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SHEFFER POSETS AND R- SIGNED PERMUTATIONS

Richard Ehrenborg et Margaret A. Readdy

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Sheffer posets and r-signed permutations

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Abstract

We generalize the notion of a binomial poset to a larger class of posets, which we call Sheffer posets. There are two interesting subspaces of the incidence algebra of such a poset. These spaces behave like a ring and a module and are isomorphic to certain generating functions.

We also generalize the concept of R-labelings to linear edge-labelings, and prove a result analogous to a theorem of Björner and Stanley about R-labelings. This, together with the theory of rank-selections from a Sheffer poset, enables us to study augmented r-signed permutations. As a special case, we obtain the generating function $\sec(rx) \cdot (\sin(px) + \cos((r-p)x))$, which enumerates alternating augmented r-signed permutations.

1 Introduction

The theory of binomial posets, developed by Doubilet, Rota, and Stanley [6], explains why generating functions of the form

$$\sum_{n \geq 0} a_n \frac{t^n}{B(n)} \qquad (a_n \in \mathbb{Z})$$

occur in enumerative combinatorics for certain sequences $B(n)$. Such sequences include $B(n) = 1$, $B(n) = n!$, and $B(n) = [n]!$. Stanley continued the development of the theory of binomial posets by studying rank-selections of binomial posets [13]. As an application, he showed why the exponential generating function of the Euler numbers has the form $\tan(x) + \sec(x)$. Recall that Euler numbers enumerate alternating permutations in the symmetric group.

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We develop Sheffer posets, an extension of binomial posets, to explain the existence of a larger class of generating functions. We define a *Sheffer poset* to be an infinite graded poset, such that the number of maximal chains $D(n)$ in an n-interval $[\hat{0}, y]$ only depends on $\rho(y)$, the rank of the element y, and the number of maximal chains $B(n)$ in an n-interval $[x, y]$, where $x \neq \hat{0}$, only depends on $\rho(x, y) = \rho(y) - \rho(x)$. The two functions $B(n)$ and $D(n)$ are called the *factorial functions* of the Sheffer poset.

Let P be a Sheffer poset. Let $R(P)$ be the subalgebra consisting of those elements f in the incidence algebra of P whose values $f(x, y)$ only depend on the length of the interval $[x, y]$ and the support of f is contained in $\{[x, y] : x \neq \hat{0}\}$. Similarly, let $M(P)$ consist of those elements in the incidence algebra that only depend on the length of the interval and have their support contained in $\{[\hat{0}, y] : y \neq \hat{0}\}$. $M(P)$ may be considered as an $R(P)$-module, where the multiplication is the usual convolution in the incidence algebra $I(P)$. The main result of Section 4 is that the algebra-module pair $(R(P), M(P))$ is isomorphic to generating functions of the form $\sum_{n \geq 0} f(n) \frac{t^n}{B(n)}$ and $\sum_{n \geq 1} g(n) \frac{t^n}{D(n)}$.

Examples of Sheffer posets include binomial posets, "upside-down trees," the r-cubical lattice, the poset of partial permutations, the poset of partial isomorphisms, and the lattice of isotropic subspaces. For more details, see Examples **a**, **c**, **d**, **h**, and **j** in Section 3. An in-depth study of the r-cubical lattice will appear in [7].

In the spirit of Stanley's results for binomial posets, we develop identities concerning the ν function (a generalization of the Möbius function μ) of rank-selections from Sheffer posets. We recently learned that Reiner [12] has anticipated these identities in the case where $\overline{\eta}(n) \equiv 1$, that is, when the ν function is the Möbius function.

Before developing rank-selection theory of Sheffer posets, we extend results by Björner and Stanley on R-labelings to linear edge-labelings. With respect to a given linear edge-labeling, we define the functions η and ν. These two functions for a poset with linear edge-labeling might be viewed as a natural extension of the the zeta-function ζ and Möbius function μ for a poset with an R-labeling. Our analogue of the Björner and Stanley result is given in Proposition 2. It says that the number of maximal chains with descent set S in a poset having a linear edge-labeling λ is given by $(-1)^{|S|-1} \cdot \nu_S(\hat{0}, \hat{1})$.

