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Parker's Permutation Problem Involves the Catalan Numbers

preprint

August 3, 1992

The late E. T. Parker submitted the following problem to the MONTHLY:

> Let a_i, $1 \le i \le n$ be n integers whose sum is a multiple of $n+1$. Must there be permutations $\{x_i\}$ and $\{y_i\}$ of 1, 2, ..., n such that, for each i,
>
> $$x_i + y_i \equiv a_i \bmod (n+1)?$$

Neither the proposer nor the referees gave a solution, so the problem was not used in the regular Problems section.

John Selfridge recasts the problem in the following form:

Write out the addition table for the numbers 1 up to n, mod $n+1$; e.g., for $n=5$:

	5	4	3	2	1
1	0	5	4	3	2
2	1	0	5	4	3
3	2	1	0	5	4
4	3	2	1	0	5
5	4	3	2	1	0

where there are n entries 0 and $n-1$ of each of the others. Given any n of these residues, with repetitions allowed, but with zero sum mod $n+1$, can they be chosen just one from each row and one from each column?

It is not difficult to verify that this can always be done if $n \leq 5$ and a computer could probably go twice as far. But after looking at the problem for a while, you'll feel sure that the answer is "yes" – but can you prove it?

Clearly there are $n!$ choices from the table, but these include duplicates. Just how many multisets of n residues are there, mod $n+1$?

It is easy to make the rough estimate that strongly suggests that the answer to Parker's problem is affirmative. Precisely the number is

$$\sum_{k=0}^{n-1} p(k(n+1); n, n)$$

where $p(m; n, n)$ is the number of partitions of m into at most n parts each no bigger than n. Here is a table of the summand for $1 \leq n \leq 7$.

n														Total
1							1							1
2						1		1						2
3					1		3		1					5
4				1		5		7		1				14
5			1		9		20		11		1			42
6		1		13		48		51		18		1		132
7	1		20		100		169		112		26		1	429

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Reminiscent of the Stirling numbers of the second kind, except that the present numbers are a bit smaller, and the row sums, instead of being Bell numbers, are the Catalan numbers,

$$\frac{1}{n+1}\binom{2n}{n}.$$

To prove that this is so, Ira Gessel recalls that the number of of partitions of k with at most m parts, each no bigger than n, is the coefficient of q^k in the q-binomial coefficient

$$\begin{bmatrix} m+n \\ n \end{bmatrix} = \frac{(q)_{m+n}}{(q)_m (q)_n}$$

where $(q)_i = (1-q)(1-q^2)\cdots(1-q^i)$. What we want is the sum of the coefficients of q^k, for k divisible by $n+1$, in $\begin{bmatrix} 2n \\ n \end{bmatrix}$.

Suppose that $P(q)$ is any polynomial and let r be a positive integer. To find the sum of the coefficients of q^k for k divisible by r in $P(q)$, let ζ be a primitive rth root of unity. Then the desired sum is easily seen to be

$$\frac{1}{r}\left(P(1)+P(\zeta)+P(\zeta^2)+\cdots+P(\zeta^{r-1})\right).$$

(A similar formula can be used to find the sum of the coefficients of q^k in $P(q)$ for k in a given residue class modulo r.)

Return to our original problem and let ζ be a primitive $(n+1)$st root of unity and let $P(q) = \begin{bmatrix} 2n \\ n \end{bmatrix}$. It is sufficient to show that for $i = 1, \ldots, n$, $P(\zeta^i) = 0$. But ζ^i is some primitive dth root of unity for some divisor d of $n+1$. So it is enough to show that if d is a divisor of $n+1$ with $d > 1$ and ξ is a primitive dth root of unity, then $P(\xi) = 0$. Let $\phi_d(q)$ be the dth cyclotomic polynomial. It is sufficient to show that $P(q)$ is divisible by $\phi_d(q)$. Equivalently we may show that the power of $\phi_d(q)$ dividing $(q)_{2n}$ is strictly greater than the power of $\phi_d(q)$ dividing $(q)_n^2$. The power of $\phi_d(q)$ dividing $(q)_i$ is $\lfloor i/d \rfloor$ so we need only show that

$$\left\lfloor \frac{2n}{d} \right\rfloor > 2\left\lfloor \frac{n}{d} \right\rfloor.$$

But, if $n+1 = td$ with $d > 1$, then the left side is $2t-1$ and the right side is $2t-2$.

