A commutative diagram of triangular arrays

Peter Bala, September 06 2015

We consider how several triangular arrays in the OEIS are related via logarithmic differentiation and the binomial transform.

\- - - - - - - - - - - - - - - - - -

Let
$$A(x,t) = \sum_{n\ge 0} a_n(t)x^n$$
be the bivariate generating function of a lower triangular array, where $a_n(t)$ are the polynomial row generating functions of the array. Suppose that $a_0(t) = 1$. We define a modified logarithmic differentation operator $\mathcal{L}$ by

$$\mathcal{L}(A(x,t)) \quad = \quad 1 + x\frac{A^{'}(x,t)}{A(x,t)}, \tag{1}$$

where the prime indicates differentiation with respect to x. If we write

$$\mathcal{L}(A(x,t)) \quad = \quad \sum_{n\ge 0} b_n(t)x^n \tag{2}$$

then the $b_n(t)$ are polynomials in t with $b_0(t) = 1$. By an abuse of notation we will write (2) as
$$\mathcal{L}(a_n(t)) = b_n(t).$$
Inverting (1) gives the relation between $a_n(t)$ and $b_n(t)$ in the form

$$\sum_{n\ge 0} a_n(t)x^n \quad = \quad \exp\left(\sum_{n\ge 1} b_n(t)\frac{x^n}{n}\right). \tag{3}$$

An entry dated Oct 13 2010 by Paul D. Hanna in A001263 - the triangle of Narayana numbers, is equivalent to the statement that $\mathcal{L}$ maps the bivariate generating function of A001263 to the bivariate generating function of A008459 - the triangle of the squares of the binomial coefficients. Again abusing notation slightly we write this as

$$\mathcal{L}(A001263) \quad = \quad A008459.$$

We can extend Hanna's observation to the following commutative diagram of triangular arrays (n.l. indicates the array is not currently listed in the OEIS and | | denotes the unsigned version of the array).

$$
\begin{array}{ccccccc}
n.l. & \xrightarrow{Bin} & A107131 & \xrightarrow{Bin} & A001263 & \xrightarrow{Bin} & A126182 \\
\downarrow{\scriptstyle \mathcal{L}} & & \downarrow{\scriptstyle \mathcal{L}} & & \downarrow{\scriptstyle \mathcal{L}} & & \downarrow{\scriptstyle \mathcal{L}} \\
(-1)^n A099037 & \xrightarrow{Bin} & A105868 & \xrightarrow{Bin} & A008459 & \xrightarrow{Bin} & n.l. \\
\downarrow{\scriptstyle \mathcal{L}} & & \downarrow{\scriptstyle \mathcal{L}} & & \downarrow{\scriptstyle \mathcal{L}} & & \downarrow{\scriptstyle \mathcal{L}} \\
n.l & \xrightarrow{Bin} & |A117411| & \xrightarrow{Bin} & A086645 & \xrightarrow{Bin} & n.l.
\end{array}
$$

Fig. 1

In the diagram, Bin denotes the binomial transform of an array, which has the effect of premultiplying an array by Pascal's triangle A007318. At the generating function level

$$\mathrm{Bin}\,(A(x,t)) = \frac{1}{1-x}A\left(\frac{x}{1-x},t\right) \tag{4}$$

gives the generating function for the transformed array.

Using (1) and (4) it is easy to verify that the operator $\mathcal{L}$ commutes with the binomial transform

$$\mathcal{L} \text{ o } \mathrm{Bin} = \mathrm{Bin} \text{ o } \mathcal{L}. \tag{5}$$

Fig. 1 can be extended indefinitely to the left and to the right using the operator Bin and downwards using the operator $\mathcal{L}$ to give other integral triangular arrays, but these arrays are not listed in the OEIS. We could also extend the diagram upwards using the inverse operator $\mathcal{L}^{-1}$ but the resulting arrays will no longer be integral.

