2 left to do from Stewart

green - new - work coded

7376
7377
7457
7459
2863
3626
3604
2021
6037
→ 7402

12 October 1993

Neil James Alexander Sloane
% Room 2C-376, Mathematics Research Center
AT&T Bell Telephone Laboratories Inc.
600 Mountain Avenue
Murray Hill, New Jersey 07974-0636
908+582-3000, ext. 2005

Subject: A Handbook of Integer Sequences

Dear Dr. Sloane,

Sir, following are the various notes and additions that I have made to your handbook under the existing entries enumerated below.

Sloane's Sequence Number

8. $x^3 - y^2 = k$, the above is $|k|$.

15. Smallest solutions of X when $A \cdot X^2 + 1 =$ a square number.

30. Twice this Seq. is SSN 108.

33. primes $4 \cdot k+1 = A^2 + B^2$, A>B these are the Bs. see SSN 169.

36. $A_n = \text{Int}(\Phi \cdot n) - \text{Int}(\Phi \cdot (n-1))$ Pickover, Computer and the Imagination, p24.

68. ···, 9233, 45752, 285053, 1846955, Am Sci v240n6p25.

79a. 1, 2, 2, 2, 2, 4, 4, 2, 2, ··· ., the nbr. of ways of expressing the positive integer $\mathbf{n} \geq 1$ as the sum of two (ordered pair) squarefree positive integers relative prime to **n**. AMM v99n6p573, June/July 92. — new A7457

83. of the primes. see SSN 85.

84. by primes only.

85. of the primes. see SSN 83.

89. $A_n = \text{Int}(\frac{1}{2} + \sqrt{2 \cdot n})$, Graham p97. — nicer than what I have now

91. A Self-Generating Describing Sequence by Solomon Golomb, Concrete Math, p66.

103a. Related to Edward Waring's problem, 1, 2, 2, 3, 4, 7, 8, 34, 32, 102, 51, 135, 150, 166, 181, ···, Ian Stewart, Games, Set, and Math, p124, see SSN 1353. — new

look at book

108. half of this Sequence is SSN 30.

110. Robert Daniel Carmichael's λ(n) function.

136. Companion Pell numbers, $A_0 = 2$ & $A_1 = 2$.

~~169. primes 4k+1 = $A^2 + B^2$, A>B these are the As. see SSN 33.~~

173a. 1, 2, 3, 4, 5, 6, 7, 8, 9, 1, 0, 1, 1, 1, 2, 1, 3, 1, 4, 1, 5, 1, 6, 1, 7, 1, 8, 1, 9, 2, 0, 2, 1, 2, 2, 2, 3, 2, 4, 2, 5, 2, 6, 2, 7, 2, 8, 2, 9, 3, 0, 3, 1, 3, 2, 3, 3, 3, 4, 3, 5, 3, 6, 3, 7, 3, 8, 3, 9, 4, $\cdots$, the almost-Natural numbers, JRM v?n?p?, Math Mag v61n2p131, April 88. A7376

~~180. $\Phi(n)$, see SSN 370 & 371.~~

~~185. P^{α}, P is a prime and $\alpha \geq 1$.~~

185a. 1, 2, 3, 4, 5, 6, 7, 8, 9, 13, 14, 15, 16, 18, 19, 24, 25, 27, 28, 31, 32, 33, 34, 35, 36, 37, 39, 49, 51, 67, 72, 76, 77, 81, 86, $\cdots$, and no others < 57134. Powers of two lacking the digit 0 in its expansion, Joseph S. Madachy, Mathematics on Vacation, p126-8. Mada66 A7377

187. Joe Roberts, Lure of the Integer, p22. see SSN 1685. maybe this explains A0028

188. Index to SSN 1828.

200. $\sigma(A_{n+1}) > \sigma(A_n)$.

201. Stanislaw Ulam Summation Seq., see SSN 231 & 909.

206. $A_1 = 1$, $A_{n+1} = \lfloor \sqrt{2 \cdot A_n \cdot (A_n + 1)} \rfloor = \lfloor \sqrt{2} \cdot (A_n + \frac{1}{2}) \rfloor$, Am Math Mo. v95n8p705 Oct88, & Math Mag. v64n3p168 June91.

207. $A_1=1$, $A_2=2$ & $A_3=3$, Am Math Mo. v95n8p705 Oct88.

213. see SSN 248.

214. $\cdots, (\cdots) = 2^{512} = 2^{2^9}, \cdots,$

223. $A_n = n + \text{Int}(\sqrt{n} + \frac{1}{2}) = n + \text{Int}(\sqrt{n} + \text{Int}(\sqrt{n}))$, Math Mag v63n1p53-5 Feb90.

