Ordering the Levels L_k and L_{k+1} of $\mathcal{B}_{2k+1}$

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Abstract

A system of numeration in which every k, with $0 < k \in \mathbb{Z}$, appears as a restricted growth string, or *RGS*, has the k-th Catalan number as the RGS 10^k. This induces a canonical ordering of the vertices of the dihedral quotients of the middle-levels graphs.

1 Restricted Growth Strings

Let $0 \le m, k \in \mathbb{Z}$ and let $n = 2k+1$. In this paper, each such an m is represented [1, 8] as a *restricted growth string* (or *RGS*) $\alpha = \alpha(m)$, related to the Catalan numbers $C_k = \frac{1}{n}\binom{n}{k}$ ([9] A000108) in that $\alpha(C_k) = 10\cdots 0 = 10^k$ ([1] pg. 325). These RGS s α form a system of numeration $\mathcal{S}$ ([9] A239903) that encodes the vertices of the quotient graph R_k of the middle-levels graph M_k (Section 3) under action of the dihedral group D_{2n} of order $2n$. In fact, the RGS s encode n-strings $F(\alpha)$ (via "castling", Section 2) that become in Section 7 the vertices of R_k (via "un-castling", Section 5). Moreover, R_k has its vertices in 1-1 correspondence with the first C_k RGS s. This arises from the 1-factorization of R_k given via the lexical matchings of [5], as shown from Section 6 on, yielding a canonical setting for R_k and therefore for M_k.

Entering into details, the non-negative integers m in their natural order can be represented by the successive members of a sequence $\mathcal{S}$ of RGS s that starts with

$$0, 1, 10, 11, 12, 100, 101, 110, 111, 112, 120, 121, 122, 123, 1000, \ldots \qquad (1)$$

and that has the RGS s $1, 10, 100 = 10^2, 1000 = 10^3, \ldots, 10\cdots 0 = 10^t$ $(t \ge 0)$ etc. corresponding respectively to the numbers $C_1 = 1$, $C_2 = 2$, $C_3 = 5$, $C_4 = 14$, $\ldots$, $C_{t+1} = \frac{1}{2t+3}\binom{2t+3}{t+1}$, etc., where symbolic powers are used. To visualize the continuation of $\mathcal{S}$ in (1), each RGS in $\mathcal{S}$ is transformed for adequate $k > 1$ into a k-string $a_{k-1}a_{k-2}\cdots a_2a_1$ by prefixing to it enough zeros if necessary. Then, the following definition allows the said continuation by excising all zero entries previous to the leftmost nonzero entry of such $a_{k-1}a_{k-2}\cdots a_2a_1$. Letting $1 < k \in \mathbb{Z}$, a *k-germ* is a $(k-1)$-string $a_{k-1}a_{k-2}\cdots a_2a_1$ satisfying: **1.** The leftmost position in $a_{k-1}a_{k-2}\cdots a_2a_1$, namely position $k-1$, contains a digit $a_{k-1} \in \{0, 1\}$. **2.** Given a position $i > 1$ with $i < k$ in $a_{k-1}a_{k-2}\cdots a_2a_1$, then to the immediate right of the corresponding digit a_i, the digit a_{i-1} (meaning at position $i-1$) satisfies $0 \le a_{i-1} \le a_i + 1$.

The reader may compare these strings with the essentially similar *Catalan* RGSs of Section 15.2 [1], or with the mixed radix systems [2], including the factorial number, or factoradic, system [3], [4], [6] pg. 192, [7] pg. 12, or [9] A007623. We refer as well to Stanley's interpretation of Catalan numbers [10], Exercise (u), as mentioned in [9] A239903.

Every k-germ $a_{k-1}a_{k-2}\cdots a_2a_1$ yields a $(k+1)$-germ $a_ka_{k-1}a_{k-2}\cdots a_2a_1 = 0a_{k-1}a_{k-2}\cdots a_2a_1$. A k-germ $a_{k-1}a_{k-2}\cdots a_2a_1 \neq 00\cdots 0$ stripped of the null digits to the left of the leftmost position containing digit 1 becomes a nonzero RGS. We also consider the RGS 0 corresponding to the null k-germs, where $0 < k \in \mathbb{Z}$.

The k-germs are ordered as follows: Given any two k-germs, say $\alpha = a_{k-1}\cdots a_2a_1$ and $\beta = b_{k-1}\cdots b_2b_1$, where $\alpha \neq \beta$, we say that α precedes β, written $\alpha < \beta$, whenever either **(i)** $a_{k-1} < b_{k-1}$ or **(ii)** $a_j = b_j$, for $k-1 \leq j \leq i+1$, and $a_i < b_i$, for some $1 \leq i < k-1$.

The order defined this way on k-germs of RGSs $\alpha(m)$ $(m \leq C_{k+1})$ is said to be their *stair-wise* order, corresponding biunivocally (via the assignment $m \to \alpha(m)$) with the natural order on m. Thus, there are exactly C_{k+1} k-germs $\alpha = \alpha(m) < 10^k$, for every $k > 0$.

To determine the RGS corresponding to a given decimal integer x_0, or vice versa, we employ *Catalan's triangle* Δ, namely a triangular arrangement composed by positive integers starting with the following rows Δ_j, for $j = 0, \ldots, 8$:

1								
1	1							
1	2	2						
1	3	5	5					
1	4	9	14	14				
1	5	14	28	42	42			
1	6	20	48	90	132	132		
1	7	27	75	165	297	429	429	
1	8	35	110	275	572	1001	1430	1430

with a linear reading as that of the sequence A009766 [9]. The numbers τ_i^j in Δ_j $(0 \leq j \in \mathbb{Z})$, given by $\tau_i^j = (j+i)!(j-i+1)/(i!(j+1)!)$, are characterized as well by four items:
1. $\tau_0^j = 1$, for every $j \geq 0$;
2. $\tau_1^j = j$ and $\tau_j^j = \tau_{j-1}^j$, for every $j \geq 1$;
3. $\tau_i^j = \tau_i^{j-1} + \tau_{i-1}^j$, for every $j \geq 2$ and $i = 1, \ldots, j-2$;
4. $\sum_{i=0}^j \tau_i^j = \tau_j^{j+1} = \tau_{j+1}^{j+1} = C_j$, for every $j \geq 1$.

The determination of the RGS corresponding to a decimal integer x_0 proceeds as follows. Let $y_0 = \tau_k^{k+1}$ be the largest member of the second diagonal of Δ with $y_0 \leq x_0$. Let $x_1 = x_0 - y_0$. If $x_1 > 0$, then let $Y_1 = \{\tau_{k-1}^j\}_{j=k}^{k+b_1}$ be the largest set of successive terms in the $(k-1)$-column of Δ with $y_1 = \sum(Y_1) \leq x_1$. Either $Y_1 = \emptyset$, in which case we take $b_1 = -1$, or not, in which case $b_1 = |Y_1| - 1$. Let $x_2 = x_1 - y_1$. If $x_2 > 0$, then let $Y_2 = \{\tau_{k-2}^j\}_{j=k}^{k+b_2}$ be the largest set of successive terms in the $(k-2)$-column of Δ with $y_2 = \sum(Y_2) \leq x_2$. Either $Y_2 = \emptyset$, in which case we take $b_2 = -1$, or not, in which case $b_2 = |Y_3| - 1$. Iteratively, we arrive at a null x_k. Then the RGS corresponding to x_0 is $a_{k-1}a_{k-2}\cdots a_1$, where $a_{k-1} = 1$, $a_{k-2} = 1 + b_1$, …, and $a_1 = 1 + b_k$.

For example, if $x_0 = 38$, then $y_0 = \tau_3^4 = 14$, $x_1 = x_0 - y_0 = 38 - 14 = 24$, $y_1 = \tau_2^3 + \tau_2^4 = 5 + 9 = 14$, $x_2 = x_1 - y_1 = 24 - 14 = 10$, $y_2 = \tau_1^2 + \tau_1^3 + \tau_1^4 = 2 + 3 + 4 = 9$, $x_3 = x_2 - y_2 = 10 - 9 = 1$, $y_3 = \tau_0^1 = 1$ and $x_4 = x_3 - y_3 = 1 - 1 = 0$, so that $b_1 = 1$, $b_2 = 2$, and $b_3 = 0$, taking to $a_4 = 1$, $a_3 = 1 + b_1 = 2$, $a_2 = 1 + b_2 = 3$ and $a_1 = 1 + b_3 = 1$, determining the 5-germ of 38 to be $a_4a_3a_2a_1 = 1231$. If $x_0 = 20$, then $y_0 = \tau_3^4 = 14$, $x_1 = x_0 - y_0 = 20 - 14 = 6$, $y_1 = \tau_2^3 = 5$, $x_2 = x_1 - y_1 = 1$, $y_2 = 0$ is an empty sum (since its possible summand $\tau_1^2 > 1 = x_2$), $x_3 = x_2 - y_2 = 1$, $y_3 = \tau_0^1 = 1$ and $x_4 = x_3 - x_3 = 1 - 1 = 0$, determining the 5-germ of 20 to be $a_4a_3a_2a_1 = 1101$. Moreover, if $x_0 = 19$, then $y_0 = \tau_3^4 = 14$, $x_1 = x_0 - y_0 = 19 - 14 = 5$, $y_1 = \tau_2^3 = 5$, $x_2 = x_1 - y_1 = 5 - 5 = 0$, determining the 5-germ $a_4a_3a_2a_1 = 1100$.

