_{t+1}}.$$

$$\cong \int_{s_{t-1}-k+1} g(x)\,dx$$

If done correctly, get $$\int_{1-\frac{k-1}{s}}^{1} g(x)\,dx$$

Then differentiate to get

(2) $$g(1 - g(x))\,g'(x) = \frac{s_{t+1}}{s_t\, s_{t-1}} \to \text{constant}$$

$\therefore \log s_{t+1} = \log s_t + \log s_{t-1} + \cancel{\text{const}}\ c$

$\therefore \log s_{t+1} + c = \log s_t + c + \log s_{t-1} + c$

$\therefore A_t = (\log s_t + c)$ satisfies Fib. eqn

$$\therefore \underline{\log s_t \sim \phi^t}$$

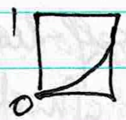
(2) looks a bit nicer if set $y = g(x)$, $x = h(y)$, get

$$1 - y = h\left(\frac{h'(y)}{h'(0)}\right)$$

— but still can't solve (1) or (2) exactly.

Q: what is soln to

$$g(1 - g(x))\, g'(x) = C$$

which is pos & monotonic in , ?

Cont. p 164

Theorem Bjorn Poonen on Levine's sequence

cont from page 155

$\log S_n \sim c\phi^n$

Proof (E) $L_n :=$ left of $n^{th} = S_{n+1}$; $S_n = n^{th}$ term ;

$L_{n+1} = \sum_n :=$ sum of $n^{th} = S_{n+2}$

claim (i) $L_{n+1} \leq L_n S_n$ ✓ (Easy)

claim (ii) $\sum_{n+1} \geq \underbrace{\underbrace{1+1+\cdots+1}_{S_n} + \cdots + \underbrace{2+2\cdots+2+\cdots+2}_{S_n}}_{L_{n+1}}$

$\geq \frac{1}{2}\left(\frac{L_{n+1}}{S_n}\right)^2 S_n$

$\therefore S_{n+3} \geq \frac{1}{2}\frac{S_{n+2}^2}{S_n}$

$\frac{S_{n+3}}{2S_{n+2}} \geq \left(\frac{S_{n+2}}{2S_{n+1}}\right)\left(\frac{S_{n+1}}{2S_n}\right)$

Hence $\log \frac{S_{n+3}}{2S_{n+2}}$ satisfies Fib. recurrence $\square$