228. this is **n**, see SSN 1051.

230a. 2, 3, 5, 7, 1, 1, 1, 3, 1, 7, 1, 9, 2, 3, $\cdots$, an irrational decimal fraction, Am Math Mo. v100n8p779 Oct 93, HW1 5th Ed. p139.

231. Stanislaw Ulam Summation Seq., see SSN 201 & 909.

235. also see SSN 982.

241. the Sieve of Eratosthenes.

242a. 1, 2, 3, 5, 7, 11, 31, 379, 1019, 1021, 2657, $\cdots$, p "Primorials" = $(2\cdot3\cdot5\cdot7\cdot11\cdot13\cdot17\cdot \cdots \cdot p)+1$ is prime.

246a. 2, 3, 5, 7, 11, 13, 19, 23, $\cdots$, Higg's primes in $\sum_2$, Am Math Mo v100n3p233, March 93. A7459

248. $\cdots$, 19937, 21701, 23209, 44497, 86243, 110503, 132049, 216091, $\cdots$? $\cdots$, 756839, these are the **n**s of primes of the form $2^{\mathbf{n}} - 1$.

256. $(\Phi^n - \hat{\Phi}^n)/\sqrt{5}$. see SSN 1679.

266. $A_n = 2 \cdot A_{n-1} - 1$.

322a. 1, 1, 2, 3, 7, 21, 49, 165, 552, 2176, 9988, ⋯, Knots, Math Mag v61n1p7 Feb 88.

326a. 1, 2, 3, 7, 23, 59, 314, 1529, 8209, ⋯, $An = (A_{n-1} * A_{n-3} + A_{n-2}^2) / A_{n-4}$, for $n > 3$, Coll. Math. Journal v24n4p393.

327. ⋯, 17051707, 20831323, Math of Compt v52n185p221-4 & Knuth v3p402, see SSN 984.

346. $\prod_{d=1,\ d \mid n}^{n} d$

358. ⋯, 353, 503, 613, 617, 863, by Index.

370. see SSN 180 & 371.

371. see SSN 180 & 370.

385. Ramanujan's Super Abundant $d(A_{n+1}) > d(A_n)$, Math Mag v64n5p343-6 Dec91.

391. the Lazy Caterer's Seq.

423. $A_n = A_{n-1} + A_{n-2} + A_{n-3} + A_{n-4}$.

427. $n + C(\frac{n}{4}) + C(\frac{n-1}{2})$.

429. $A_n = A_{n-1} + A_{n-2} + A_{n-3} + A_{n-4} + A_{n-5}$.

431. $A_n = A_{n-1} + A_{n-2} + A_{n-3} + A_{n-4} + A_{n-5} + A_{n-6}$.

432. $A_n = 2 \cdot A_{n-1} = \sum_{k=0}^{n} C(\frac{n}{k})$, Graham p231.

497. also see SSN 990.

509. $\mathrm{Int}(n / \Phi^2)$ see SSN 917.

511. Leonard Euler's Gen. Pentagonal Nbrs. $\frac{n}{2} \cdot (3 \cdot n - 1)$, $n \geq \pm 1$.

522. $= C(\frac{n}{2}) - n$, the nbr of diagonals of a regular convex n-agon, Math. Mag v61n1p28.

529. $X^2 - D \cdot Y^2 = -1$, SI1 2nd p334.

531a. 1, 2, 5, 10, 17, 28, ⋯, Surds of period 1, $= n^2 + 1$.

552. $[(1+\sqrt{2})^n - (1-\sqrt{2})^n]/\sqrt{8}$, Knuth v2p605, David Wells p90 & SI1 p327.

561. Whys & Wherefores p92. Ball&Coxeter p112, Computers in Nbr. Theory p363.

569. JRM v13n2p158.

577. $A_n = \frac{(2\cdot n)!}{n!(n+1)!} = \frac{1}{n+1}\cdot C(\frac{2\cdot n}{n}) = \frac{2}{n+1}\cdot C(\frac{2n+1}{n}) = \frac{1}{n+1!}\cdot 2\cdot 6\cdot 10\cdot \cdots \cdot(4\cdot n-10) = [A_{n-1}\cdot(4\cdot n-6)]/n$,
$B_{n+1} = B_0\cdot B_n + B_1\cdot B_{n-1} + B_2\cdot B_{n-2} + \cdots + B_{n-1}\cdot B_1 + B_n\cdot B_0$.
Math Mag v61n4p211 Oct88.

589. $A_0 = 0$, $A_{n+1} = n\cdot A_n + 1$.