Given an RGS or a k-germ $a_{k-1} \cdots a_1$, the considerations above can easily be played backwards to recover the corresponding decimal integer x_0.

2 Castling

Theorem 1. *Each k-germ $\alpha \neq 0^{k-1}$ determines a k-germ $\beta(\alpha) = b_{k-1} \cdots b_1 < \alpha$ with $b_i = a_i - 1$, where a_i is the rightmost nonzero entry of α, and $a_j = b_j$ for $j \neq i$. Now, the k-germs form a tree $\mathcal{T}_k$ rooted at 0^{k-1} in which each k-germ $\alpha \neq 0^{k-1}$ is a child of $\beta(\alpha)$.*

Proof. This is immediate, illustrated in the first three columns of Table I, (table which as a whole is detailed below and serves as illustration to the proof of Theorem 2). □

TABLE I

m	α	β	$F(\beta)$	i	$W^i \mid X \mid Y \mid Z^i$	$W^i \mid Y \mid X \mid Z^i$	$F(\alpha)$	α
0	0	–	–	–	–	–	210**	0
1	1	0	210**	1	$2 \mid 1 \mid 0* \mid *$	$2 \mid 0* \mid 1 \mid *$	20*1*	1
0	00	–	–	–	–	–	3210***	00
1	01	00	3210***	1	$3 \mid 2 \mid 10** \mid *$	$3 \mid 10** \mid 2 \mid *$	310**2*	01
2	10	00	3210***	2	$32 \mid 1 \mid 0* \mid **$	$32 \mid 0* \mid 1 \mid **$	320*1**	10
3	11	10	320*1**	1	$3 \mid 20* \mid 1* \mid *$	$3 \mid 1* \mid 20* \mid *$	31*20**	11
4	12	11	31*20**	1	$3 \mid 1*2 \mid 0* \mid *$	$3 \mid 0* \mid 1*2 \mid *$	30*1*2*	12
0	000	–	–	–	–	–	43210****	000
1	001	000	43210****	1	$4 \mid 3 \mid 210*** \mid *$	$4 \mid 210*** \mid 3 \mid *$	4210***3*	001
2	010	000	43210****	2	$43 \mid 2 \mid 10** \mid **$	$43 \mid 10** \mid 2 \mid **$	4310**2**	010
3	011	010	4310**2**	1	$4 \mid 310** \mid 2* \mid *$	$4 \mid 2* \mid 310** \mid *$	42*310***	011
4	012	011	42*310***	1	$4 \mid 2*3 \mid 10** \mid *$	$4 \mid 10** \mid 2*3 \mid *$	410**2*3*	012
5	100	000	43210****	3	$432 \mid 1 \mid 0* \mid ***$	$432 \mid 0* \mid 1 \mid ***$	4320*1***	100
6	101	100	4320*1***	1	$4 \mid 3 \mid 20*1** \mid *$	$4 \mid 20*1** \mid 3 \mid *$	420*1**3*	101
7	110	100	4320*1***	2	$43 \mid 20* \mid 1* \mid **$	$43 \mid 1* \mid 20* \mid **$	431*20***	110
8	111	110	431*20***	1	$4 \mid 31* \mid 20** \mid *$	$4 \mid 20** \mid 31* \mid *$	420**31**	111
9	112	111	420**31**	1	$4 \mid 20**3 \mid 1* \mid *$	$4 \mid 1* \mid 20**3 \mid *$	41*20**3*	112
10	120	110	431*20***	2	$43 \mid 1*2 \mid 0* \mid **$	$43 \mid 0* \mid 1*2 \mid **$	430*1*2**	120
11	121	120	430*1*2**	1	$4 \mid 30*1* \mid 2* \mid *$	$4 \mid 2* \mid 30*1* \mid *$	42*30*1**	121
12	122	121	42*30*1**	1	$4 \mid 2*30* \mid 1* \mid *$	$4 \mid 1* \mid 2*30* \mid *$	41*2*30**	122
13	123	122	41*2*30**	1	$4 \mid 1*2*3 \mid 0* \mid *$	$4 \mid 0* \mid 1*2*3 \mid *$	40*1*2*3*	123

By representing $\mathcal{T}_k$ with each k-germ β having its children α enclosed between parentheses after β, and separating siblings with commas, we can write
$\mathcal{T}_4 = 000(001, 010(011(012)), 100(101, 110(111(121)), 120(121(122(123)))))$.

The procedure of three steps in Theorem 2 below will be called *castling* procedure.

Theorem 2. *To each k-germ $\alpha = a_{k-1}\cdots a_1$ corresponds an n-string $F(\alpha) = f_0f_1\cdots f_{2k}$ whose entries are k asterisks $(*)$ and the numbers $0,1,\ldots,k$ (once each), and such that $F(0^{k-1}) = k(k-1)(k-2)\cdots 210*\cdots*$. If $\alpha \neq 0^{k-1}$, then $F(\alpha)$ is obtained from the parent $F(\beta) = F(\beta(\alpha)) = h_0h_1\cdots h_{2k}$ of α in $\mathcal{T}_k$ by means of the following* castling procedure *steps:*

1. *let $W^i = h_0h_1\cdots h_{i-1} = f_0f_1\cdots f_{i-1}$ and $Z^i = h_{2k-i+1}\cdots h_{2k-1}h_{2k} = f_{2k-i+1}\cdots f_{2k-1}f_{2k}$ be respectively the initial and terminal substrings of length i in $F(\beta)$;*

2. *let $\Omega > 0$ be the leftmost entry of the substring $U = F(\beta)\setminus(W^i \cup Z^i)$ and consider the concatenation $U = X|Y$, with Y starting at entry $\Omega - 1$;*

3. *by noticing that $F(\beta) = W^i|X|Y|Z^i$, set $F(\alpha) = W^i|Y|X|Z^i$.*

As a result: **(a)** *the leftmost entry of each $F(\alpha)$ is k;* **(b)** *each number to the immediate right of a number $b \in \{1,\ldots,k\}$ in such $F(\alpha)$ is less than b;* **(c)** *$0*$ is a substring of $F(\alpha)$, but $*0$ is not;* **(d)** *W^i is a number i-substring;* **(e)** *Z^i is formed by i of the k asterisks.*

Proof. Let α be a k-germ $\alpha = a_{k-1}\cdots a_1 \neq 0^{k-1}$. In the sequence of applications of items 1-3 along the path in $\mathcal{T}_k$ from its root 0^{k-1} to α, unit augmentation of a_i for larger values of i, $(0 < i < k)$, must occur earlier, and then in strictly descending order of the entries i of the intermediate k-germs. Thus, the length of the inner substring $X|Y$ is maintained non-decreasing after each application. This is illustrated in Table I above, where the order of the appearing substrings X and Y, that have their first elements being respectively Ω and $\Omega - 1$, is reversed in successively decreasing steps. In the process, items (a)-(e) in the statement are seen to be satisfied.

In Table I, the k-germs α are presented in stair-wise order (see first column) for $k = 2,3,4$, both on the second and ninth columns; their corresponding images under F are shown on the eighth column. The three successive listings in the table have C_k rows each, where $C_2 = 2$, $C_3 = 5$ and $C_4 = 14$; the remaining columns in the table are filled, from the third row on, as follows: **(i)** β as arising in item (c) of Theorem 2; **(ii)** $F(\beta)$, taken from the eighth column in the previous row; **(iii)** the length i $(k-1 \geq i \geq 1)$ of W^i and Z^i; **(iv)** the decomposition $W^i|Y|X|Z^i$ of $F(\beta)$; **(vi)** the decomposition $W^i|X|Y|Z^i$ of $F(\alpha)$, re-concatenated in the following, or eighth column as $F(\alpha)$, with $\alpha = F^{-1}(\alpha)$ in the ninth column. □

To each $F(\alpha)$ corresponds a binary n-string $\phi(\alpha)$ of weight k obtained by replacing each number by 0 and each asterisk $*$ by 1. By attaching the entries of $F(\alpha)$ as subscripts to the corresponding entries of $\phi(\alpha)$, a subscripted binary n-string $\bar{\phi}(\alpha)$ is obtained, as on the left of Table II. Let $\aleph(\phi(\alpha))$ be given by the *reverse complement* of $\phi(\alpha)$, that is

$$\text{if } \phi(\alpha) = a_0a_1\cdots a_{2k}, \text{ then } \aleph(\phi(\alpha)) = \bar{a}_{2k}\cdots\bar{a}_1\bar{a}_0, \tag{2}$$

where $\bar{0} = 1$ and $\bar{1} = 0$. A subscripted version $\bar{\aleph}$ of $\aleph$ is immediately obtained for $\bar{\phi}(\alpha)$. Each image under $\aleph$ is an n-string of weight $k+1$ and has the 1 s with number subscripts and the 0 s with asterisk subscripts. The number subscripts reappear from Section 6 to Section 8 as *lexical colors* [5]. Table II illustrates the notions just presented, for $k = 2,3$.