Row polynomials

We can use the operators $\mathcal{L}$ and Bin to propagate information from one array in Fig 1. to other arrays in the diagram. As an example, we consider how the row polynomials of the arrays in the diagram are related. It turns out that a good place to start is with the row polynomials of A105868. The entries in the lower triangular array A105868 are defined as

$$A105868(n,k) = \binom{n}{k}\binom{k}{n-k}.$$

The n-th row polynomial $R_{A105868}(n,t)$ of the array is thus equal to

$$R_{A105868}(n,t) = \sum_k \binom{n}{k}\binom{k}{n-k}t^k.$$

We can express the row polynomials in the form

$$R_{A105868}(n,t) = [x^n](1+tx+tx^2)^n \tag{6}$$

since

$$\begin{aligned}
[x^n](1+tx+tx^2)^n &= [x^n](1+tx(1+x))^n \\
&= [x^n]\left(\sum_k \binom{n}{k} t^k x^k (1+x)^k\right) \\
&= [x^n]\left(\sum_{k,i} \binom{n}{k}\binom{k}{i} t^k x^{k+i}\right) \\
&= \sum_k \binom{n}{k}\binom{k}{n-k} t^k.
\end{aligned}$$

The following (easily proved) result relates the row polynomials of arrays in the same row of Fig 1.

Proposition 1. Let $F(x)$ and $G(x)$ be formal power series. If $a(n) = [x^n]F(x)G(x)^n$ then for $k \in \mathbb{Z}$,

$$\text{Bin}^k(a(n)) = [x^n]F(x)(kx+G(x))^n. \quad \square$$

Proposition 1 combined with (6) leads to expressions for the row polynomials of the other arrays in the middle row of Fig. 1. We find

$$R_{A008459}(n,t) = [x^n](1+(t+1)x+tx^2)^n \tag{7}$$

and

$$\begin{aligned}
R_{A099307}(n,t) &= (-1)^n \quad [x^n](1+(t-1)x+tx^2)^n \\
&= \quad [x^n](-1+(1-t)x-tx^2)^n.
\end{aligned} \tag{8}$$

The next result can be used to relate the row polynomials of the arrays belonging to the same column of Fig. 1.

Proposition 2. Let $a(n)$ be a sequence with $a(0) = 1$. Let $A(x) = \sum_{n\ge 0} a(n)x^n$ denote the generating function of the sequence. Then

$$\mathcal{L}(a(n)) = [x^n]\left(\frac{x}{\text{Rev}(xA(x))}\right)^n,$$

where Rev denotes the series reversion with respect to x.

Proof. Set

$$b(n) = \mathcal{L}(a(n)).$$

By (3)

$$A(x) = \sum_{n\geq 0} a(n)x^n = \exp\left(\sum_{n\geq 1} b(n)\frac{x^n}{n}\right). \qquad (9)$$

By [2, Proposition 2], which is stated there for integer sequences but this restriction is not essential, there exists a formal power series $G(x)$ such that

$$b(n) = [x^n]G(x)^n,$$

where

$$G(x) = \frac{x}{\text{Rev}(f(x))}$$

with

$$\begin{aligned} f(x) &= x\exp\left(\sum_{n\geq 1} b(n)\frac{x^n}{n}\right) \\ &= xA(x) \end{aligned}$$

by (9). The result now follows. □

If we apply Proposition 2 to (6) and (7) in turn we find, after a short calculation, the following expressions for the row polynomials of the indicated arrays:

$$R_{|A117411|}(n,t) = [x^n](tx + \sqrt{1+4tx^2})^n \qquad (10)$$

$$R_{A086645}(n,t) = [x^n]((1+t)x + \sqrt{1+4tx^2})^n. \qquad (11)$$

RELATED NOTES

[1] P. Bala, Notes on logarithmic differentiation, the binomial transform and series reversion - uploaded to A100100

[2] P. Bala, Representing a sequence as [x^n]G(x)^n - uploaded to A066398