603. Convergents to Lehmers Constant $\xi - 1$, see SSN 1230.

617. $\cdots$, 272, 306, Kissing nbrs. David Wells p84, also called Newton, Contact, Coordination and Ligancy numbers.

625. $A_n = 2\cdot A_{n-1} + 2$.

651. $\cdots$, 275750636070, 1633292229030, 9737153323590, $\cdots$.

668. see SSN 1081.

695. $Int(e^n)$.

731. Ball&Coxeter p112.

742. $2^n\cdot n!$.

766. $!n = P!(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots \pm \frac{1}{P!})$, David Wells p122.

773. $\prod_{k=0}^{n} C(\frac{n}{k})$, Hyperfactorial / Superfactorial, SSN 1514/SSN 811, Graham p231.

775. Kraitchik p250.

786a. 2, 11, 13, 17, $\cdots$, they nor their square can be expressed as $a^2 + 5\cdot b^2$, Beiler p285, see SSN 2264a.

A3626

787. Am Math Mag v97n7p625.

789. $S[\frac{n+2}{n}]$.

811. $\sum_{k=1}^{n} k!$, Superfactorial, Graham p231.

891. see SSN 1336.

909. $\cdots$, 176, 187, 192, 196, $\cdots$, Stanislaw Ulam Summation Seq., see SSN 201 & 231.

911. $\cdots$, 571, $\cdots$? $\cdots$, 2971, 4723, 5387, by Index, Pickover Mazes p350.

917. $A_n = Int(n/\Phi) = Int(n\cdot\Phi)$, see SSN 509.

921. $\sigma(n)$.

924. $A_0 = 2$ & $A_1 = 1$, $\Phi^n + \hat{\Phi}^n = \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n$, see SSN 1679.

941. Polyiamond.

973a. ~~1, 1, 1, 1, 1, 1, 3, 5, 9, 23, 75, 421, 1103, 5047, 41783, 281527, 2534423, 14161887, 232663909, 3277905290,~~ ···, $A_n = (A_{n-1} * A_{n-5} + A_{n-2} * A_{n-4} + A_{n-3}^2) / A_{n-6}$, Somos, ~~Mathematical Intelligencer v13n1p40,~~ Pickover, Mazes p351.

982. see SSN 235.

984. Math of Compt. v52n185p221-4, see SSN 327.

988. Pascal's Triangle (mod 2) read in decimal, Math Mag v63n1p3 Feb90.

990. see SSN 497.

1002. Sum of the Integers, $C(\frac{n}{2})$, $A_{n+1} = [n \cdot A_n + 2 \cdot (n+1)^2]/(n+2)$.

1035. Sieve of Stanislaw Ulam.

1039. Gaussian Primes of real nbrs.

1048. Josephus Flavius sieve.

1051. $= n^2 + n + 1$, see SSN 228.

1059. Stirling Nrbs. of the 2nd Kind $S\{\frac{n}{2}\}$, $A_n = 2 \cdot A_{n-1} + 1$.

1064. $\frac{1}{2}[(1+\sqrt{2})^n + (1-\sqrt{2})^n]$.

1070. ···, 377379369, 1105350729, ···, $\sum_{j=0}^{n/3} (-1)^j \cdot C(\frac{n}{j}) \cdot C(\frac{2 \cdot n - 1 - 3 \cdot j}{n-1})$.

1071. $\mathrm{Int}(e^n + \frac{1}{2})$.

1072. Polyhexes.

1072a. 3, 7, 23, ···, primes which cannot be expressed in the form $a^2 + 5 \cdot b^2$ although their squares can. Beiler p284, see SSN 2264a.

1081. SSN 688 (the "Primorials") +1 = $(2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdots \cdot p) + 1$, p is prime, AmMathMo v95n8p699, see SSN 242a.

1084. "decimated" Lucus Seq., Number Theory, Spring-Verlag v7p73 & SI1 2nd Ed.

1087. the periodicities of the Fibonacci sequence modulus n, Lure of the Integers p162.

1089. N.L.Gilbreath hypothesis 1958, conjecture, $A_{n+1} > A_n - 2$.

1101. ⋯, 1836311903, ⋯, =Int[$\frac{1}{\sqrt{5}}\cdot\left(\frac{3+\sqrt{5}}{2}\right)^{n}$]. Niven&Zuckerman, 4th Ed. p123.

1130. $= \frac{3}{n+1}\cdot C\binom{2\cdot n-2}{n}$.

1132a. π (n!) for n >1, = 1, 3, 9, 30, 128, 675, 4231, 30969, 258689, 2428956, 25306287, 289620751, 3610490805, ⋯. 3604 (McKay)

1142. = Int($\pi^{n} + \frac{1}{2}$).