Not all the n-strings satisfying items (a)-(e) in Theorem 2 happen along a descending rooted path of $\mathcal{T}_k$ via successive application of the castling procedure steps (1)-(3), (but those that do end up representing in Section 7 the vertices of the graph R_k cited in Section 1). For example, $F(01)$ yields, with $i = 1$, the 7-tuple $F' = 30**21*$. However, $\phi(F') = 0011001$ is already represented cyclically in Table I by $\phi(F(11)) = 1100100$, as needed in what follows.

TABLE II

m	α	$\phi(\alpha)$	$\bar{\phi}(\alpha)$	$\bar{\aleph}(\phi(\alpha))=\aleph(\bar{\phi}(\alpha))$	$\aleph(\phi(\alpha))$
0	0	00011	$0_2 0_1 0_0 1_* 1_*$	$0_* 0_* 1_0 1_1 1_2$	00111
1	1	00101	$0_2 0_0 1_* 0_1 1_*$	$0_* 1_1 0_* 1_0 1_2$	01011
0	00	0000111	$0_3 0_2 0_1 0_0 1_* 1_* 1_*$	$0_* 0_* 0_* 1_0 1_1 1_2 1_3$	0001111
1	01	0001101	$0_3 0_1 0_0 1_* 1_* 0_2 1_*$	$0_* 1_2 0_* 0_* 1_0 1_1 1_3$	0100111
2	10	0001011	$0_3 0_2 0_0 1_* 0_1 1_* 1_*$	$0_* 0_* 1_1 0_* 1_0 1_2 1_3$	0010111
3	11	0010011	$0_3 0_1 1_* 0_2 0_0 1_* 1_*$	$0_* 0_* 1_0 1_2 0_* 1_1 1_3$	0011011
4	12	0010101	$0_3 0_0 1_* 0_1 1_* 0_2 1_*$	$0_* 1_2 0_* 1_1 0_* 1_0 1_3$	0101011

3 The Middle-Levels Graphs

Let $1 < n \in \mathbb{Z}$. The *n-cube graph* H_n is the Hasse diagram of the Boolean lattice $\mathcal{B}_n$ on the coordinate set $[n] = \{0, \ldots, n-1\}$. Each vertex of H_n is referred in three different ways, as:
(a) the subset $A = \{a_0, a_1, \ldots, a_{j-1}\} = a_0 a_1 \cdots a_{j-1}$ of $[n]$ it represents, for $0 < j \leq n$;
(b) the characteristic n-vector $B_A = (b_0, b_1, \ldots, b_{n-1})$ over the field $\mathbb{F}_2$ that the subset A in (a) represents, meaning it is given by $b_i = 1$ if and only if $i \in A$, $(i \in [n])$, and represented for short by $B_A = b_0 b_1 \cdots b_{n-1}$;
(c) the polynomial $\epsilon_A(x) = b_0 + b_1 x + \cdots + b_{n-1} x^{n-1}$ associated with the vector B_A in (b).

A subset A as in (a) is said to be the *support* of the vector B_A in (b). For each $j \in [n]$, the *j-level* L_j is the vertex subset in H_n formed by those $A \subseteq [n]$ with $|A| = j$.

For $1 \leq k \in \mathbb{Z}$, the *middle-levels graph* M_k is defined as the subgraph of H_n induced by $L_k \cup L_{k+1} = V(M_k)$. This is the set of vertices of M_k. By viewing these vertices as polynomials as in item (c) above, an equivalence relation π on $V(M_k)$ is given by:

$$\epsilon_A(x) \pi \epsilon_{A'}(x) \Longleftrightarrow \exists\, i \in \mathbb{Z} \text{ such that } \epsilon_{A'}(x) \equiv x^i \epsilon_A(x) \pmod{1 + x^n},$$

where $A, A' \in V(M_k)$. There exists a quotient graph M_k/π induced by the following regular (i.e., free and transitive) action Υ' of $\mathbb{Z}_n$ on $V(M_k)$:

$$\Upsilon' : \mathbb{Z}_n \times V(M_k) \to V(M_k) \text{ such that } \Upsilon'(i, v) = v(x) x^i \pmod{1 + x^n} \quad (3)$$

to be used in the proof of Theorem 4 and presented again in polynomial terms, where $v \in V(M_k)$ and $i \in \mathbb{Z}_n$. Now, M_k/π is the graph whose vertices are the equivalence classes under π of those of M_k and whose edges are the equivalence classes that π induces on the edge set $E(M_k)$ of M_k.

4 Reflection-Symmetry Bijections

The definition of $\aleph$ in display (2) is extended to a bijection $\aleph : L_k \to L_{k+1}$. The image of each element $v \in L_k$ through this bijection $\aleph$ is said to be the *reverse complement* of v.

Let $\rho_i : L_i \to L_i/\pi$ be the canonical projection given by assigning $b_0 b_1 \cdots b_{n-1} \in L_i$ to the class $(b_0 b_1 \cdots b_{n-1})$ of $b_0 b_1 \cdots b_{n-1}$ in $\in L_i/\pi$, for $i = k, k+1$. Let $\aleph_\pi : L_k/\pi \to$

L_{k+1}/π be given by $\aleph_\pi((b_0 b_1 \cdots b_{n-1})) = (\bar{b}_{n-1} \cdots \bar{b}_1 \bar{b}_0)$. Then, $\aleph_\pi$ is a bijection and there are commutative identities $\rho_{k+1}\aleph = \aleph_\pi \rho_k$ and $\rho_k \aleph^{-1} = \aleph_\pi^{-1} \rho_{k+1}$.

We list vertically the vertex parts L_k and L_{k+1} of M_k (resp., L_k/π and L_{k+1}/π of M_k/π), displaying a splitting of $V(M_k) = L_k \cup L_{k+1}$ (resp., $V(M_k)/\pi = L_k/\pi \cup L_{k+1}/\pi$) into pairs, each pair contained in a corresponding horizontal line, its two composing vertices equidistant from a vertical line ℓ (resp., ℓ/π) like the dashed line ℓ/π in Figure 1, Section 5 below, for M_2/π. Each resulting horizontal vertex pair in M_k (resp., M_k/π) must be of the form $(B_A, \aleph(B_A))$ (resp., $((B_A), (\aleph(B_A)) = \aleph_\pi((B_A))))$, disposed from left to right, at both sides of ℓ. A non-horizontal edge of M_k/π is said to be a *skew edge*.

Theorem 3. *To each skew edge $e = (B_A)(B_{A'})$ of M_k/π corresponds another skew edge $\aleph_\pi((B_A))\aleph_\pi^{-1}((B_{A'}))$ obtained from e by reflection on the line ℓ/π. Then:* **(i)** *the skew edges of M_k/π appear in pairs, with the endpoints in each pair forming two pairs of horizontal vertices equidistant from ℓ/π;* **(ii)** *the horizontal edges of M_k/π have multiplicity ≤ 2.*

Proof. The skew edges $B_A B_{A'}$ and $\aleph^{-1}(B_{A'})\aleph(B_A)$ of M_k are reflection of each other about ℓ. They have the pairs $(B_A, \aleph(B_{A'}))$ and $(\aleph^{-1}(B_A), B_{A'})$ of endpoints lying on horizontal lines. Now, ρ_k and ρ_{k+1} extend together to a covering graph map $\rho : M_k \to M_k/\pi$, since the edges accompany the projections correspondingly, as for $k = 2$:

$$\aleph((00011))= \quad \aleph(\{00011,10001,11000,01100,00110\})=\{00111,01110,11100,11001,10011\}=(00111),$$
$$\aleph^{-1}((01011))=\aleph^{-1}(\{01011,10110,10110,11010,10101\})=\{00101,10010,01001,10100,01010\}=(00101),$$

showing the order of the elements in the images of the classes mod π under $\aleph$ and $\aleph^{-1}$, (presented backwards, i.e. from right to left, cyclically between braces, and continuing on the right once one reaches a leftmost brace). This behavior holds for every $k > 2$:

$$\aleph((b_0 \cdots b_{2k}))= \quad \aleph(\{b_0 \cdots b_{2k},\, b_{2k} \ldots b_{2k-1},\, \ldots,\, b_1 \cdots b_0\})=\{\bar{b}_{2k} \cdots \bar{b}_0,\, \bar{b}_{2k-1} \cdots \bar{b}_{2k},\, \ldots,\, \bar{b}_1 \cdots \bar{b}_0\}=(\bar{b}_{2k} \cdots \bar{b}_0),$$
$$\aleph^{-1}((\bar{b}'_{2k} \cdots \bar{b}'_0))=\aleph^{-1}(\{\bar{b}'_{2k} \cdots \bar{b}'_0,\, \bar{b}'_{2k-1} \cdots \bar{b}'_{2k},\, \ldots,\, \bar{b}'_1 \cdots \bar{b}'_0\})=\{b'_0 \cdots b'_{2k},\, b'_{2k} \cdots b'_{2k-1},\, \ldots,\, b'_1 \cdots b'_0\}=(b'_0 \cdots b'_{2k}),$$

where $(b_0 \cdots b_{2k}) \in L_k/\pi$ and $(b'_0 \cdots b'_{2k}) \in L_{k+1}/\pi$. This establishes item (i) of the statement.