As an application of Proposition 2, we are able to enumerate augmented r-signed permutations. An r-signed permutations is a permutation where each entry has been given one of r possible signs. An augmented r-signed permutation corresponds to a maximal chain in the r-cubical lattice. By the theory of rank-selections of Sheffer posets, we can now easily derive that the number of alternating r-signed permutations is given by $\sec(rx) \cdot (\sin(px) + \cos((r-p)x))$. Our method explains why this generating function takes on such a simple form. Three special cases of this generating function have been considered before: the classical Euler numbers, augmented alternating signed permutations [11], and alternating indexed permutations [16].

The isotropic subspace lattice, presented in Example (**j**), may be viewed as a q-analogue of the cubical lattice. A natural question to ask is if there are any q-analogues to the r-cubical lattice in Example (**d**). In Section 8 we construct an analogue to the 4-cubical lattice. This construction requires the field to have characteristic zero. Thus the lattice presented is not a q-analogue, but rather a "linear analogue".

The authors would like to thank Gábor Hetyei, Jacques Labelle, Simon Plouffe, and Richard Stanley for many helpful discussions.

2 Definitions

Recall the definition of a binomial poset P.

Definition 1 *A poset P is called a* binomial poset *if it satisfies the following three conditions:*

1. *P is locally finite with $\hat{0}$ and contains an infinite chain.*
2. *Every interval $[x, y]$ is graded. If $\rho(x, y) = n$, then we call $[x, y]$ an n-*interval.
3. *For all $n \in \mathbb{N}$, any two n-intervals contain the same number $B(n)$ of maximal chains. We call $B(n)$ the* factorial function *of P.*

For standard poset terminology we refer the reader to [14]. Also, for material on binomial posets see [6, 13, 14].

Classical examples of binomial posets are the linear order on $\mathbb{N} = \{0, 1, 2, \ldots\}$, the lattice of finite subsets of an infinite set (this is the boolean lattice), and the lattice of finite dimensional subspaces from an infinite dimensional vector space over the finite field $\mathbb{F}_q$. These examples have the factorial functions $B(n) = 1$, $B(n) = n!$, and $B(n) = [n]!$, respectively. Recall that $[n] = 1 + q + \cdots + q^{n-1}$ and $[n]! = [1] \cdot [2] \cdots [n]$.

The definition of a Sheffer poset is quite similar, except that we treat the intervals of the form $[\hat{0}, y]$ differently.

Definition 2 *A poset P is called a* Sheffer poset *if it satisfies the following four conditions:*

1. *P is locally finite with $\hat{0}$ and contains an infinite chain.*
2. *Every interval $[x, y]$ is graded. If $\rho(x, y) = n$, then we call $[x, y]$ an n-*interval.
3. *Two intervals $[\hat{0}, y]$ and $[\hat{0}, v]$, such that $y \neq \hat{0}$, $v \neq \hat{0}$, and $\rho(y) = \rho(v)$, have the same number $D(n)$ of maximal chains.*
4. *Two intervals $[x, y]$ and $[u, v]$, such that $x \neq \hat{0}$, $u \neq \hat{0}$, and $\rho(x, y) = \rho(u, v)$, have the same number $B(n)$ of maximal chains.*

An interval $[\hat{0}, y]$, where $y \neq \hat{0}$, is called a *Sheffer interval*, whereas an interval $[x, y]$, with $x \neq \hat{0}$, is called a *binomial interval*. Let $\mathcal{B}$ be the set of all binomial intervals, and $\mathcal{S}$ be the set of all Sheffer intervals. Formally, that is

$$\begin{aligned} \mathcal{B} &= \{[x, y] \ : \ x \neq \hat{0}\}, \\ \mathcal{S} &= \{[\hat{0}, y] \ : \ y \neq \hat{0}\}. \end{aligned}$$

We call the pair of functions $B(n)$ and $D(n)$ the *factorial functions* of the Sheffer poset P. Since we seldom speak about the one-element interval $[\hat{0}, \hat{0}]$, we never define $D(0)$. It is easy to observe that $B(0) = B(1) = D(1) = 1$ and that both $B(n)$ and $D(n)$ are weakly increasing functions.