1163. $n\cdot A_n = 3\cdot(2\cdot n-3)\cdot A_{n-1} - (n-3)\cdot A_{n-2}$, Vardi p198.

1165. S[$\frac{n}{2}$], $A_{n+1} = n\cdot A_n + (n-1)!$.

1183. Cyclic numbers, SPR p146.

1215. ⋯, 5187, 10604, 11714, 13365, 18315, 22935, 25545, 32864, 38804, 39524, 46215, 48704, ⋯, $\Phi(n) = \Phi(n+1)$, see 𝕾𝕾𝕹 1328.

1217. $\prod_{n=1} 2\cdot n-1$, $n!/(2^{\frac{n}{2}} * \frac{n}{2}!)$ for even n

1230. see 𝕾𝕾𝕹 603.

1231. $\frac{1}{2}[(1+\sqrt{2})^{2\cdot n} + (1-\sqrt{2})^{2\cdot n}]$.

1233. Graham, Knuth & Patashnik, Concrete Mathematics, p528.

1247. $A_n = 6\cdot A_{n-1} - A_{n-2} + 2 = \frac{1}{4}[(\sqrt{2}+1)^{2\cdot n+1} - (\sqrt{2}-1)^{2\cdot n+1} - 2]$, with Consecutive Legs (lesser given), Beiler p123, see 𝕾𝕾𝕹 1630.

1262. 285311670601, 98077104930805, 302875106592241, 144456088732254195, ⋯, for even n, $X = n^{n+1} - n^{n} - n + 1$, for odd n, $X = n^{n} - n + 1$. A2021

1309. Murphy sequence, JourRecMath v10n3p230.

1313. Martin Gardner, Penrose Tiles To Trapdoor Ciphers, p69.

1322. 65, 66, ⋯, eg. not the primes 𝕾𝕾𝕹 241.

1323. Semi-Primes.

1327. = Int[$\frac{1}{2}(7+\sqrt{48\cdot n+1})$], Am.Math.Mo. v98n9p874.

1328. ⋯, 1082, 1226, 1322, 1330, 1346, 1466, 1514, 1608, 1754, 1994, 2132, ⋯ $\Phi(n) = \Phi(n+2)$, see 𝕾𝕾𝕹 1215.

1336. N^{α}, $\alpha > 1$, see 𝕾𝕾𝕹 891.

1352a. Related to Edward Waring's problem, 1, 4, 9, 19, 21, 31, 45, 62, 82, 102, 120, 135, 150, 166, 181, $\cdots$, Ian Stewart, Games, Set, and Math, p123, see SSN 1353.

1353. $\sim\text{Int}[(\frac{3}{2})^k] + 2^k - 2$, Lure of the Integers, p138.

1363. the Sum of the Triangular Nbrs. $\frac{n}{6}\cdot(n+1)\cdot(n+2) = \sum_{k=1} \frac{n\cdot(n+1)}{2}$.

1385. $\cdots$, 740461601, 2012783315, 5471312310, 1487256883, $\cdots$, when k exceeds n of $\sum_{n=1}^{k} \frac{1}{n}$

Jour Rec Math v10n2p124 1977 & Math Mag v65n5p308 Dec92.

1415. $\cdots$, 1188640, 4345965, $\cdots$, $= \frac{4}{n+1}\cdot C(\frac{2\cdot n-3}{n})$.

1463. $\cdots$, 92897280, 1857945600, 40874803200, $\cdots$, $A_1 = 1$, $A_{n+1} = 2\cdot n\cdot A_n = 2^{n-1}\cdot n!$.

1514. Hyperfactorial, Graham, Knuth & Patashnik, Concrete Mathematics, p477.

1524. Wilson Quotients, not remainders.

1530a. Congruent numbers,

1558. $= n^2 - n - 1$, no index listing.

1562a. $n > 0$.

1569. $\cdots$, 411, 525, $\cdots$, $\frac{1}{16}\cdot[2\cdot n\cdot(n+2)\cdot(2\cdot n+1) + (-1)^n - 1]$, $T_0 = 0$ & $T_1 = 1$, $T_{n+1} = T_n + T_{n-1} - T_{n-2} + 3\cdot n + 1$, Coll Math Journ v20n5p370-84, Math Mag v66n1p40.

1574. the Sum of the Integers Squared, $= \frac{n}{6}\cdot(n+1)\cdot(2n+1) = \sum_{k=1}^{n} n^2$.

1578. $= \frac{n}{4!}\cdot(n+1)\cdot(n+2)\cdot(n+3)$.

1602. $\frac{5}{n+1}\cdot C(\frac{2n-4}{n})$.