Every horizontal edge $v\aleph_\pi(v)$ of M_k/π has $v \in L_k/\pi$ represented by $\bar{b}_k \cdots \bar{b}_1 0 b_1 \cdots b_k$ in L_k, (so $v = (\bar{b}_k \cdots \bar{b}_1 0 b_1 \cdots b_k)$). Thus, there are 2^k such vertices in L_k and at most 2^k corresponding vertices of L_k/π. For example, $(0^{k+1}1^k)$ and $(0(01)^k)$ are endpoints in L_k/π of two horizontal edges in M_k/π, each. To prove that this implies item (ii), we have to see that there cannot be more than two representatives $\bar{b}_k \cdots \bar{b}_1 b_0 b_1 \cdots b_k$ and $\bar{c}_k \cdots \bar{c}_1 c_0 c_1 \cdots c_k$ of a vertex $v \in L_k/\pi$, with $b_0 = c_0 = 0$. Such a v would be written as $v = (d_0 \cdots b_0 d_{i+1} \cdots d_{j-1} c_0 \cdots d_{2k})$, with $b_0 = d_i$, $c_0 = d_j$ and $0 < j - i \leq k$. A substring $\sigma = d_{i+1} \cdots d_{j-1}$ with $0 < j - i \leq k$ is said to be $(j-i)$-*feasible* if v fulfills (ii) with multiplicity at least 2. Let us see that every $(j-i)$-feasible substring σ forces in L_k/π only vertices $\omega \in L_k$ leading to two different (parallel) horizontal edges in M_k/π incident to v. In fact, periodic continuation mod n of $d_0 \cdots d_{2k}$ both to the right of $d_j = c_0$ with minimal cyclic substring $\bar{d}_{j-1} \cdots \bar{d}_{i+1} 1 d_{i+1} \cdots d_{j-1} 0 = P_r$ and to the left of $d_i = b_0$ with minimal cyclic substring $0 d_{i+1} \cdots d_{j-1} 1 \bar{d}_{j-1} \cdots \bar{d}_{i+1} = P_\ell$ yields a two-way infinite string that winds up onto a class $(d_0 \cdots d_{2k})$ containing such an ω. For example, some pairs of feasible substrings σ and resulting vertices ω are:

$(\emptyset,(\text{oo1}))$, $(0,(\text{o0o11}))$, $(1,(\text{o1o}))$, $(0^2,(\text{o00o111}))$, $(01,(\text{o01o011}))$, $(1^2, \text{o11o0}))$,
$(0^3,\text{o000o1111}))$, $(010,(\text{o010o101101}))$, $(01^2,(\text{o011o}))$, $(101,(\text{o101o}))$, $(1^3,(\text{o111o00}))$,

with ‘o’ indicating the positions $b_0 = 0$ and $c_0 = 0$, and where k has successive values $n = 1, 2, 1, 3, 3, 2, 4, 5, 2, 2, 3$. (However, the substrings $0^2 1$ and 10^2 are non-feasible). If σ is a feasible substring and $\bar{\sigma}$ is its reverse complement via $\aleph$, then the possible symmetrical substrings about $\mathrm{o}\sigma\mathrm{o} = 0\sigma 0$ in a vertex v of L_k/π are in order of ascending length:

$$\begin{gathered}
0\sigma 0,\\
\bar{\sigma}0\sigma 0\bar{\sigma},\\
1\bar{\sigma}0\sigma 0\bar{\sigma}1,\\
\sigma 1\bar{\sigma}0\sigma 0\bar{\sigma}1\sigma,\\
0\sigma 1\bar{\sigma}0\sigma 0\bar{\sigma}1\sigma 0,\\
\bar{\sigma}0\sigma 1\bar{\sigma}0\sigma 0\bar{\sigma}1\sigma 0\bar{\sigma},\\
1\bar{\sigma}0\sigma 1\bar{\sigma}0\sigma 0\bar{\sigma}1\sigma 0\bar{\sigma}1,\\
\cdots\cdots\cdots\cdots\cdots\cdots,
\end{gathered}$$

where we use again ‘0’ instead of ‘o’ for the entries immediately preceding and following the shown central copy of σ. Due to this, the finite lateral periods of the resulting P_r and P_ℓ do not allow a third horizontal edge (at v in M_k/π) up to returning to b_0 or c_0 since no entry $e_0 = 0$ of $(d_0 \cdots d_{2k})$ other than b_0 or c_0 happens such that $(d_0 \cdots d_{2k})$ has a third representative $\bar{e}_k \cdots \bar{e}_1 0 e_1 \cdots e_k$ (besides $\bar{b}_k \cdots \bar{b}_1 0 b_1 \cdots b_k$ and $\bar{c}_k \cdots \bar{c}_1 0 c_1 \cdots c_k$). Thus, those two horizontal edges are produced solely from the feasible substrings $d_{i+1} \cdots d_{j-1}$ characterized above. □

To illustrate Theorem 3, let $1 < h < n$ in $\mathbb{Z}$ be such that $\gcd(h, n) = 1$ and let $\lambda_h : L_k/\pi \to L_k/\pi$ be given by $\lambda((a_0 a_1 \cdots a_n)) \to (a_0 a_h a_{2h} \cdots a_{n-2h} a_{n-h})$. For each h with $1 < h \leq k$, there is at least one h-feasible substring σ and a resulting associated vertex $v \in L_k/\pi$ as in the proof of the theorem. For example, applying λ_h repeatedly by starting at $v = (0^{k+1} 1^k) \in L_k/\pi$ produces a number of such vertices $v \in L_k/\pi$. If we assume $h = 2h'$ with $h' \in \mathbb{Z}$, then an h-feasible substring σ has the form $\sigma = \bar{a}_1 \cdots \bar{a}_{h'} a_{h'} \cdots a_1$, so there are at least $2^{h'} = 2^{\frac{h}{2}}$ such h-feasible substrings.

5 Dihedral Actions and Quotients

Let G be a graph. An *involution* of G is a graph map $\aleph : G \to G$ such that $\aleph^2$ is the identity. Given a graph G with an involution $\aleph : G \to G$, an $\aleph$-*folding* of G is a graph H whose vertices and edges are respectively the pairs $\{v, \aleph(v)\}$ and $\{e, \aleph(e)\}$, where $v \in V(G)$, and $e \in E(G)$. Here, e has end-vertices v and $\aleph(v)$ if and only if $\{e, \aleph(e)\}$ is a loop.

Note that both maps $\aleph : M_k \to M_k$ and $\aleph_\pi : M_k/\pi \to M_k/\pi$ in Section 4 are involutions. Let us denote each pair $((B_A), \aleph_\pi((B_A)))$ of M_k/π, horizontally represented in Section 4, via the notation $[B_A]$, where $|A| = k$. An $\aleph$-folding R_k of M_k/π is obtained whose vertices are the pairs $[B_A]$ and having:

(1) an edge $[B_A][B_{A'}]$ per skew-edge pair $\{(B_A)\aleph_\pi((B_{A'})), (B_{A'})\aleph_\pi((B_A))\}$;

(2) a loop at $[B_A]$ per horizontal edge $(B_A)\aleph_\pi((B_A))$. Because of Theorem 3, there may be up to two loops at each vertex of R_k.

