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s-Additive Sequen

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s-Additive Sequences

(notes,examples,refs collected by RKG)

May 16, 1994

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An s-**additive sequence**, $\{a_i\}$, $i = 1, 2, \ldots$, is defined, given positive integers $a_1 < a_2$, by taking a_n, $n \geq 3$ to be the least integer greater than a_{n-1} which is representable in exactly s ways as the sum of two distinct earlier members of the sequence. 1-additive sequences were suggested by Ulam and the members of the sequence

1, 2, 3, 4, 6, 8, 11, 13, 16, 18, 26, 28, 36, 38, 47, 48, 53, 57, 62, 69, 72, 77, 82, 87, 97, 99, 102, 106, 114, 126, 131, 138, 145, 148, 155, 157, 160, 165, 170, 172, 189, 197, 204, 221,

have been called **U-numbers**.

0-additive sequences have also been called **sum-free sequences**. Here are some examples:

X 1, 2, 4, 7, 10, 13, 16, 19, ..., $3k+1$, ... (2 is exceptional) dull

X 1, 3, 5, 7, 9, ..., $2k+1$, ... dull

YES 1, 4, 6, 8, 11, 13, 16, 18, 23, 25, 28, 30, 35, ..., $12k+1$, $12k+4$, $12k+6$, $12k+11$, ... (8 is exceptional) use

X 1, 5, 7, 9, ..., $2k+1$, ... (3 is missing) dull

YES 1, 6, 8, 10, 12, 15, 17, 19, 24, 26, ..., $9k+1$, $9k+6$, $9k+8$, ... (12 is exceptional). This example is given by Finch, **99**(1992) 671. use

NO 2, 3, 4, 8, 9, 14, 15, 20, 21, 26, 27, ..., $6k+2$, $6k+3$, ... (4 is exceptional) dull

NO 2, 4, 5, 8, 11, 14, 17, 20, ..., $3k+2$, ... (4 is exceptional) dull

YES 2, 5, 6, 9, 10, 13, 17, 20, 21, 24, 28, ..., $15k+2$, $15k+5$, $12k+6$, $12k+9$, $12k+13$, ... (10 is exceptional) use

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Excerpt from 1975 MONTHLY updating article [1975, 998] (refs in brakets are to year

& page numbers of the MONTHLY

"Raymond Queneau writes concerning questions on a sequence of Ulam, posed by Recamán [1973, 919]. Muller (1966) computed the first 20000 terms; there is a probabilistic study by Wunderlich (1970) and Queneau observes that the sequence is a special case of an s-additive sequence. His paper (1972) does not answer any of the specific questions raised by Recamán; P. Braffort of Paris has studied s-additive sequences on a computer. Owens (1974) gives the second example, $u_{19} = 62$, $u_{20} = 69$, of two consecutive U-numbers which add to a U-number, $u_{31} = 131$; there are no others with sum $\leq 99933 = u_{7584}$. David Zeitlin's extensive comments must await later publication."

Excerpt from 1987 MONTHLY updating article [1987, 962]

"Bernardo Recamán [1973, 919; and see 1975, 998] asked several questions about Ulam's sequence, $U_1 = 1$, $U_2 = 2$, and for $n \geq 3$, U_n is the least integer expressible *uniquely* as the sum of two distinct earlier members of the sequence. A remark of Eggleton [1973, 920] shows that $U_{n+1} \leq U_n + U_{n-2}$. Hence $U_{n+1} < 2U_n$ and it follows [1977, 809] that $\{U_n\}$ is **complete**, i.e. that every positive integer is expressible as the sum of distinct U-numbers. David Zeitlin (see [1977, 815] for reference) conjectured that $\{U_n\}$ is still complete, even after the deletion of one or two members. Robert Stong (wrc) recalls his earlier proofs of this, and of another Zeitlin conjecture, that $\{U_n^*\}$ is complete, where $U_1^* = 1$, $U_2^* = 2$, and, for $n \geq 3$, $U_n^* = U_n + U_{n-2}$ (since $U_{n+1}^* = U_{n+1} + U_{n-1} < 2U_n + 2U_{n-2} = 2U_n^*$)."

Excerpts from 1993 MONTHLY updating article [1993, 946 & 948]

"Steven Finch extended Queneau's computations of "Ulam sequences" [1973, 919; 1975, 998; 1987, 962], a **(u, v)-sequence** of positive integers $\{a_i\}$ being defined by $a_1 = u$, $a_2 = v$ and, for $n > 2$, a_n is the least integer expressible *uniquely* as the sum of two distinct earlier members. Queneau showed that the (2,5)-, (2,7)- and (2,9)-sequences are **regular** in the sense that their differences are ultimately periodic. Finch (1991, 1992) proved that if the (u, v)-sequence has only finitely many even terms, then it is regular. Schmerl & Spiegel (tbp) prove that the $(2, v)$-sequence has just two even terms for any odd $v > 3$."