Gessel also observes that the same argument shows that the sum of the coefficients of $\begin{bmatrix} 2n \\ n \end{bmatrix}$ congruent to j modulo $n+1$ is also the Catalan number for any j. In some cases $n+1$ can be replaced by other numbers, in the sense that if s is the number replacing $n+1$, the sum of the coefficients of $\begin{bmatrix} 2n \\ n \end{bmatrix}$ congruent to j modulo s is independent of j, and is thus equal to $\frac{1}{s}\binom{2n}{n}$, but he hasn't investigated this further.

Apart from the Catalan numbers, none of the sequences associated with our triangular array appear to be in the second edition of Sloane's Handbook [**3**], unless a preprint of this paper reaches him in time and some of them receive his seal of approval. They can be written in terms of the partition function, but as one leaves the edges of the table, the formulas become less and less closed. For $k = 1$ and $k = n-2$ they are $p(n+1)-2$ and $p(n+2)-4$, where $p(m)$ is the number of partitions of m. If we write $P(m) = \sum_{i=1}^{m} p(i)$, then the formulas for $k = 2$ and $k = n-3$ are $p(2n+2) - 2P(n+1) + 1$ and $p(2n+3) - 2P(n+2) + 7$. Have readers seen this array in any other context?

Let us return to Parker's original problem. For $n = 1$ and 2 there are just the right number of choices. For $n = 3$ the choice 013 can be made in two ways. For $n = 4$ the choices

0014, 0023, 1234 can each be made in 3 ways;
0113, 0122, 0244, 0334 each in 2 ways; and
0000, 1112, 1144, 1333, 2224, 2233, 3444 uniquely.

Mira, ~~4~~ seqs to enter please

For $n = 5$, 01245 can be chosen in 8 ways;
each of 00123, 00345, 01344, 02235 in 6 ways;
00015, 00024, 01335, 02334, 11235, 12234, 13455, 23445 in 4 ways;
00114, 00255, 01122, 01155, 02244, 04455, 11334, 23355 in 3 ways;
eleven others in 2 ways and the remaining ten uniquely.

Clearly the number of repetitions is larger the more distinct the sizes of the parts. Is there any neat way of relating the number of repetitions to the shape of the partition?

There are so many manifestations of the Catalan numbers [**1**, **2**] that it seems likely that there are more direct combinatorial proofs awaiting discovery. And what about Parker's original problem?

REFERENCES

1. Henry W. Gould, Bell & Catalan Numbers: research bibliography of two special number sequences, 6th edition, Morgantown WV, 1985.

2. Michael J. Kuchinski, Catalan Structures and Correspondences, MSc thesis, West Virginia University, 1977.

3. Neil J. A. Sloane, The New Book of Integer Sequences, W. H. Freeman, 1993.

APPENDIX [not for publication; for information NJAS & others]

$k = 1$: 0, 1, 3, 5, 9, 13, 20, 28, 40, 54, 75, 99, 133, 174, 229, 295, 383, 488, 625, 790, 1000, 1253, 1573, 1956, 2434, 3008, 3716, 4563, 5602, 6840, 8347, ... %O 1,3 A7042

$k = n - 2$: –, 1, 3, 7, 11, 18, 26, 38, 52, 73, 97, 131, 172, 227, 293, 381, 486, 623, 788, 998, 1251, 1571, 1954, 2432, 3006, 3714, 4561, ~~5600~~, 6838, 8345, 10139, ... ~~A7043 %O 2,2~~

$k = 2$: 0, 0, 1, 7, 20, 48, 100, 194, 352, 615, 1034, 1693, 2705, 4239, 6522, 9889, 14786, 21844, 31913, 46165, 66162, 94035, 132600, 185637, 258128, 356674, 489906, 669173, 909212, 1229217, 1653993, ... A7044 %O 1,4

$k = n - 3$: –, 0, 1, 5, 20, 51, 112, 221, 411, 720, 1221, 2003, 3206, 5021, 7728, 11698, 17472, 25766, 37580, 54254, 77617, 110087, 154942, 216488, 300456, 414365, 568113, 774571, 1050572, 1417868, 1904641, ... A7045 %O 2,3

%N From a partition triangle.

%R AMM xx yyy 33-

~~%O~~