1608. $\cdots$, 5461, 21845, 87381, 349525, 1398101, 5592405, $A_1 = 1$, $A_n = 4\cdot A_{n-1} + 1 = \frac{1}{3}(4^n - 1)$.

1630. $A_n = 6\cdot A_{n-1} - A_{n-2} = \frac{1}{\sqrt{8}}[(\sqrt{2}+1)^{2\cdot n+1} + (\sqrt{2}-1)^{2\cdot n+1}]$.

with Consecutive Legs (Hypotenuses given), Beiler p123, see SSN 1247,

1636. $(A^5 - B^5)/D$, $A > B \geq D \geq 1$.

1678. the least ε such that $10^\varepsilon \equiv 1 \bmod n$, $n \nmid 10$ & $n > 6$, see SSN 1680.

1679. $\Phi = \frac{1}{2}[\sqrt{5} + 1]$ & $\hat{\Phi} = \frac{1}{2}[\sqrt{5} - 1]$.

1680. see SSN 1678.

might explain this one

1685. Joe Roberts, Lure of the Integers, p22. see $\mathfrak{SSN}$ 187.

1714. $\frac{n}{12}\cdot(n+1)^2\cdot(n+2)$

1718. $C\left(\frac{C(\frac{n}{2})+1}{2}\right)$

1719. $\frac{n}{5!}\cdot(n+1)\cdot(n+2)\cdot(n+3)\cdot(n+4)$.

1734. $S\{\frac{n}{2}\}$.

1760. Triangular numbers which are squares. The above numbers are the square roots.

1762. $S[\frac{n}{3}]$.

~~1768. QUESTION. Should not the eighth entry be 30720 instead of 30270? $2^X * (2\cdot n+1)$.~~

1779. $S[\frac{n+3}{n}]$.

1783. Joe Roberts, Lure of the Integers, p208.

1799. $A_n = \prod_1^x P_X$, $X = 2, 4, 9, 22, \cdots$, ?

1819. $\equiv 6\cdot n+1$, no primes of the form $3\cdot n-1$.

1823. Cyclic numbers of 1, SPR p143.

1828. Prime Hex. nbrs., Hexagonal clusters - "Polyhedral Dissections."

$$A_n = 1 + 6 + 12 + \cdots + 6\cdot n = n^3 - (n-1)^3 = 3\cdot n^2 + 3\cdot n + 1 = 1 + \sum_{k=1}^{n} 6\cdot n.$$

1834. $\frac{1}{2}\{[(1+\sqrt{6})^n + (1-\sqrt{6})^n]$.

1845. $S\{\frac{n+2}{n}\} = \frac{n}{4!}\cdot(n+1)\cdot(n+2)\cdot(3\cdot n+1)$.

1847. $= \frac{n}{6!}\cdot(n+1)\cdot(n+2)\cdot(n+3)\cdot(n+4)\cdot(n+5)$.

1866. $\frac{7}{n+1}\cdot C(\frac{2n-6}{n})$.

1869. $\frac{1}{2}\cdot[(1+\sqrt{2})^{2\cdot n+1} + (1-\sqrt{2})^{2\cdot n+1}]$.

1886. Denominator of successive continued fractions of $\bar{\pi}$ "Pi bar" = Pi -3,

1905. $\mathbf{n^3}$.

1907. $\frac{1}{6}\cdot n^2\cdot(n+1)\cdot(n+2)$.

1924. the above are the indexes, $T_n = \frac{1}{2} \cdot n \cdot (n+1)$, see SSN 1760.

1935. Egyptian Fraction's Denominators. ..., 162924332716605980,

1970. $\frac{n}{4!} \cdot (n+1) \cdot (n+2) \cdot (5 \cdot n - 1)$.

1981. $\frac{9}{n+1} \cdot C(\frac{2 \cdot n - 8}{n})$.

2018. $S\{\frac{n}{4}\}$.

2022. $S[\frac{n}{4}]$.

2044. $(A^5 + B^5)/D$, $A > B \geq 1$, $26 \geq D \geq 1$.

2048. $\frac{11}{n+1} \cdot C(\frac{2 \cdot n - 10}{n})$.

2091. the denominators of James Stirling's formula

2100. $(A^6 + B^6)/D$, $A > B \geq 1$, $240 \geq D \geq 1$.

2104. $\frac{13}{n+1} \cdot C(\frac{2 \cdot n - 12}{n})$.

2105. Semi-Cuban Primes, $\frac{1}{2}[n^3 - (n-2)^3]$.

2120. 14^n.

2136. $S\{\frac{n+3}{n}\}$.

2141. $S\{\frac{n}{5}\}$.