As in the theory of binomial posets, we let $\left[{n \atop k} \right]$ denote the number of elements of rank k in a binomial interval of length n. That is, for $\rho(x, y) = n$ and $x \neq \hat{0}$,

$$\begin{bmatrix} n \\ k \end{bmatrix} = \left|\{z \in [x, y] \;:\; \rho(x, z) = k\}\right|.$$

This is a well-defined number and is given by $\left[{n \atop k} \right] = \frac{B(n)}{B(k) \cdot B(n-k)}$. Similarly, for $1 \leq k \leq n$ we define $\left[\left[{n \atop k} \right]\right.$ to be the number of elements of rank k in a Sheffer interval of length n. That is, for $\rho(y) = n > 0$,

$$\left[\begin{bmatrix} n \\ k \end{bmatrix}\right. = \left|\left\{z \in [\hat{0}, y] \;:\; \rho(z) = k\right\}\right|.$$

To find a formula for $\left[\left[{n \atop k} \right]\right.$, and thus show that this number is well-defined, count the number of maximal chains from $\hat{0}$ to y ($\rho(y) = n$) which pass through an element z of rank k. Since $k \geq 1$, $[\hat{0}, z]$ is a Sheffer interval and has $D(k)$ maximal chains. The interval $[z, y]$ is a binomial interval, thus having $B(n-k)$ maximal chains. Hence there are $D(k) \cdot B(n-k)$ maximal chains through the element z. Since the total number of maximal chains in the interval $[\hat{0}, y]$ is $D(n)$, we conclude

$$\left[\begin{bmatrix} n \\ k \end{bmatrix}\right. = \frac{D(n)}{D(k) \cdot B(n-k)}.$$

Observe that we do not define $\left[\left[{n \atop k} \right]\right.$ when $k = 0$.

Next we define $A(n)$ to be the number of atoms in a binomial interval of length n. That is, $A(n)$ is equal to $\left[{n \atop 1} \right]$. One then obtains that $A(n) = \frac{B(n)}{B(n-1)}$ and $B(n) = A(n) \cdots A(1)$. Also, it is easy to see that $A(n)$ is an increasing sequence, which implies that the numbers $B(n)$ form a log-convex sequence. (See [14, Exercise 3.78a].) Similarly, the sequence $\left[\left[{n \atop 1} \right]\right. = \frac{D(n)}{B(n-1)}$ is an increasing sequence. This implies the inequality $D(n+1) \cdot B(n-1) \geq D(n) \cdot B(n)$. Also, define $C(n)$ to be the number of coatoms in a Sheffer interval of length n. Thus

$$C(n) = \left[\begin{bmatrix} n \\ n-1 \end{bmatrix}\right. = \frac{D(n)}{D(n-1)}.$$

It follows directly that $D(n) = C(n) \cdots C(1)$. Observe that the statement "$C(n)$ is an increasing sequence" and "$D(n)$ is log-convex" are equivalent. (These statements are not true in general; see Example **(c)**.)

Finally, if $D(n)$ is log-convex then the Whitney numbers of the second kind for a Sheffer interval are log-concave. (To obtain $\left[\left[{n \atop k} \right]\right.^2 \geq \left[\left[{n \atop k-1} \right]\right. \cdot \left[\left[{n \atop k+1} \right]\right.$, multiply the two inequalities $D(k)^2 \leq D(k-1) \cdot D(k+1)$ and $B(n-k)^2 \leq B(n-k+1) \cdot B(n-k-1)$, take the inverse on both sides and multiply with $D(n)^2$.)

An open problem we pose is to determine which pairs of functions $B(n)$ and $D(n)$ may be factorial functions of Sheffer posets. This is most likely a very hard problem. To the authors' knowledge it has not yet been determined which functions are factorial functions of binomial posets. (See [14, Exercise 3.78b].)