Theorem 4. *R_k is a quotient graph of M_k under an action $\Upsilon : D_{2n} \times M_k \to M_k$.*

Proof. To define Υ, recall that D_{2n} is the semidirect product $\mathbb{Z}_n \rtimes_\varrho \mathbb{Z}_2$ via the group homomorphism $\varrho : \mathbb{Z}_2 \to \mathrm{Aut}(\mathbb{Z}_n)$ such that $\varrho(1)$ is the automorphism assigning $i \in \mathbb{Z}_n$ to $(n - i) \in \mathbb{Z}_n$ and such that $\varrho(0)$ as the identity. If $* : D_{2n} \times D_{2n} \to D_{2n}$ indicates group multiplication and $i_1, i_2 \in \mathbb{Z}_n$, then $(i_1, 0) * (i_2, j) = (i_1 + i_2, j)$ and $(i_1, 1) * (i_2, j) = (i_1 - i_2, 1 + j)$, for $j \in \mathbb{Z}_2$. Set $\Upsilon((i,j),v) = \Upsilon'(i, \aleph^j(v))$, for $i \in \mathbb{Z}_n$ and $j \in \mathbb{Z}_2$, where Υ' was defined in display (3). It is easy to see that Υ is a well-defined action of D_{2n} on M_k. By writing $(i,j) \cdot v = \Upsilon((i,j),v)$ and $v = a_0 \cdots a_{2k}$, we have $(i,0) \cdot v = a_{n-i+1} \cdots a_{2k} a_0 \cdots a_{n-i} = v'$ and $(0,1) \cdot v' = \bar{a}_{i-1} \cdots \bar{a}_0 \bar{a}_{2k} \cdots \bar{a}_i = (n-i,1) \cdot v = ((0,1) * (i,0)) \cdot v$, leading to the compatibility condition $((i,j) * (i',j')) \cdot v = (i,j) \cdot ((i',j') \cdot v)$ that a group action must satisfy, (together with the identity condition). □

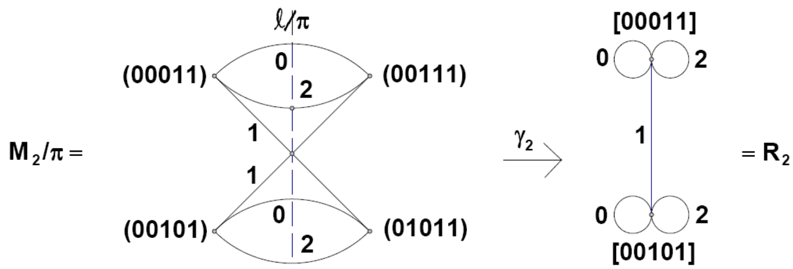

Figure 1: Reflection symmetry of M_2/π about a line ℓ/π and resulting graph map γ_2

Let the graph map $\gamma_k : M_k/\pi \to R_k$ be the projection corresponding to the action Υ as represented for $k = 2$ in Figure 1. This map is associated with reflection symmetry of M_2/π about the dashed vertical line ℓ/π acting as symmetry axis. In the figure, R_2 is represented as the image of γ_2 and contains two vertices and just one (vertical) edge between them, with each vertex incident to two loops. Both the representations of M_2/π and R_2 in the figure have their edges indicated with colors 0,1,2, as arising from Section 6.

6 Lexical Procedure (or LP)

Let us see now how each vertex v of L_k/π has its incident edges enumerated via the *lexical colors* $0, 1, \ldots, k$ arising from the treatment of [5].

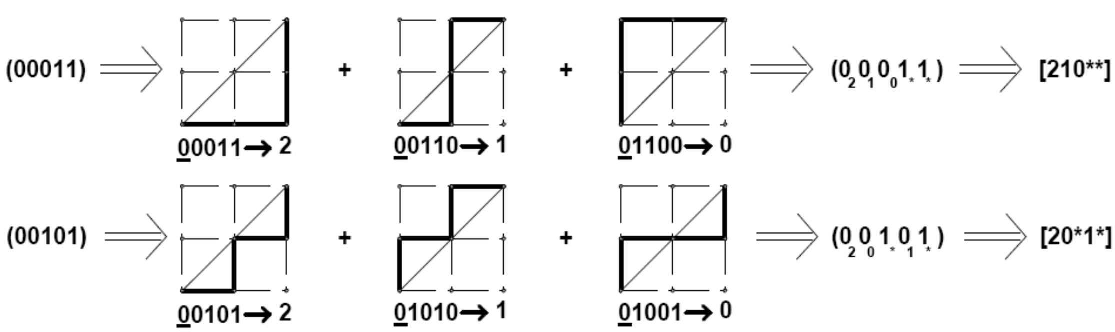

Figure 2: Representing the color assignment for $k = 2$

Let P_{k+1} be the subgraph of the unit-distance graph of the real line $\mathbb{R}$ induced by the set $[k+1] \subset \mathbb{Z} \subset \mathbb{R}$. We represent the grid $\Gamma = P_{k+1} \square P_{k+1}$ in the plane with a diagonal Δ traced from the lower-left vertex placed at $(0,0)$ to the upper-right vertex placed at (k,k). For each $v \in L_k/\pi$ there are $k+1$ n-tuples of the form $b_0 b_1 \cdots b_{n-1} = 0 b_1 \cdots b_{n-1}$ that represent v

with $b_0 = 0$. For each such representative n-tuple, we construct a $2k$-path D in Γ from $(0,0)$ to (k,k) in $2k$ steps indexed from $i = 0$ to $i = 2k-1$ as follows. (See Figure 2 with examples of D in dark trace, further commented in Section 7). Initially, let $i = 0$, $w = (0,0)$ and D contain solely w and no edges. Repeat the following sequence of steps (1)-(3) $2k$ times, and then perform the subsequent steps (4)-(5):

(1) If $b_i = 0$ (resp., $b_i = 1$), then set $w' := w + (1,0)$ (resp., $w' := w + (0,1)$).

(2) Reset $V(D) := v(D) \cup \{w'\}$, $E(D) := E(D) \cup \{ww'\}$, $i := i+1$ and $w := w'$.

(3) If $w \neq (k,k)$, or equivalently, if $i < 2k$, then go back to step (1).

(4) Set $\bar{v} \in L_{k+1}/\pi$ as a vertex of M_k/π adjacent to v and obtained from the representative n-tuple $b_0 b_1 \cdots\ b_{n-1} = 0b_1 \cdots b_{n-1}$ of v by replacing the entry b_0 by $\bar{b}_0 = 1$ in $\bar{v}$, keeping the entries b_i unchanged in $\bar{v}$, where $i > 0$.

(5) Set the *color* of the edge $v\bar{v}$ to be the number c of horizontal (alternatively, vertical) arcs of D below the diagonal Δ of Γ.

A proof of the original version of this in [5] uses the numbers $k+1-c$ with $c \in [k+1]$. In fact, if addition and subtraction in $[n]$ are taken modulo n, then by writing $[y,x) = \{y, y+1, y+2, \ldots, x-1\}$, for $x, y \in [n]$, and $S^c = [n] \setminus S$, for $S = \{i \in [n] : b_i = 1\} \subseteq [n]$, the cardinalities of the sets $\{y \in S^c \setminus x : |[y,x) \cap S| < |[y,x) \cap S^c|\}$ yield all the numbers $k+1-c$ in 1-1 correspondence with our colors c, where $x \in S^c$ varies.

The lexical procedure (or *LP*) just presented yields 1-factorizations not only of M_k/π but also of R_k and M_k, clarified by the end of the next section.

7 Un-Castling

In this section, a color notation $\delta(v)$ is attached to each vertex v of R_k (i.e., in L_k/π), so that there is a unique k-germ $\alpha = \alpha(v)$ with $[F(\alpha)] = \delta(v)$. We start by representing the lexical color assignment suggested for $k = 2$ in Figure 2, with the LP (indicated by arrows "$\Rightarrow$") departing from $v = [00011]$ (top) and $v = [00101]$ (bottom), then passing to working sketches of Γ (separated by plus signs, "+"), one sketch per representative of v of the form $b_0 b_1 \cdots b_{n-1} = 0b_1 \cdots b_{n-1}$ (shown under the sketch, with the entry $b_0 = 0$ underscored, and pointing via an arrow "$\to$" to its color $c \in [k+1]$, acquired as in step (5) of the LP) in which to trace the edges of $D \subset \Gamma$, (where c is the number of horizontal arcs of D below Δ).

In each of the two cases in Figure 2, to the right of the three shown sketches, a second arrow "$\Rightarrow$" points to a modification v_* of $b_0 b_1 \cdots b_{n-1} = 0b_1 \cdots b_{n-1}$ obtained by setting as a subindex of each entry 0 the color c obtained from its corresponding sketch, and an asterisk "$*$" to each entry 1. Further to the right, a third arrow "$\Rightarrow$" points to the n-tuple $\delta(v)$ formed by the string of subindexes of entries of v_* in the order they appear from left to right. The following procedure will be called *un-castling* procedure.

Theorem 5. *To each $\delta(v)$ as above corresponds a unique k-germ $\alpha = \alpha(v)$ with $[F(\alpha)] = \delta(v)$ obtained as follows. Given $v \in L_k/\pi$, let $W^i = k(k-1)\cdots(k-i)$ be the maximal initial number $(i+1)$-substring of $\delta(v)$, where $0 \le i \le k$. Let $\alpha(v^0) = a_{k-1} \cdots \alpha_1 = 00\cdots0$. If $i = k$, then let $\alpha(v) = \alpha(v^0)$; else, set $m = 0$ and proceed as follows:*

1. *set $\delta(v^m) = [W^i|X|Y|Z^i]$, where Z^i is the terminal j_m-substring of $\delta(v^m)$, with $j_m = i+1$, and X, Y (in that order) start at contiguous numbers Ω and $\Omega + 1 \le k - i$;*

2. *set* $\delta(v^{m+1}) = [W^i|Y|X|Z^i]$*;*
3. *let* $\alpha(v^{m+1})$ *be obtained from* $\alpha(v^m)$ *just by increasing its entry* a_{j_m} *by 1;*
4. *if* $\delta(v^{m+1}) = [k(k-1)\cdots 210 * \cdots *]$*, then stop; else, increase* m *by 1 and go to step 1.*

Proof. This is a procedure inverse to that of castling (Section 2). □

This un-castling procedure leads to a finite sequence $\delta(v^0), \delta(v^1), \ldots, \delta(v^s)$ of n-strings in L_k/π with parameters $j_0 \geq j_1 \geq \cdots \geq j_s$ and k-germs $\alpha(v^0), \alpha(v^1), \ldots, \alpha(v^s)$. It also leads from $\alpha(v^0)$ to $\alpha(v) = \alpha(v^s)$ by unit incrementation of a_{j_i}, for $i = 0, \ldots, s$, with each incrementation yielding the corresponding $\alpha(v^i)$. Observe that F is a bijection between the set $V(\mathcal{T}_k)$ of k-germs and the set L_k/π, both being of cardinality C_k. Thus, to deal with $V(R_k)$ it is enough to deal with $V(\mathcal{T}_k$, a fact useful in interpreting Theorem 6 below. For example $\delta(v) = \delta(v^0) = \delta[40*1*2*3*] = [\,4\,|\,0*\,|\,1*2*3\,|\,*] = [W^0|X|Y|Z^0]$ with $i = 0$ and $\alpha(v^0) = 000$, to be continued in Table III with $\delta(v^1) = [W^0|Y|X|Z^0]$ to finally arrive at $\alpha(v) = \alpha(v^s) = \alpha(v^6) = 123$.

