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Montgomery

"Kevin Browns...."

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KEVIN BROWN'S ENUMERATION PROBLEM

Let Σ_n denote the symmetric group on the numbers $\{0, 1, \ldots, n-1\}$. For $\pi \in \Sigma_n$, form the sum

$$S(\pi) = \sum_{i=0}^{n-1} (\pi(i) + \pi(i+1))^2. \tag{1}$$

Here we consider π to have period n, so that $\pi(n+1) = \pi(1)$. Let $v(n)$ denote the number of distinct values that $S(\pi)$ takes on, as π runs over all $n!$ members of Σ_n. Kevin Brown (`kevin2003@delphi.com`) notes the values of $v(n)$ for $1 \le n \le 10$, and then continues

> ... Can anyone supply more terms of this sequence? Or give an asymptotic formula for the kth term? Does the sum of the inverses of these numbers converge? This seems like a difficult enumeration problem ...

Put

$$T(\pi) = \sum_{i=0}^{n-1} \pi(i)\pi(i+1). \tag{2}$$

Clearly,

$$S(\pi) = 2T(\pi) + 2\sum_{i=0}^{n-1} i^2. \tag{3}$$

¿From this we note that all values of $S(\pi)$ are even, and that the values of $S(\pi)$ and of $T(\pi)$ are in one-to-one correspondence. Consequently it suffices to study $T(\pi)$.

To measure the range of values of $T(\pi)$, we put

$$m(n) = \min_{\pi \in \Sigma_n} T(\pi), \qquad M(n) = \max_{\pi \in \Sigma_n} T(\pi). \tag{4}$$

First we note that

$$M(n) = \frac{2n^3 - 3n^2 - 11n + 18}{6} \qquad (n \ge 2). \tag{5}$$

This is achieved by the permutation

$$\pi(i) = \begin{cases} 2i & (0 \le i \le [\frac{n-1}{2}]), \\ 2n - 2i - 1 & ([\frac{n-1}{2}] < i \le n-1). \end{cases} \tag{6}$$

To demonstrate that $M(n)$ can be no larger, it suffices to prove that

$$M(n+1) \le M(n) + n^2 - 2, \tag{7}$$

and then induct. To establish the above, suppose that $\pi \in \Sigma_n$, and that the number n is to be inserted between two adjacent members (say x and y) of π. When n is introduced, the quantity T is increased by $n(x+y)$, and decreased by xy (since x and y are no longer adjacent). Thus the total change is $nx + ny - xy$. This is an increasing function of x, and also of y, and hence it is maximized when x and y are as large a possible—that is, when x and y are $n-1$ and $n-2$ in either order. Thus we have (7).

As for $m(n)$, it seems that

$$m(n) = \begin{cases} \frac{n^3-3n^2+5n-3}{6} & (n \text{ odd}, n \geq 1), \\ \frac{n^3-3n^2+5n-6}{6} & (n \text{ even}, n \geq 2). \end{cases} \tag{8}$$

When n is odd, this is achieved by the permutation

$$\pi(i) = \begin{cases} 0 & (i = 0), \\ n-i & (i \text{ odd}, 1 \leq i \leq n-2), \\ i-1 & (i \text{ even}, 2 \leq i \leq n-1). \end{cases} \tag{9}$$

When n is even it is achieved by the permutation

$$\pi(i) = \begin{cases} 0 & (i = 0), \\ n-i & (i \text{ odd}, 1 \leq i \leq n/2), \\ i-1 & (i \text{ even}, 2 \leq i \leq n/2), \\ i-1 & (i \text{ odd}, n/2 < i \leq n-1), \\ n-i & (i \text{ even}, n/2 < i \leq n-2). \end{cases} \tag{10}$$

It should be possible to show that $m(n)$ is not smaller than given by (8). In any case, (8) is not far off the mark, since it is easy to show that

$$m(n) \geq \frac{n^3 - 3n^2 + 2n}{6} \qquad (n \geq 1).$$

To prove this it suffices to note that Cauchy's inequality gives

$$n^2(n-1)^2 = \left(\sum_{i=0}^{n-1} \pi(i) + \pi(i+1) \right)^2 \leq nS(\pi),$$

and then apply (3).

Clearly

$$v(n) \leq M(n) - m(n) + 1.$$

I conjecture that equality holds here, for all $n \geq 7$. That is, I conjecture that

$$v(n) = \begin{cases} \frac{n^3-16n+27}{6} & (n \text{ odd}, n \geq 7), \\ \frac{n^3-16n+30}{6} & (n \text{ even}, n \geq 8). \end{cases} \tag{11}$$

Our conjectures are supported by the values in the following table.

n	$m(n)$	$M(n)$	$v(n)$
1	0	0	1
2	0	0	1
3	2	2	1
4	5	9	3
5	12	23	8
6	22	46	21
7	38	80	43
8	59	127	69
9	88	189	102
10	124	268	145
11	170	366	197
12	225	485	261
13	292	627	336

To discern the statistical distribution of the numbers $T(\pi)$, we think of this quantity as a sum of n random variables. Each summand has expectation $n^2/4+O(n)$, and variance $7n^4/144 + O(n^3)$. The summands are not independent, but they are nearly so, and hence it ought to be possible to use tools of probability to establish a form of the central limit theorem for this situation. That is, it should be possible to show that if α and β are fixed, $\alpha < \beta$, then the proportion of $\pi \in \Sigma_n$ for which

$$\frac{n^3}{4} + \alpha\sqrt{\frac{7}{144}}n^{5/2} \le T(\pi) \le \frac{n^3}{4} + \beta\sqrt{\frac{7}{144}}n^{5/2}$$

tends to

$$\frac{1}{\sqrt{2\pi}}\int_\alpha^\beta e^{u^2/2}\,du = \Phi(\beta) - \Phi(\alpha)$$

as n tends to infinity. In particular, by this approach it should be possible to show that

$$\frac{v(n)}{n^{5/2}} \to \infty$$

as $n \to \infty$.

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