We selected the name Sheffer for this class of posets for the following reason. Recall that a sequence of polynomials $\{p_n(x)\}_{n\geq 0}$ is said to be of *binomial type* if $\deg(p_n(x)) = n$, $p_n(0) = \delta_{n,0}$, and the following identity holds:

$$p_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} p_k(x) p_{n-k}(y).$$

Given a binomial sequence $\{p_n(x)\}_{n\geq 0}$, a sequence of polynomials $\{s_n(x)\}_{n\geq 0}$ is a *Sheffer sequence* associated to the binomial sequence if for each $n \in \mathbb{N}$, the polynomials $s_n(x)$ satisfy $\deg(s_n(x)) = n$ and the identity

$$s_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} s_k(x) p_{n-k}(y).$$

(These two equations may be easily expressed in terms of Hopf algebras.) Thus the main motivation for our choice of terminology is that a Sheffer sequence has a similar relation to a binomial sequence as a Sheffer poset to a binomial poset. This will become more apparent when we develop the theory of the incidence algebra of Sheffer posets in greater detail.

Finally, we would like to point out that both binomial and Sheffer posets are special cases of triangular posets. A graded poset is *triangular* if the number of maximal chains in the interval $[x, y]$ only depends upon $\rho(x)$ and $\rho(y)$. The interested reader is referred to [6] and [14, Exercise 3.79] for more properties of triangular posets.

3 Examples

Let us now consider a few examples of Sheffer posets.

a. Let P be a binomial poset with factorial function $B(n)$. Then P is directly a Sheffer poset, with factorial functions $B(n)$ and $D(n) = B(n)$, for $n \geq 1$.

b. Let P be a binomial poset with factorial function $B(n)$. By adjoining a new minimal element $\widehat{-1}$ to P (or more generally to several disjoint copies of P), we obtain a Sheffer poset with factorial functions $B(n)$ and $D(n) = B(n-1)$, for $n \geq 1$.

c. Let $E_1, E_2, \ldots$ be an infinite sequence of disjoint nonempty finite sets, where E_n has cardinality e_n. Consider the set

$$\{\hat{0}\} \cup \bigcup_{n\geq 1} \prod_{i\geq n} E_i,$$

where $\prod$ stands for cartesian product. We make this into a ranked poset by letting $\hat{0}$ be the minimal element, and defining the cover relation by

$$(x_n, x_{n+1}, x_{n+2}, \ldots) \prec (x_{n+1}, x_{n+2}, \ldots),$$

where $x_i \in E_i$. Thus the elements of $\prod_{i\geq n} E_i$ have rank n. This poset is a Sheffer poset with the factorial functions $B(n) = 1$ and $D(n) = \prod_{i=1}^{n-1} e_i$. We may view this poset as an "upside-down tree."

d. Let r be a positive integer. Consider ordered r-tuples $A = (A_1, A_2, \ldots, A_r)$ of subsets from an infinite set I, such that $A_i \cap A_j = \emptyset$ when $i \neq j$, and $A_1 \cup A_2 \cup \cdots \cup A_r$ is cofinite with respect to the set I. Define the order relation by $A \leq B$ if $A_i \supseteq B_i$ for all $i = 1, 2, \ldots, r$, and adjoin a minimum element $\hat{0}$. We call this Sheffer poset the ***r*-cubical lattice**. It has the factorial functions $B(n) = n!$ and $D(n) = r^{n-1} \cdot (n-1)!$. These posets have been studied in [10]. When $r = 1$, this is the boolean algebra with a new minimal element (see Example **(b)**). For $r = 2$, we obtain the cubical lattice.

e. Let P be a Sheffer poset. Define $P^{(k)}$ to be poset on the set $\{x \in P : k \text{ divides } \rho(x)\}$. It is easy to see that $P^{(k)}$ is a Sheffer poset with the factorial functions $B_k(n) = \frac{B(n \cdot k)}{B(k)^n}$ for $n \geq 0$, and $D_k(n) = \frac{D(n \cdot k)}{D(k) \cdot B(k)^{n-1}}$ for $n \geq 1$.