TABLE III

j_0=0	$\delta(v^1)$	=	$[4\vert 1{*}2{*}3\vert 0{*}\vert{*}]$	=	$[41{*}2{*}30{*}{*}]$	=	$[4\vert 1{*}\vert 2{*}30{*}\vert{*}]$	$\alpha(v^1)$=001	$[F(001)]=\delta(v^1)$
j_1=0	$\delta(v^2)$	=	$[4\vert 2{*}30{*}\vert 1{*}\vert{*}]$	=	$[42{*}30{*}1{*}{*}]$	=	$[4\vert 2{*}\vert 30{*}1{*}\vert{*}]$	$\alpha(v^2)$=011	$[F(011)]=\delta(v^2)$
j_2=0	$\delta(v^3)$	=	$[4\vert 30{*}1{*}\vert 2{*}\vert{*}]$	=	$[430{*}1{*}2{*}{*}]$	=	$[43\vert 0{*}\vert 1{*}2\vert{*}{*}]$	$\alpha(v^3)$=012	$[F(012)]=\delta(v^3)$
j_3=1	$\delta(v^4)$	=	$[43\vert 1{*}2\vert 0{*}\vert{*}{*}]$	=	$[431{*}20{*}{*}{*}]$	=	$[43\vert 1{*}\vert 20{*}\vert{*}{*}]$	$\alpha(v^4)$=112	$[F(112)]=\delta(v^4)$
j_4=1	$\delta(v^5)$	=	$[43\vert 20{*}\vert 1{*}\vert{*}{*}]$	=	$[4320{*}1{*}{*}{*}]$	=	$[432\,\vert\,0{*}\vert 1\vert{*}{*}]$	$\alpha(v^5)$=122	$[F(122)]=\delta(v^5)$
j_5=2	$\delta(v^6)$	=	$[432\vert 1\vert 0{*}\vert{*}{*}{*}]$	=	$[43210{*}{*}{*}{*}]$			$\alpha(v^6)$=123	$[F(123)]=\delta(v^6)$

A pair of skew edges $(B_A)\aleph_\pi((B_{A'}))$ and $(B_{A'})\aleph((B_A))$ in M_k/π is said to be a *skew reflection edge pair*, (or *SREP*). This provides a color notation for any $v \in L_{k+1}/\pi$ such that in each particular edge class mod π:
(I) each edge receives a common color regardless of the endpoint on which the LP (or its modification, see below) for $v \in L_{k+1}/\pi$ is applied;
(II) the two edges in each SREP in M_k/π are assigned a common color in $[k+1]$.
The modification in step (I) consists in replacing in Figure 2 each v by $\aleph_\pi(v)$ so that on the left we have now instead (00111) (top) and (01011) (bottom) with respective sketch subtitles

$$0011\underline{1}\to 2, \quad 1001\underline{1}\to 1, \quad 1100\underline{1}\to 0,$$
$$0101\underline{1}\to 2, \quad 1010\underline{1}\to 0, \quad 0110\underline{1}\to 1,$$

resulting in similar sketches when the steps (1)-(5) of the LP are taken with right-to-left reading-and-processing of the entries on the left side of the subtitles (before the arrows "→"), where now the values of each b_i must be taken complemented.

Since an SREP in M_k determines a unique edge ϵ of R_k (and vice versa), the color received by this pair can be attributed to ϵ, too. Clearly, each vertex of either M_k or M_k/π or R_k defines a bijection from its incident edges onto the color set $[k+1]$. The edges obtained via $\aleph$ or $\aleph_\pi$ from these edges have the same corresponding colors because of the LP.

Theorem 6. *A* 1*-factorization of* M_k/π *by the edge colors* $0, 1, \ldots, k$ *is obtained via the LP that can be lifted to a covering* 1*-factorization of* M_k *and collapsed onto a folding* 1*-factorization of* R_k *inducing the color notation* $\delta(v)$ *on each of its vertices* v*. Moreover, for each* $v \in V(R_k)$ *and notation* $\delta(v)$*, there is a unique* k*-germ* $\alpha = \alpha(v)$ *such that* $[F(\alpha)] = \delta(v)$*.*

Proof. As pointed out in item (II) above in this section, each SREP in M_k/π has its edges with a common color of $[k+1]$. Thus, the $[k+1]$-coloring of M_k/π induces a well-defined $[k+1]$-coloring of R_k. This yields the claimed collapsing to a folding 1-factorization of R_k. The lifting to a covering 1-factorization in M_k is immediate. The arguments above determine that the collapsing 1-factorization in R_k induces the k-germs $\alpha(v)$ claimed in the statement. □

8 Color-Adjacency Tables

From now on, the vertices $v = [F(\alpha)]$ of R_k are presented in stair-wise order via their notation α, with no parenthetical or bracketed enclosures, and further denoted $\delta(v)$ as in Section 7. Thus, we view R_k as the graph whose vertices are the k-germs α and whose adjacency is inherited from that of their δ-notation in R_k via pullback by F^{-1} (namely, via un-castling).

In Table IV, examples of such disposition are shown for $k = 2$ and 3, where m, $\alpha = \alpha(m)$ and $F(\alpha)$ are shown in the first three columns, for $0 \leq m < C_k$. The neighbors of $F(\alpha)$ in the central columns are presented as $F^0(\alpha)$, $F^1(\alpha)$, ..., $F^k(\alpha)$ respectively for the colors $0, 1, \ldots, k$ of the edges incident to them, where the notation is given via the effect of the function $\aleph$. The last four columns yield the k-germs α^0, α^1, ..., α^k associated via F^{-1} respectively with the listed neighbor vertices $F^0(\alpha)$, $F^1(\alpha)$, ..., $F^k(\alpha)$ of $F(\alpha)$ in R_k.

TABLE IV

m	α	$F(\alpha)$	$F^0(\alpha)$	$F^1(\alpha)$	$F^2(\alpha)$	$F^3(\alpha)$	α^0	α^1	α^2	α^3
0	0	210∗∗	210∗∗	20∗1∗	10∗∗2	–	0	1	0	–
1	1	20∗1∗	1∗20∗	210∗∗	0∗1∗2	–	1	0	1	–
0	00	3210∗∗∗	3210∗∗∗	320∗1∗∗	310∗∗2∗	210∗∗∗3	00	10	01	00
1	01	310∗∗2∗	2∗310∗∗	2∗30∗1∗	3210∗∗∗	1∗20∗∗3	01	12	00	11
2	10	320∗1∗∗	31∗20∗∗	3210∗∗∗	30∗1∗2∗	20∗1∗∗3	11	00	12	10
3	11	31∗20∗∗	320∗1∗∗	20∗∗31∗	31∗20∗∗	10∗∗2∗3	10	11	11	01
4	12	30∗1∗2∗	1∗2∗30∗	2∗310∗∗	320∗1∗∗	0∗1∗2∗3	12	01	10	12

For $k = 4$, observe in Table V a similar resulting stair-wise adjacency disposition. Generalizing this Color-Adjacency Table (or CAT(k), with $k = 4$), the following statement of Theorem 7 is observed, as indicated in the doubly aggregated row under the table, citing the sole (non-asterisk) number column order from right to left that is equal in columns α and α^i ($i = 0, 2, \ldots, k$) and other properties, including that the columns α^0 in all CAT(k)s ($k > 1$) integrate into an integer sequence not present as of now in [9].