2142. $S[\frac{n}{5}]$.

2178. $\cdots$, 65617, 66161, $\cdots$, $A^4 + B^4$, $A > B \geq 1$.

2215. $S\{\frac{n}{6}\}$.

2216. $S[\frac{n}{6}]$.

2239. $S[\frac{n+4}{n}]$.

2262. $\sum_{n=0} (2 \cdot n + 1)^3$.

2263. $S\{\frac{n}{7}\}$.

2264. $S[\frac{n}{7}]$.

2264a. 29, 41, 61, $\cdots$, primes which can be expressed as $A^2 + 5 \cdot B^2$, Beiler p284, see SSN 786a & 1072a.

2272. $S\{\frac{n+4}{n}\}$.

2277. n^5.

2292. SPR 98, & Stewart p29.

2293a. Surds of a periodicities of 3: 41, 130, 269, 370, 458, 697, 986, ⋯,
$= (5\cdot k+1)^2 + 4\cdot k + 1$, SI1 p323-4.

2294. $\frac{1}{2}(A^4 + B^4)$, $A > B \geq 1$.

2311. SciAm. v242n6p30.

(6037)

2323a. Primitive Weird numbers: 70, 836, 4030, 5830, 7192, 7912, 9272, 10792, 17272, 45356, 73616, 83312, 91388, 113072, 243892, 254012, 338572, 343876, 388076, 519712, 539744, 555616, 682592, 786208, 1188256, 1229152, 1713592, 1901728, 2081824, 2189024, 3963968, 4128448, 4145216, 4199030, 4486208, 4559552, 4632896, 4790812, 4960448, 5440192, 5568448, 6460864, 6621632, 7354304, 7470272, 8000704, 8134208, 8812312, 9928792, 11339816, 11547352, ⋯, Stephen P. Richard p85.

2333a. tri-Perfect numbers: 120, 672, 523776, 459818240, 1476304896, 31001180160, ⋯, David Wells p135.

2347a. Prime Cullen numbers by Index $C_n = n \cdot 2^n + 1$, for **n**s : 1, 141, 4713, 5795, 6611, 18496, and no others ≤ 20000. Paulo Ribenboim, p175.

2352. see SSN 2363.

2363. see SSN 2352.

2365. base Two - PseudoPrimes, $2^{p-1} \equiv 1 \bmod p$.

A6052

2372a. the number of Magic Squares of order n: 1, 0, 1, 880, 275305224, ⋯, David Wells p190.

(GA 88 216) illustrated

additional Bibliography

American Scientific Magazine v240n6 & v242n6.
Ball&Coxeter, Mathematical Recreations and Essays, 13th Ed. Dover Publ., NY, 1987.
College Math. Journal v24n4.
Martin Gardner, Whys & Wherefores, The University of Chicago Press, 1989.
Martin Gardner, Penrose Tiles To Trapdoor Ciphers, W.H. Freeman & Co., NY, 1989.
Graham, Knuth & Patashnik, Concrete Mathematics, A Foundation for Computer Science, Addison-Wesley, Reading, MA, 1989.
HW1 5thEd.
Donald Ervin Knuth, The Art of Computer Programming, 2nd Ed., v2&3, Addison-Wesley,
Maurice Kraitchik, Mathematical Recreations, Chpt Ten "The Problem of the Queens," Dover, 2nd revised Ed, NY, 1953.
Joseph S. Madachy, Math. on Vacation,
Niven&Zuckerman, 4th Ed.
Clifford A. Pickover, Computers and the Imagination, Visual Adventures Beyond the Edge, St. Martin's Press, NY, 1991.
Clifford A. Pickover, Mazes for the Mind, St. Martin's Press, NY, 1992.
Paulo Ribenboim, The Little Book of BIG Primes, Springer-Verlag, NY, 1991.
Stephen P. Richards, A Number For Your Thoughts, Publd. by S.P. Richards, New

(SPR)

Providence, NJ, 1982.
Joe Roberts, Lure of the Integer,
Ian Stewart, Games, Set, and Math, Penguin Books, London, 1989.
Ian Stewart, The Problems of Mathematics, OPUS books, Oxford Un. Press, Oxford, 1987.
Ilan Vardi, Computational Recreations in Mathematica®, Addison-Wesley, Redwood City, CA, 1991.
~~David Wells,~~ Dictionary of Curious and Interesting Numbers, ~~Penguin Books, London, 1986.~~
Computers in Nbr. Theory
Number Theory, Spring-Verlag v7
SI1 2nd Ed.

additional Index

If at some future date, I run across an addition, I will forward the same to you.