f. Let P and Q be Sheffer posets. Define the poset $P * Q$ on the set

$$P * Q = \{(x, y) \in P \times Q \ : \ \rho(x) = \rho(y)\},$$

where the order relation is $(x, y) \leq (u, v)$ if $x \leq_P u$ and $y \leq_Q v$. The factorial functions of $P * Q$ are $B_P(n) \cdot B_Q(n)$ and $D_P(n) \cdot D_Q(n)$. The product of two Sheffer posets is not a lattice in general. In fact if there is an $x \in P$ covered by $u, z \in P$ and a $w \in Q$ covering $y, v \in Q$, such that $\rho(x) + 1 = \rho(w)$ then in $P * Q$ we have $(x, y) \prec (u, w)$, $(x, v) \prec (u, w)$, $(x, y) \prec (z, w)$, and $(x, v) \prec (z, w)$, showing that $P * Q$ is not a lattice. We may use this observation to construct two nonisomorphic Sheffer posets with identical factorial functions. Let P be the boolean poset with a new minimal element adjoined. (See Example **(b)**.) Let Q be the poset in Example **(c)** with $e_n = r$ for all $n \geq 1$. Then the factorial functions of the product $P * Q$ are $B(n) = n!$ and $D(n) = r^{n-1} \cdot (n-1)!$, that is, the same factorial functions as the r-cubical lattice in Example **(d)**. These two posets are not isomorphic, even though they have the same factorial functions, since the r-cubical lattice is indeed a lattice, while $P * Q$ is not.

g. Let E be a nonempty finite set of cardinality a, and r a nonnegative integer. Consider the set

$$\{\hat{0}\} \cup (\mathbb{P} \times E^r),$$

where $\mathbb{P} = \{1, 2, \ldots\}$. Define the cover relation $(n, x_1, x_2, \ldots, x_r) \prec (n+1, x_2, \ldots, x_r, x_{r+1})$, where $x_i \in E$, and that $\hat{0}$ is the minimal element. Then this is a Sheffer poset with the factorial functions $B(n) = \max(1, a^{n-r})$ and $D(n) = a^{n-1}$.

h. Let I be an infinite set. Consider all triplets (A, B, π) such that A and B are cofinite subsets of I satisfying $|I - A| = |I - B| < \infty$, and π is a bijection between A and B. Introduce the order relation $(A, B, \pi) \leq (C, D, \sigma)$ if $A \supseteq C$, $B \supseteq D$, and $\pi|_C = \sigma$, and adjoin a minimum element $\hat{0}$. This poset is the lattice of all partial permutations with cofinite support on an infinite set. It is a Sheffer poset with the factorial functions $B(n) = n!$ and $D(n+1) = n!^2$. There is also a linear version of this poset, namely the lattice of all partial vector space isomorphisms. Let W be an infinite linear space over the finite field $\mathbb{F}_q$. Let the poset consists of all triplets (U, V, π), where U and V are subspaces of W, such that $\dim(W/U) = \dim(W/V) < \infty$ and $\pi : U \longrightarrow V$ is an isomorphism. As before, the order relation is given by restriction and by adjoining a minimum element $\hat{0}$. That is, $(U, V, \pi) \leq (X, Y, \sigma)$ if $U \supseteq X$, $V \supseteq Y$, and $\pi|_X = \sigma$. The factorial functions of this lattice are $B(n) = [n]!$ and $D(n+1) = (q-1)^n \cdot q^{\binom{n}{2}} \cdot [n]!^2$.

i. Consider the 120-cell, that is, the four dimensional regular polytope with Schläfi symbol $\{5, 3, 3\}$; see [5]. Its face lattice is a graded poset of rank 5. This poset is not a Sheffer poset, since it is not infinite, but it has the other properties of a Sheffer posets. In fact, the factorial functions are $B(n) = n!$ for $n \leq 4$, and $D(1) = 1$, $D(2) = 2$, $D(3) = 10$, $D(4) = 120$, and $D(5) = 14400$.