"Neil Calkin notes the relevance of Peter Cameron's survey article (1987) and his own thesis (1988) to Steven Finch's 0-additive sequences problem [1992, 671]. Finch has calculated $1\frac{1}{2}$ million terms of the sequence $\{a_n\}$, where $\{a_1, \ldots, a_6\} = \{3, 4, 6, 9, 10, 17\}$ and for $n \geq 6$, a_{n+1} is the least integer greater than a_n which is *not* of the form $a_i + a_j$, $i < j$; without detecting any regularity (ultimate periodicity of the differences). Finch believes that this may be due to a massive initial segment of irregular values, while Calkin suspects that there may be counterexamples to Finch's conjecture. They are preparing a joint paper."

Examples, in addition to Ulam's sequence, of 1-additive sequences (hastily hand-calculated, and probably containing errors):

1, 3, 4, 5, 6, 8, 13, 16, 18, 20, 25, 30, 35, 39, 44, 46, 53, 56, 58, 63, 65, 67, 75, 86, 89, 98, 101, 103, 108, 110, 112, 136, 141, 143, 148, 158, 162, 167, 172, 174, 184, 193, 200, 205, 207, 212, 217, ...

1, 4, 5, 6, 7, 8, 10, 16, 18, 19, 21, 31, 32, 33, 42, 46, 56, 57, 59, 69, 70, 71, 81, 82, 83, 84, 93, ...

1, 5, 6, 7, 8, 9, 10, 18, 23, 25, 27, 29, 38, ...

2, 5, 7, 9, 11, 12, 13, 15, 19, 23, 27, 29, 35, 37, 41, 43, 45, 49, 51, 55, 61, 67, 69, 71, 79, 83, 85, 87, 89, 95, 99, 107, 109, 119, 131, 133, 135, 137, 139, 141, 145, 149, ... (periodic, modulo 126, with 2 & 12 the only exceptions). A7300

2, 7, 9, 11, 13, 15, 16, 17, 19, 21, 25, 27, 31, 35, 41, 47, 49, 57, 59, 61, 65, 67, 69, 71, 75, 81, 85, 91, 93, 95, 101, 103, 105, 109, 117, 121, 123, 133, 135, 139, 141, 143, 145, 147, 151, ...

3, 4, 7, 10, 11, 13, 15, 16, 21, 22, 27, 30, 35, 36, 41, 44, 50, 53, 55, 61, 69, 70, 75, 78, 84, 87, 92, 93, 107, 112, 121, 132, 140, 146, 149, 152, 166, ...

4, 5, 9, 13, 14, 17, 19, 21, 24, 25, 27, 35, 37, 43, 45, 47, 57, 67, 69, 73, 77, 83, 93, 101, 105, 109, 113, 115, 123, 125, 133, 149, 153, 163, 173, 197, 201, 205, 209, 213, 217, ... (periodic, modulo 192, with 4, 14 & 24 the only exceptions). A6844

4, 7, 11, 15, 18, 19, 23, 25, 27, 31, 32, 33, 35, 41, 47, 49, 55, 57, 61, 63, 75, 87, 89, 91, 105, 119, 121, 125, 129, 133, 139, 147, 153, 161, 185, 189, 203, 206, 213, 225, 233, 235, ... (periodic, modulo 11301098, with 4, 18, 32 & 206 the only even values).

REFERENCES

Neil J. Calkin, Sum-free sets and measure spaces, PhD thesis, Univ. of Waterloo, 1988. [1992, 671]

Neil J. Calkin & Steven R. Finch, Necessary and sufficient conditions for a sum-free set to be ultimately periodic. [1992, 671]

Peter J. Cameron, Portrait of a typical sum-free set, Surveys in Combinatorics 1987, *London Math. Soc. Lecture Notes*, **123**(1987) Cambridge Univ. Press, 13–42. [1992, 671]

Steven R. Finch, Conjectures about s-additive sequences, *Fibonacci Quart.*, **29** (1991) 209–214. [1973, 919]

Steven R. Finch, On the regularity of certain 1-additive sequences, *J. Combin. Theory Ser.*

A, **60**(1992) 123–130. [1973, 919]

Steven R. Finch, Patterns in 1-additive sequences, *Experimental Math.*, **1**(1992) 57–63. [1973, 919]

Steven R. Finch, Are 0-additive sequences always regular?, **99**(1992) 671–673.

P. N. Muller, On properties of unique summation sequences, Master's thesis, SUNY at Buffalo, 1966.

Frank W. Owens, Sums of consecutive U-numbers, Abstract 74T-A74, *Notices Amer. Math. Soc.*, **21**(1974) A-298.

Raymond Queneau, Sur les suites s-additives, *J. Combin. Theory*, **12**(1972) 31–71.

Bernardo Recamán, Questions on a sequence of Ulam, **80**(1973) 919–920.

James H. Schmerl & Eugene Spiegel (tbp), The regularity of some 1-additive sequences, (1994) [1973, 919]

S. M. Ulam, *Problems in Modern Mathematics*, Interscience, New York, 1964, p. ix.

M. C. Wunderlich, The improbable behaviour of Ulam's summation sequence, in A. O. L. Atkin & B. J. Birch, *Computers in Number Theory* (Proc. 2nd Atlas Conf., Oxford, 1970, Academic Press, 1971, 249–257.

David Zeitlin, Ulam's sequence $\{U_n\}$, $U_1 = 1$, $U_2 = 2$, is a complete sequence, Abstract 75T-A267, *Notices, Amer. Math. Soc.*, **22**(1975) A-707.