Sequentially yours,

Robert G. Wilson

Robert G. Wilson v
Ph.D., ATP/CF&GI

RGWv:T4400c
sloane26

f94 → 7403

Calk94

A CURIOUS BINOMIAL IDENTITY

NEIL J. CALKIN

In this note we shall prove the following curious identity of sums of powers of the partial sum of binomial coefficients.

1. An identity

Theorem. $\sum_{l=0}^{n}\left(\sum_{k=0}^{l}\binom{n}{k}\right)^3 = n2^{3n-1} + 2^{3n} - \frac{3n}{4}2^n\binom{2n}{n}$.

Proof. Define $f_n = \sum_{l=0}^{n}\left(\sum_{k=0}^{l}\binom{n}{k}\right)^3$. It is sufficient to show that

$$f_{n+1} - 8f_n = 4\ 2^{3n} - 3\ 2^n\binom{2n}{n}$$

Write $A_l = \sum_{k=0}^{l}\binom{n}{k}$. Then $f_n = \sum_{l=0}^{n} A_l^3$.

✓f94

$$\begin{aligned}
f_{n+1} &= \sum_{l=0}^{n+1}\left(\sum_{k=0}^{l}\binom{n+1}{k}\right)^3 \\
&= 2^{3n+3} + \sum_{l=0}^{n}\left(\sum_{k=0}^{l}\binom{n+1}{k}\right)^3 \\
&= 2^{3n+3} + \sum_{l=0}^{n}\left(\sum_{k=0}^{l}\binom{n}{k} + \binom{n}{k-1}\right)^3 \\
&= 2^{3n+3} + \sum_{l=0}^{n}\left(2A_l - \binom{n}{l}\right)^3 \\
f_{n+1} - 8f_n &= 2^{3n+3} + \sum_{l=0}^{n}\left(2A_l - \binom{n}{l}\right)^3 - (2A_l)^3 \\
&= 2^{3n+3} - \sum_{l=0}^{n} 12A_l^2\binom{n}{l} + \sum_{l=0}^{n} 6A_l\binom{n}{l}^2 - \sum_{l=0}^{n}\binom{n}{l}^3
\end{aligned}$$

1991 *Mathematics Subject Classification.* 05A10.

Observation 1:

$$\sum_{l=0}^{n} A_l \binom{n}{l}^2 = \frac{1}{2} 2^n \binom{2n}{n} + \frac{1}{2} \sum_{l=0}^{n} \binom{n}{l}^3$$

Indeed;

$$\begin{aligned}\sum_{l=0}^{n} A_l \binom{n}{l}^2 &= \sum_{l=0}^{n} A_{n-l} \binom{n}{n-l}^2 \\ &= \sum_{l=0}^{n} A_{n-l} \binom{n}{l}^2\end{aligned}$$

and since

$$A_l + A_{n-l} = 2^n + \binom{n}{l}$$

we have

$$\begin{aligned}\sum_{l=0}^{n} A_l \binom{n}{l}^2 &= \frac{1}{2} \sum_{l=0}^{n} \left(2^n + \binom{n}{l}\right) \binom{n}{l}^2 \\ &= \frac{1}{2} \sum_{l=0}^{n} 2^n \binom{n}{l}^2 + \frac{1}{2} \sum_{l=0}^{n} \binom{n}{l}^3 \\ &= \frac{1}{2} 2^n \binom{2n}{n} + \frac{1}{2} \sum_{l=0}^{n} \binom{n}{l}^3\end{aligned}$$

Observation 2:

$$\sum_{l=0}^{n} A_l^2 \binom{n}{l} = \frac{2^{3n}}{3} + \frac{1}{2} 2^n \binom{2n}{n} + \frac{1}{6} \sum_{l=0}^{n} \binom{n}{l}^3$$

Indeed,

$$\begin{aligned}
2^{3n} = A_n^3 &= \sum_{l=0}^{n} A_l^3 - A_{l-1}^3 \\
&= \sum_{l=0}^{n} A_l^3 - \left(A_l - \binom{n}{l}\right)^3 \\
&= \sum_{l=0}^{n} 3A_l^2\binom{n}{l} - \sum_{l=0}^{n} 3A_l\binom{n}{l}^2 + \sum_{l=0}^{n}\binom{n}{l}^3 \\
&= \sum_{l=0}^{n} 3A_l^2\binom{n}{l} - \frac{3}{2}2^n\binom{2n}{n} - \frac{1}{2}\sum_{l=0}^{n}\binom{n}{l}^3
\end{aligned}$$

Hence

$$\sum_{l=0}^{n} A_l^2\binom{n}{l} = \frac{2^{3n}}{3} + \frac{1}{2}2^n\binom{2n}{n} + \frac{1}{6}\sum_{l=0}^{n}\binom{n}{l}^3$$