TABLE V

m	α	α^0	α^1	α^2	α^3	α^4	m	α	α^0	α^1	α^2	α^3	α^4
–	––	––	––	––	––	––	–	––	––	––	––	––	––
0	000	000	100	010	001	000	7	110	100	111	110	012	010
1	001	001	101	012	000	011	8	111	111	110	122	011	111
2	010	011	121	000	112	110	9	112	101	122	112	010	112
3	011	010	120	011	111	001	10	120	122	011	100	123	120
4	012	012	123	001	110	122	11	121	121	010	121	122	101
5	100	110	000	120	101	100	12	122	120	112	111	121	012
6	101	112	001	123	100	121	13	123	123	012	101	120	123
–	––	––	––	––	––	––	–	––	––	––	––	––	––
		3∗∗	∗∗∗	3∗∗	∗2∗	∗∗1			3∗∗	∗∗∗	3∗∗	∗2∗	∗∗1

Theorem 7. *Let $k > 1$. Then each column α^i in $\mathrm{CAT}(k)$, for $i = 2, 3 \ldots, k$, preserves the respective $j(\alpha^i)$-th entry, where $j(\alpha^i) = k-1, \ldots, 2, 1$ respectively for $i = 2, 3 \ldots, k-1, k$. In other words, $j(\alpha^2) = k-1$, $j(\alpha^3) = k-2$, $\ldots$, $j(\alpha^{k-1}) = 2$, $j(\alpha^k) = 1$. Such an entry-invariance rule does not exist for column α^1. However, $j(\alpha^0) = k-1$. Also, the germs α^i $(0 < i < k)$ in each row of $\mathrm{CAT}(k)$ for $k > 1$ equal the terminal $(k-2)$-substrings of α^{i+1} in the corresponding row in $\mathrm{CAT}(k+1)$. Moreover, all columns α^0 in the tables $\mathrm{CAT}(k)$, for every $k > 1$, form an RGS sequence and thus a corresponding integer sequence, too.*

Proof. Let $\alpha = a_{k-1} \cdots a_2 a_1$ be a k-germ. Then α shares with α^0 all the entries to the left of its leftmost entry 1. This guarantees the last assertion of the theorem. The rest of the proof is developed in Subsection 8.1. The cited integer sequence is not yet in [9]. □

8.1 Adjacency via Specific Colors

Given a k-germ $\alpha = a_{k-1} \cdots a_1$ and a substring $\alpha' = a_{k-j} \cdots a_{k-i}$ of α, where $0 < j \leq i < k$, let $\psi(\alpha') = a_{k-i} \cdots a_{k-j}$ be the reverse string of α'. We consider two special substrings of α, namely: **(a)** the *straight ascent* $\alpha_1 = a_{k-1} \cdots a_{k-i_1}$ of α is maximal ascending substring; **(b)** the *landing ascent* $\alpha'_1 = a_{k-1} \cdots a_{k-i_1}$ of α is maximal non-descending substring with at most two equal terms, unless $a_{k-1} = 0$ in which case α'_1 equals α_1 in (a). In any case, $0 < i_1 < k$.

To get α^0, let $A_1 = ||\alpha_1||$ be the length of the straight ascent $\alpha_1 = a_{k-1} \cdots a_{k-i_1}$ of α. Let $B_1 = A_1 + a_{k-1}$. Set $\beta = b_{k-1} \cdots b_1 = \alpha^0$. Then β has straight ascent $\beta_1 = b_{k-1} \cdots b_{k-i_1} = \alpha_1$ and $\alpha_1 + \psi(\beta_1) = B_1 \cdots B_1$. If $\alpha \neq \alpha_1$, then let $A_2 = ||\alpha_2||$ be the length of the largest continuation substring α_2 of α_1 in α for which $\alpha_2 + \psi(\beta_2) = B_2 \cdots B_2$ with $B_2 = A_1 + A_2 + a_{k-1} - 2$. If possible, let $A_3 = ||\alpha_3||$ be the length of the largest continuation substring α_3 of the concatenated substring $\alpha_1|\alpha_2$ in α for which $\alpha_3 + \psi(\beta_3) = B_3 \cdots B_3$ with $B_3 = A_2 + A_3 - 2$. If possible, let $A_4 = ||\alpha_4||$ be the length of the largest continuation substring α_4 of $\alpha_1|\alpha_2|\alpha_3$ in α for which $\alpha_4 + \psi(\beta_4) = B_4 \cdots B_4$ with $B_4 = A_3 + A_4 - 2$. And so on inductively: if possible, let $A_r = ||\alpha_r||$ be the length of the largest continuation substring α_r of $\alpha_1| \cdots |\alpha_{r-1}$ in α for which $\alpha_r + \psi(\beta_r) = B_r \cdots B_r$ with $B_r = A_{r-1} + A_r - 2$. This procedure yields α^0 from α, for any k-germ α.

Top get α^1, let $A'_1 = ||\alpha'_1||$ be the length of the landing ascent $\alpha'_1 = a'_{k-1} \cdots a'_{k-i_1}$ of α. Set $\beta = b_{k-1} \cdots b_1 = \alpha^1$. Then β has landing ascent $\beta_1 = b_{k-1} \cdots b_{k-i_1}$ such that $\alpha'_1 + \psi(\beta_1) = B_1 \cdots B_1$ with $B_1 = i_1$. If $\alpha \neq \alpha'_1$, then let $A'_2 = ||\alpha'_2||$ be the length of the largest continuation substring α'_2 of α'_1 in α for which $\alpha'_2 + \psi(\beta_2) = B_2 \cdots B_2$ with $B_2 = A'_1 + A'_2 - 2$. If possible, let $A'_3 = ||\alpha'_3||$ be the length of the largest continuation substring α'_3 of $\alpha'_1|\alpha'_2$ in α for which $\alpha'_3 + \psi(\beta_3) = B_3 \cdots B_3$ with $B_3 = A'_2 + A'_3 - 2$. If possible, let $A'_4 = ||\alpha'_4||$ be the length of the largest continuation substring α'_4 of $\alpha'_1|\alpha'_2|\alpha'_3$ in α for which $\alpha'_4 + \psi(\beta_4) = B_4 \cdots B_4$ with $B_4 = A'_3 + A'_4 - 2$. And so on inductively: if possible, let $A'_r = ||\alpha'_r||$ be the length of the largest continuation substring α'_r of $\alpha'_1| \cdots |\alpha'_{r-1}$ in α for which $\alpha'_r + \psi(\beta_r) = B_r \cdots B_r$ with $B_r = A'_{r-1} + A'_r - 2$. This procedure yields α^1 from α, for any k-germ α.

To get α^2, let $\beta = b_{k-1} \cdots b_1 = \alpha^2$ and note $b_{k-1} = a_{k-1}$. Let $\alpha' = \alpha \setminus \{a_{k-1}\}$. Let $A'_1 = ||\alpha'_1||$ be the length of the landing ascent $\alpha'_1 = a_{k-2} \cdots a_{k-i_1}$ of α'. Then $\beta' = \beta \setminus \{b_{k-1}\}$ has landing ascent $\beta'_1 = b_{k-2} \cdots b_{k-i_1}$ such that $\alpha'_1 + \psi(\beta'_1) = B'_1 \cdots B'_1$ with $B'_1 = i_1 - 1 + a_{k-1}$. If $\alpha' \neq \alpha'_1$, then let $A'_2 = ||\alpha'_2||$ be the length of the largest continuation substring α'_2 of α'_1 in

α' for which $\alpha'_2+\psi(\beta'_2)=B'_2\cdots B'_2$ with $B'_2=A'_1+A'_2-2$. If possible, let $A'_3=||\alpha'_3||$ be the length of the largest continuation substring α'_3 of $\alpha'_1|\alpha'_2$ in α' for which $\alpha'_3+\psi(\beta'_3)=B'_3\cdots B'_3$, where $B'_3=A'_2+A'_3-2$. If possible, let $A'_4=||\alpha'_4||$ be the length of the largest continuation substring α'_4 of $\alpha'_1|\alpha'_2|\alpha'_3$ in α' for which $\alpha'_4+\psi(\beta'_4)=B'_4\cdots B'_4$ with $B'_4=A'_3+A'_4-2$. And so on inductively: if possible, let $A'_r=||\alpha'_r||$ be the length of the largest continuation substring α'_r of $\alpha'_1|\cdots|\alpha'_{r-1}$ in α' for which $\alpha'_r+\psi(\beta'_r)=B'_r\cdots B'_r$ with $B'_r=A'_{r-1}+A'_r-2$. This procedure yields α^1 from α, for any k-germ α.