j. Let $\mathbb{F}$ be a field, and consider $\mathbb{F}^{2n}$ as a vector space of dimension $2n$ over $\mathbb{F}$. Let A be the two by two matrix

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

and let B be the $2n$ by $2n$ matrix defined by $B = A \otimes I_n$. Note that since A is a skew-symmetric matrix, B is also skew-symmetric. Define a skew-symmetric bilinear form on $\mathbb{F}^{2n}$ by

$$\langle \mathbf{x} \mid \mathbf{y} \rangle = \mathbf{x}^* B \mathbf{y}.$$

For a subspace V of $\mathbb{F}^{2n}$, define

$$V' = \{\mathbf{y} \in \mathbb{F}^{2n} \ : \ \forall \mathbf{x} \in V, \ \langle \mathbf{x} \mid \mathbf{y} \rangle = 0\}.$$

The subspace V is *isotropic* if $V \subseteq V'$. That is, "V isotropic" is equivalent to "for all $\mathbf{x}, \mathbf{y} \in V$ we have $\langle \mathbf{x} \mid \mathbf{y} \rangle = 0$."

Lemma 1 *If V is an isotropic subspace of $\mathbb{F}^{2n}$ then* $\dim(V') = 2n - \dim(V)$.

The proof of this lemma is quite similar to the proof of Proposition 5.

Lemma 2 *If V is an isotropic subspace of $\mathbb{F}_q^{2n}$ of dimension k then there are $1 + q + \cdots + q^{2(n-k)-1} = [2(n-k)]$ isotropic subspaces of dimension $k+1$ containing V.*

Proof: A subspace W of dimension $k+1$ containing V is of the form $W = V + \mathbb{F}\mathbf{w}$ for some vector $\mathbf{w} \notin V$. The condition that W is isotropic is equivalent to that $\langle \mathbf{w} \mid \mathbf{x} \rangle = 0$ for all $x \in V$. Hence $w \in V'$, and we obtain $W \subseteq V'$. Thus we would like to choose W such that $V \subseteq W \subseteq V'$ and $\dim(W) = k+1$. This is equivalent to choosing a 1-dimensional subspace of V'/V which has dimension $2(n-k)$. This may be done in $[2(n-k)]$ possible ways. □

Define $I_n(\mathbb{F})$ to be the poset of all isotropic subspaces of the vector space $\mathbb{F}^{2n}$ over the field $\mathbb{F}$, with the order relation $V \leq W$ if $V \supseteq W$, and adjoin a minimal element $\hat{0}$. This poset is graded and has rank $n+1$. $I_n(\mathbb{F}_q)$ is not a Sheffer poset, since it does not contain an infinite chain. But as for the poset in Example **(i)**, we may compute the factorial functions. Since every interval $[x, y]$, where $x \neq \hat{0}$, is isomorphic to the subspace lattice, we have that $B(k) = [k]!$ for $k \leq n$. By applying Lemma 2 we can count the number of maximal chains in an interval $[\hat{0}, y]$, and thus obtain $D(k+1) = [2k] \cdot [2k-2] \cdots [2] = [2k]!!$ for $k \leq n$.

We may view a subspace V of $\mathbb{F}^{2n}$ as a subspace of $\mathbb{F}^{2n+2}$ by adding two coordinates which are set to equal to zero. Thus we can define order-preserving surjective maps $\phi_n : I_n(\mathbb{F}) \longrightarrow I_{n+1}(\mathbb{F})$ by $\phi_n(\hat{0}) = \hat{0}$ and $\phi_n(V) = V + \mathbb{F} \cdot \mathbf{e}_{2n+2}$. Let $I(\mathbb{F})$ to be the direct limit of the posets $I_n(\mathbb{F})$. Then $I(\mathbb{F}_q)$ is a Sheffer poset, and has the factorial functions $B(n) = [n]!$ and $D(n+1) = [2n]!!$. Hence, we may consider $I_n(\mathbb{F}_q)$ as the q-analogue of the cubical lattice in Example **(d)**.

