Zeckendorf *vs.* Wythoff representations: Comments on A007895.

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1 Introduction

Let g (golden ratio) be the positive root of $x^2 = x + 1$; let h be the other root. So $g = (1 + \sqrt{5})/2$, $h = (1 - \sqrt{5})/2$. Then the n'th Fibonacci number is

$$F_n = \frac{g^n - h^n}{\sqrt{5}}$$

A linear combination of powers of those roots makes an additive (Lucas) sequence; and $F_0 = 0$, $F_1 = 1$, so it's the right one.

Thus one can compute the n'th Fibonacci number directly. Furthermore, one can relate F_a, F_b, and F_{a+b}:

$$\begin{aligned} g^b F_a + h^a F_b &= \frac{g^{b+a} - g^b h^a}{\sqrt{5}} + \frac{h^a g^b - h^{a+b}}{\sqrt{5}} \\ &= \frac{g^{a+b} - h^{a+b}}{\sqrt{5}} \\ &= F_{a+b} \end{aligned}$$

In particular,

$$\begin{aligned} F_{n+1} &= gF_n + h^n \\ F_{n+2} &= g^2 F_n + h^n \end{aligned}$$

2 Base-Fibonacci representations

A non-negative integer can be represented as sums of distinct Fibonacci numbers; for example $17 = 13 + 3 + 1 = F_7 + F_4 + F_2$. (One needn't use $F_0 = 0$ nor F_1 which equals F_2.) There can be many such representations, but one can find a representation wherein no two consecutive indices appear. (If there are two consecutives, change the largest $F_a + F_{a+1}$ to F_{a+2}; repeat until none.)

One can use binary-like numerals: $17 = 100101_F$. (The rightmost position is for F_2.)

Adding one within this representation is easy: change the final '0' to '1', or change the final '01' to '10'. Then remove consecutives as explained above. (One can thereby show that each non-negative integer has a representation.)

2.1 Pseudo-doubling

In the base-2 representation, appending zero to a numeral doubles the number. What does that accomplish in base-Fibonacci?

If $n = \sum_k F_{a_k}$, then $n'0' = \sum_k F_{1+a_k}$.

$$\begin{aligned} n'0' &= \sum_k F_{1+a_k} \\ &= \sum_k (gF_{a_k} + h^{a_k}) \\ &= \left(\sum_k gF_{a_k}\right) + \left(\sum_k h^{a_k}\right) \\ &= gn + \sum_k h^{a_k} \end{aligned}$$

Now, h is a small negative value: the sum of those powers of h is bounded above by $h^2 + h^4 + h^6 + \ldots = -h$, and below by $h^3 + h^5 + h^7 + \ldots = -h^2$. So $n'0' = gn - \epsilon$, where $h < \epsilon < h^2$. That error-range is $h^2 - h = 1$, and $n'0' = \lfloor gn - h \rfloor = \lceil gn - h^2 \rceil$.

Pseudo-quadruple $n'00'$ has the same error-range; $n'00' = \lfloor g^2 n - h \rfloor = \lceil g^2 n - h^2 \rceil$.

3 Fibonacci, Beatty, and Wythoff

Let $G_n = \lfloor gn \rfloor$ (for $n > 0$). This sequence goes `1,3,4,6,8,9,11,12,14,16,17,19,`... (A000201 [1]). Because g is irrational and greater than one, this is a Beatty sequence; and $g/(g-1) = g^2$ generates the complementary Beatty sequence: $J_n = \lfloor g^2 n \rfloor =$ `2,5,7,10,13,15,18,20,`... (A001950

[1]). The two sequences partition the positive integers.

These sequences are used to produce the Wythoff representation of a positive integer a. a appears in exactly one of the sequences, at position n; so:

- If $a = 1$, the representation of a is the empty sequence of digits.
- Otherwise, if $a = G_n$, the representation of a is '0' preceded by the representation of n.
- If $a = J_n$, the representation of a is '1' preceded by the representation of n.

To find the meaning of, say, 1011_W, change $0 \to G$, $1 \to J$, and apply the functions left-to-right, starting with argument 1. $J(1) \to 2$, $G(2) \to 3$, $J(3) \to 7$, $J(7) \to 18$. So $1011_W = 18$.

(Observe that a non-empty representation begins with '1': there is no need to use $G(1) \to 1$.)

4 Wythoff versus Fibonacci

The functions $G_n = \lfloor gn \rfloor$ and $J_n = \lfloor g^2 n \rfloor$ are similar to pseudo-double $n'0' = \lfloor gn - h \rfloor$ and pseudo-quadruple $n'00' = \lfloor g^2 n - h \rfloor$. Indeed,

$$\begin{aligned}
G_{n+1} &= \lfloor g(n+1) \rfloor \\
&= \lfloor gn + g \rfloor \\
&= \lfloor gn + 1 - h \rfloor \\
&= \lfloor gn - h \rfloor + 1 \\
&= n'0' + 1 \\
J_{n+1} &= \lfloor g^2(n+1) \rfloor \\
&= \lfloor g^2 n + g^2 \rfloor \\
&= \lfloor g^2 n + 2 - h \rfloor \\
&= \lfloor g^2 n - h \rfloor + 2 \\
&= n'00' + 2 = n'01' + 1
\end{aligned}$$

Therefore, the Wythoff representation of $n + 1$ is like the base-Fibonacci representation of n, except that each '1' of Wythoff maps to a '01' of Fibonacci. (Each '0' maps to '0'.) For examples,

n	Wyth$(n+1)$	Fibo(n)
1	1	01
2	10	010
3	100	0100
4	11	0101
5	1000	01000
6	101	01001
7	110	01010
8	10000	010000
9	1001	010001
10	1010	010010
11	1100	010100
12	111	010101
13	100000	0100000
14	10001	0100001
15	10010	0100010
16	10100	0100100
17	1011	0100101
18	11000	0101000
19	1101	0101001
20	1110	0101010

The mapping preserves the number of ones.

References

[1] Neil Sloane, *The Online Encyclopedia of Integer Sequences*, `http://www.research.att.com/~njas/sequences`