To get α^3, let $\beta=b_{k-1}\cdots b_1=\alpha^3$ and note $b_{k-2}=a_{k-2}$. If $a_{k-2}\in\{0,2\}$ then $b_{k-1}=a_{k-1}$. If $a_{k-2}=1$ then $b_{k-1}=1-a_{k-1}$. Let $\alpha'=\alpha\setminus\{a_{k-1},a_{k-2}\}$ and let $A'_1=||\alpha'_1||$ be the length of the landing ascent $\alpha'_1=a_{k-3}\cdots a_{k-i_1}$ of α'. Then $\beta'=\beta\setminus\{b_{k-1},b_{k-2}\}$ has landing ascent $\beta'_1=b_{k-3}\cdots b_{k-i_1}$ such that $\alpha'_1+\psi(\beta'_1)=B'_1\cdots B'_1$ with $B'_1=i_1-1+a_{k-2}$. If $\alpha'\neq\alpha'_1$, then let $A'_2=||\alpha'_2||$ be the length of the largest continuation substring α'_2 of α'_1 in α' for which $\alpha'_2+\psi(\beta'_2)=B'_2\cdots B'_2$ with $B'_2=A'_1+A'_2-2$. If possible, let $A'_3=||\alpha'_3||$ be the length of the largest continuation substring α'_3 of $\alpha'_1|\alpha'_2$ in α' for which $\alpha'_3+\psi(\beta'_3)=B'_3\cdots B'_3$, where $B'_3=A'_2+A'_3-2$. If possible, let $A'_4=||\alpha'_4||$ be the length of the largest continuation substring α'_4 of $\alpha'_1|\alpha'_2|\alpha'_3$ in α' for which $\alpha'_4+\psi(\beta'_4)=B'_4\cdots B'_4$ with $B'_4=A'_3+A'_4-2$. And so on inductively: if possible, let $A'_r=||\alpha'_r||$ be the length of the largest continuation substring α'_r of $\alpha'_1|\cdots|\alpha'_{r-1}$ in α' for which $\alpha'_r+\psi(\beta'_r)=B'_r\cdots B'_r$ with $B'_r=A'_{r-1}+A'_r-2$. This procedure yields α^1 from α, for any k-germ α.

For $\beta=\alpha^3$, observe that the substrings $\alpha_{1,3}=a_{k-1}a_{k-2}a_{k-3}=000,010,100,110,120$ have respectively $\beta_{1,3}=b_{k-1}b_{k-2}b_{k-3}=001,112,101,012,123$. For $\beta=\alpha^4$, the substrings $\alpha_{1,4}=a_{k-1}a_{k-2}a_{k-3}a_{k-4}$ have respectively $\beta_{1,3}=b_{k-1}b_{k-2}b_{k-3}b_{k-4}$ as follows:

$\alpha_{1,3}$	000	010	100	110	120									
$\beta_{1,3}$	001	112	101	012	123									
$\alpha_{1,4}$	0000	0010	0100	0110	0120	1000	1010	1100	1110	1120	1200	1210	1220	1230
$\beta_{1,4}$	0001	0112	1101	0012	1223	1001	1212	0101	1112	1123	1201	1012	0123	1234

To get α^k $(1<k)$, let $\beta=\alpha^k$ and note $b_1=a_1$. If $a_1=0$ then $a_2a_1=b_2b_1$. If $a_1=1$ then $a_3a_2a_1+\psi(b_3b_2b_1)$ is a constant string BBB and $a_3=b_3$. If $a_1=2$, then $a_4a_3a_2a_1+\psi(b_4b_3b_2b_1)$ is a constant string $BBBB$ and $a_4=b_4$. In general, $a_{a_1+2}\cdots a_1+\psi(b_{b_1+2}\cdots b_1)$ is a constant string and $a_{a_1+2}=b_{b_1+2}$. We could express all numbers a_i and b_i above in this paragraph as a_i^0 and b_i^0, respectively, to keep an inductive approach. Let $a_1^1=a_{a_1+2}$ and if possible, let $a_2^1=a_{a_1+3}$, etc. In this case, let $b_1^1=b_{b_1+2}$, $b_2^1=b_{a_1+3}$, etc. Now, if $a_1^1=0$, then $a_2^1a_1^1=b_2^1b_1^1$. If $a_1^1=1$, then $a_3^1a_2^1a_2^1+\psi(b_3^1b_2^1b_1^1)$ is a constant string, and so on. In general, $a^1_{a_1^1+2}\cdots a_1^1+\psi(b^1_{b_1^1+2}\cdots b_1^1)$ is a constant string and $a^1_{a_1^1+2}=b^1_{b_1^1+2}$. The continuation of this procedure produces a subsequent string a_1^2, etc., until what remains to reach the leftmost entry of α is smaller than the needed space for the procedure itself, in which case, a remaining initial (or leftmost) straight ascent is shared by both α and β.

To get α^p, for $p=4,\ldots,k-1$, a similar treatment is adapted to the left of the entry $a_{k-p+2}=b_{k-p+2}$, while to the right of that entry, the treatment previously considered is adapted as well.

9 Catalan Binary Tree

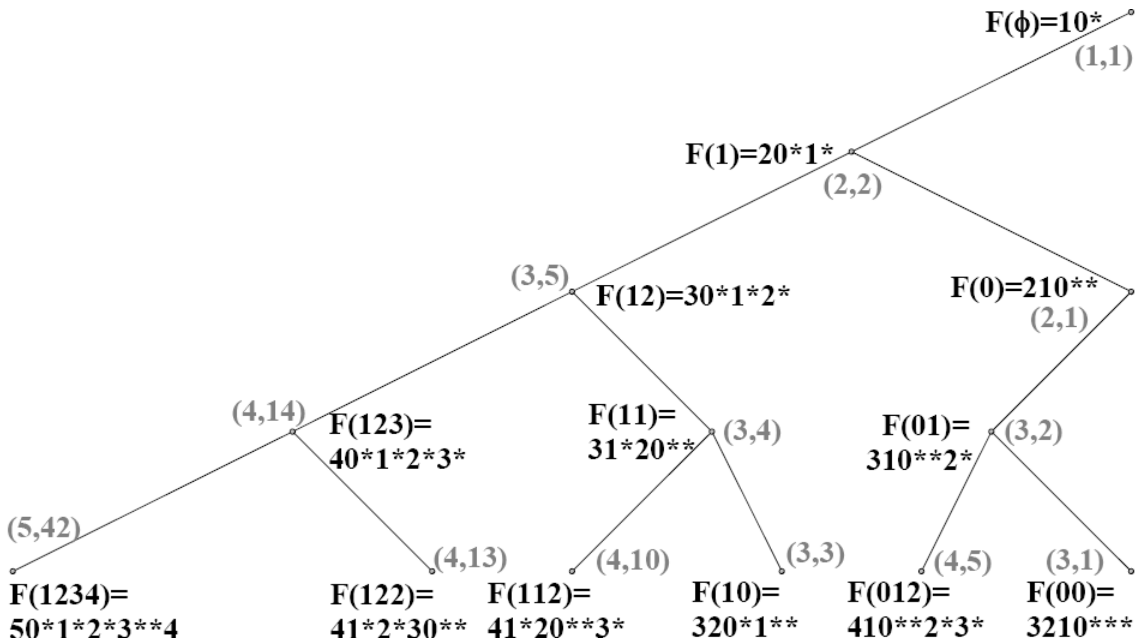

Figure 3: Restriction of T to its five initial levels

Even though the graphs R_k treated from Section 5 on were taken with $k > 1$, note that for $k = 1$ the graph R_k is defined and has just one vertex 001 with $\delta(001) = 10*$ (as in Section 7) and two loops. Thus, the only vertex of such R_1 is denoted $10*$ and the correspondence F of Section 2 can be extended by declaring $F(\emptyset) = 10*$. This is the root of a binary tree T that has $\cup_{k=1}^{\infty} V(R_k)$ as its node set and is defined as follows, where $||X||$ indicates the length of a string X: **(A)** the root of T is $10*$; **(B)** the left child of a node $\delta(v) = k|X$ in T with $||X|| = 2k$ is always defined and equals $(k+1)|X|k|*$; **(C)** unless $\delta(v) = k(k-1)\cdots 210 * * \cdots *$, it is always $\delta(v) = k|X|Y|*$, where X and Y are strings respectively starting with $j < k-1$ and $j+1$; only in that case there is a right child of $\delta(v)$, namely $k|Y|X|*$, by *un-castling* of Section 7.

Observe that T, with its nodes set in terms of k-germs, has each node $a_{k-1}a_{k-2}\cdots a_2a_1$ as a parent as follows: its left child is of the form $b_kb_{k-1}\cdots b_1 = a_{k-1}a_{k-2}\cdots a_2a_1(a_1+1)$ while its right child exists only if $a_1 > 0$ and in that case is of the form $c_{k-1}c_{k-2}\cdots c_2c_1 = a_{k-1}a_{k-2}\cdots a_2(a_1-1)$. Figure 3 represents the first five levels of T with its nodes expressed in terms of k-germs via the correspondence F, in black color. The figure also assigns to each node a (dark gray colored) ordered pair of positive integers (i,j), where $j \leq C_i$. The root, expressed by the 3-string $F(\emptyset) = 10*$, is assigned $(i,j) = (1,1)$. The left child of a node assigned (i,j) is assigned a pair $(k,j') = (i+1,j')$, where j' is the order of appearance of the corresponding k-germ α (to (k,j')) in its presentation via castling in Figure 1 and continuation for fixed k, (α becomes the RGS corresponding to j' in A239903 once the extra zeros to the left of its leftmost nonzero entry are eliminated; note $j' = j'(j)$ arises from the series associated to A076050, deducible from items 1-4 in Section 1). The right child of a node assigned (i,j) is defined only if $j > 1$ and in that case is assigned the pair $(i, j-1)$.

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