4 The incidence algebra of Sheffer posets

Let $I(P)$ be the incidence algebra of a Sheffer poset P over a field of complex numbers. Consider the following two subspaces of $I(P)$:

$$\begin{aligned} R(P) &= \{f \in I(P) \ : \ f(x,y) = 0 \text{ if } [x,y] \notin B,\ f(x,y) = f(u,v) \text{ if } \rho(x,y) = \rho(u,v)\}, \\ M(P) &= \{g \in I(P) \ : \ g(x,y) = 0 \text{ if } [x,y] \notin S,\ g(\hat{0},y) = g(\hat{0},v) \text{ if } \rho(y) = \rho(v)\}. \end{aligned}$$

That is, an element f in $R(P)$ may only take on non-zero values on a binomial interval, whereas an element g in $M(P)$ may only take on non-zero values on a Sheffer interval. Moreover, the values of a function in either $R(P)$ or $M(P)$ depend only on the length of the interval. Thus for $f \in R(P)$ we denote by $f(n)$ the value of the function f applied to a binomial interval of length n. Also, we let $g(n)$ denote the value of the function $g \in M(P)$ applied to a Sheffer interval of length n.

It is easy to see that $R(P)$ is closed under convolution. More interestingly, if $g \in M(P)$ and $f \in R(P)$ then $gf \in M(P)$. Hence we may view $M(P)$ as a $R(P)$-module.

Recall that $\mathbb{C}[[t]]$ denotes the ring of formal power series in the variable t whose coefficients are complex numbers. Let $\mathbb{C}_0[[t]]$ be the module of formal power series without a constant coefficient. That is, $\mathbb{C}_0[[t]] = t \cdot \mathbb{C}[[t]]$. Define two linear maps $\phi : R(P) \longrightarrow \mathbb{C}[[t]]$ and $\psi : M(P) \longrightarrow \mathbb{C}_0[[t]]$ by

$$\begin{aligned} \phi(f) &= \sum_{n \geq 0} f(n) \frac{t^n}{B(n)}, \\ \psi(g) &= \sum_{n \geq 1} g(n) \frac{t^n}{D(n)}. \end{aligned}$$

Proposition 1 *The pair of linear maps (ϕ, ψ) is an isomorphism between the algebra-module pair $(R(P), M(P))$ and the algebra-module pair $(\mathbb{C}[[t]], \mathbb{C}_0[[t]])$. That is, for $f, f' \in R(P)$ and $g \in M(P)$ we have*

$$\begin{aligned} \phi(ff') &= \phi(f) \cdot \phi(f'), \\ \psi(gf) &= \psi(g) \cdot \phi(f). \end{aligned}$$

Proof: One sees directly that ϕ and ψ are isomorphisms between vector spaces. Thus it is enough to prove the two identities in the statement of the proposition. Let $[x,y]$ be a binomial interval of length n in the Sheffer poset P. Then

$$\begin{aligned} ff'(n) &= (ff')(x,y) \\ &= \sum_{x \leq z \leq y} f(x,z) f'(z,y) \\ &= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} f(k) f'(n-k). \end{aligned}$$

Multiply this identity by $\frac{t^n}{B(n)}$, and sum over all $n \geq 0$ gives

$$\begin{aligned}
\sum_{n\geq 0} ff'(n) \cdot \frac{t^n}{B(n)} &= \sum_{n\geq 0}\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} f(k)f'(n-k) \cdot \frac{t^n}{B(n)} \\
&= \sum_{n\geq 0}\sum_{k=0}^{n} f(k) \cdot \frac{t^k}{B(k)} f'(n-k) \cdot \frac{t^{n-k}}{B(n-k)} \\
&= \left(\sum_{k\geq 0} f(k) \cdot \frac{t^k}{B(k)}\right) \cdot \left(\sum_{m\geq 0} f'(m) \cdot \frac{t^m}{B(m)}\right),
\end{aligned}$$

which is equivalent to $\phi(ff') = \phi(f) \cdot \phi(f')$.

