A quick count of plane (or planar embedded) labeled trees

DAVID CALLAN

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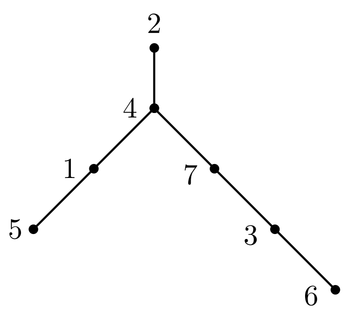
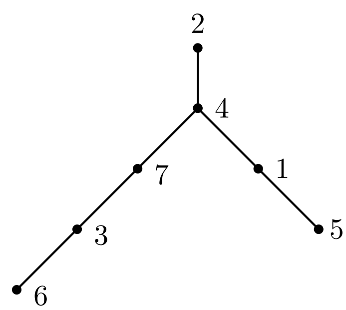
We start with a distinction between plain and plane. The two trees pictured below are the same considered as plain labeled trees (same vertices, same edges) but distinct considered as plane labeled trees (you can't interchange the branches below vertex 4 without leaving the plane).





A plain labeled tree is often called a *Cayley* tree since such trees are counted by Cayley's famous formula n^{n-2}. A plane labeled tree[1] can be defined as a Cayley tree enriched with a cycle structure on the neighbors of each vertex (no enrichment for the 1-vertex tree, which lacks neighbors). This cycle structure captures the "planar" specification that, for each vertex v, its neighboring vertices (along with their subtrees) may be rotated around v but any other rearrangement of the neighbors results in a different tree. Leroux and Miloudi [1] use the theory of combinatorial species to show (among many other things) that there are $\frac{(2n-3)!}{(n-1)!}$ (A006963) plane labeled trees on $[n] = \{1, 2, \ldots, n\}$, $n \geq 2$. Here is a simple direct proof that explains the simple answer. Henceforth, we assume $n \geq 2$. First, we need a count of compositions with "0" parts allowed.

Proposition 1. $\left| \{(i_1, \ldots, i_n) \,:\, i\text{'s nonnegative integers, } i_1 + \cdots + i_n = n-2\} \right| = \binom{2n-3}{n-2}$.

Proof. Arrange $n-2$ 1's and $n-1$ dividers in a row, for example,

$$1 \Big| 1 \Big| \Big| 1\,1 \Big| 1 \Big| \Big| \Big| 1\,.$$

The lengths of the runs of 1's separated by the dividers correspond to the sequences being counted. □

[1]not to be confused with the plane tree of Stanley's *Enumerative Combinatorics*, Vol. 2, which might more properly be called an ordered tree.

We also need a well known refinement of Cayley's formula (proved using the Prüfer construction). The *degree sequence* of a Cayley tree on $[n]$ is $(d_1, \ldots, d_n)$ where d_j is the degree (number of neighbors) of j. The degree sequence always satisfies $d_1 + \cdots + d_n = 2(n-1)$ since the tree has $n-1$ edges; equivalently, with $i_j := d_j - 1,\ 1 \leq j \leq n$, the i's are nonnegative integers satisfying $i_1 + \cdots + i_n = n-2$.

Proposition 2. [2, p. 6] *The number of Cayley trees on* $[n]$ *with degree sequence* $(d_1, \ldots, d_n)$ *is the multinomial coefficient* $\binom{n-2}{d_1-1,\,\ldots,\,d_n-1}$. □

A Cayley tree with degree sequence $(d_1, \ldots, d_n)$ generates $(d_1-1)! \cdots (d_n-1)!$ plane labeled trees. So, summing over degree sequences $\boldsymbol{d} = (d_1, \ldots, d_n)$, the desired count is

$$
\begin{aligned}
&\sum_{\boldsymbol{d}} \binom{n-2}{d_1-1,\ \ldots,\ d_n-1} (d_1-1)! \cdots (d_n-1)! \\
=\ &\sum_{\substack{(i_1,\ldots,i_n)\,:\,i\text{'s}\geq 0 \\ i_1+\cdots+i_n=n-2}} \binom{n-2}{i_1,\ \ldots,\ i_n} i_1! \cdots i_n! \\
=\ &\sum_{\substack{(i_1,\ldots,i_n)\,:\,i\text{'s}\geq 0 \\ i_1+\cdots+i_n=n-2}} (n-2)! \\
\overset{\text{Prop. 1}}{=}\ &\binom{2n-3}{n-2}(n-2)! \\
=\ &\frac{(2n-3)!}{(n-1)!}\,.
\end{aligned}
$$

References

[1] Pierre Leroux and Brahim Miloudi, Généralisations de la Formule d'Otter, *Annales des sciences mathématiques du Québec*, **16** (1), 1992, 53–80, http://www.labmath.uqam.ca/~annales/volumes/16-1/PDF/053-080.pdf.

[2] J.W. Moon, Counting Labelled Trees, Issue 1 of Canadian mathematical monographs, Canadian Mathematical Congress, 1970.