Putting these together, we indeed find that

$$f_{n+1} - 8f_n = 4\ 2^{3n} - 3\ 2^n\binom{2n}{n}$$

as required. □

2. An application

In this section we shall discuss an application of this to order statistics. Observe that the expected value of the maximum of three independent Bernoulli random variables $B(n, \frac{1}{2})$ is

$$\begin{aligned}
\sum_{l=0}^{n}\left(1 - \left(\sum_{k=0}^{l} 2^{-n}\binom{n}{k}\right)^3\right) &= n - 2^{-3n} f_n \\
&= \frac{n}{2} + \frac{3}{4}n2^{-2n}\binom{2n}{n}.
\end{aligned}$$

Hence, by the central limit theorem, the expected value m_3 of the maximum of three independent normal $N(0,1)$ random variables is

$$m_3 = \lim_{n\to\infty}\frac{\frac{3}{4}n2^{-2n}\binom{2n}{n}}{\frac{\sqrt{n}}{2}} = \frac{3}{2\sqrt{\pi}}$$

subracting off the mean, dividing by the standard deviation and applying Stirling's formula for the asymptotics of $n!$

School of Mathematics, Georgia Institute of Technology, Atlanta, Ga 30332
E-mail address: calkin@math.gatech.edu

fax COPY → 7403

AT&T
Bell Laboratories

Colin L. Mallows
Statistical Models and Methods Research Department

Room 2C-265
600 Mountain Avenue
P.O. Box 636
Murray Hill, NJ 07974-0636
voice: (908) 582-3851
fax: (908) 582-3340
e-mail: clm@research.att.com

May 12, 1994

Neil J. Calkin
Carnegie Mellon University
Schenley Park
Pittsburgh
PA 15213

Dear Dr. Calkin,

I have just seen your "Curious Binomial Identity", and observe that another derivation is possible. As you remark, we have

$$2^{-3n} f_n = n - E(\max(X_1, X_2, X_3))$$

where the X's are independent Bernoulli $(n, 1/2)$. So by symmetry the result is equivalent to

$$E(\max(X_1, X_2, X_3) - \min(X_1, X_2, X_3)) = \frac{3n}{2} \frac{1}{2^{2n}} \binom{2n}{n}$$

But

$$\max(x,y,z) - \min(x,y,z) = \frac{1}{2}(\max(x,y) - \min(x,y) + \max(x,z) - \min(x,z) + \max(y,z) - \min(y,z))$$

so it is sufficient to show

$$E(\max(X_1, X_2) - \min(X_1, X_2)) = \frac{n}{2^{2n}} \binom{2n}{n} \qquad (1)$$

Here is a cute way to do this. The expectation is

$$\sum_{k=0}^{n} \sum_{l=0}^{n} \frac{1}{2^{2n}} \binom{n}{k} \binom{n}{l} |k-l|$$

$$= \sum_{k=0}^{n} \sum_{l=0}^{n} \frac{1}{2^{2n}} \binom{n}{k} \binom{n}{l} \frac{1}{2\pi} \int_{-\infty}^{\infty} (1 - e^{2ix(k-l)}) \frac{dx}{x^2}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} (1 - \left[\frac{1+e^{2ix}}{2}\right]^n \left[\frac{1+e^{-2ix}}{2}\right]^n) \frac{dx}{x^2}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} (1 - \cos^{2n} x) \frac{dx}{x^2} = I_n \quad \text{say}$$

To evaluate this, consider the generating function

$$\sum_{n=1}^{\infty} I_n y^{n-1} = \frac{1}{2\pi} \frac{1}{1-y} \int_{-\infty}^{\infty} \frac{1-\cos^2 x}{1-y\cos^2 x} \frac{dx}{x^2}$$

$$= \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \int_{k\pi}^{k\pi+\pi} = \frac{1}{2\pi} \frac{1}{1-y} \int_{0}^{\pi} \frac{1-\cos^2 x}{(1-y\cos^2 x)} \sum_{k=-\infty}^{\infty} \frac{1}{(x+k\pi)^2}$$

$$= \frac{1}{2\pi} \frac{1}{1-y} \int_0^{\pi} \frac{1}{1 - y\cos^2 x} dx$$

$$= \frac{1}{2}(1-y)^{-3/2} = \frac{d}{dy}(1-y)^{-1/2} = \frac{d}{dy} \sum_{n=0}^{\infty} \frac{y^n}{2^{2n}} \binom{2n}{n}$$

and this proves (1).

Sincerely,

Colin L. Mallows