Similarly, let y be an element of rank $n > 0$ in the poset P. Then

$$\begin{aligned}
gf(n) &= (gf)(\hat{0}, y) \\
&= \sum_{\hat{0}<z\leq y} g(\hat{0}, z) f(z, y) \\
&= \sum_{k=1}^{n} \left[\begin{bmatrix} n \\ k \end{bmatrix}\right] g(k) f(n-k).
\end{aligned}$$

As before, we multiply the above identity by $\frac{t^n}{D(n)}$, and sum over all $n \geq 1$. Thus

$$\begin{aligned}
\sum_{n\geq 1} gf(n) \cdot \frac{t^n}{D(n)} &= \sum_{n\geq 1}\sum_{k=1}^{n} \left[\begin{bmatrix} n \\ k \end{bmatrix}\right] g(k)f(n-k) \cdot \frac{t^n}{D(n)} \\
&= \sum_{k\geq 1}\sum_{n\geq k} g(k) \cdot \frac{t^k}{D(k)} f(n-k) \cdot \frac{t^{n-k}}{B(n-k)} \\
&= \left(\sum_{k\geq 1} g(k) \cdot \frac{t^k}{D(k)}\right) \cdot \left(\sum_{m\geq 0} f(m) \cdot \frac{t^m}{B(m)}\right),
\end{aligned}$$

which is equivalent to $\psi(gf) = \psi(g) \cdot \phi(f)$. □

Let $\mu(n)$ be the Möbius function of a binomial n-interval. Similarly let $\overline{\mu}(n)$ be the Möbius function of a Sheffer n-interval for $n \geq 1$. Since the Möbius function may be computed by Philip Hall's formula, which just involves counting chains, one may conclude that both $\mu(n)$ and $\overline{\mu}(n)$ are well-defined. We may view μ as an element of $R(P)$ and $\overline{\mu}$ as an element of $M(P)$. Also define the delta and zeta functions $\delta, \zeta \in R(P)$ and $\overline{\zeta} \in M(P)$ by $\delta(n) = \delta_{n,0}$ for $n \geq 0$, $\zeta(n) = 1$ for $n \geq 0$ and $\overline{\zeta}(n) = 1$ for $n \geq 1$. Observe that δ is the unit in $R(P)$.

Since $\mu\zeta = \delta$, we conclude by Proposition 1 that

$$\sum_{n\geq 0} \mu(n) \frac{t^n}{B(n)} = \left(\sum_{n\geq 0} \frac{t^n}{B(n)}\right)^{-1}.$$

Similarly for the Möbius function of a Sheffer interval we obtain the identity

$$-\overline{\mu} = \overline{\zeta}\mu.$$

In fact, let $y \in P$ be of rank $n > 0$. Then we have that

$$\begin{aligned}
-\overline{\mu}(\hat{0}, y) &= \sum_{\hat{0}<z\leq y} \mu(z, y) \\
&= \sum_{\hat{0}<z\leq y} \overline{\zeta}(\hat{0}, z)\mu(z, y) \\
&= \overline{\zeta}\mu(\hat{0}, y).
\end{aligned}$$

By applying ψ to this identity, we obtain the following:

Lemma 3

$$-\sum_{n\geq 1} \overline{\mu}(n)\frac{t^n}{D(n)} = \left(\sum_{n\geq 1} \frac{t^n}{D(n)}\right) \cdot \left(\sum_{n\geq 0} \frac{t^n}{B(n)}\right)^{-1}.$$

Let us now apply this lemma to a few examples. With a poset P as in Example **(a)**

$$\begin{aligned}
\sum_{n\geq 1} \overline{\mu}(n)\frac{t^n}{B(n)} &= -\left(\sum_{n\geq 1} \frac{t^n}{B(n)}\right) \cdot \left(\sum_{n\geq 0} \frac{t^n}{B(n)}\right)^{-1} \\
&= -1 + \left(\sum_{n\geq 0} \mu(n)\frac{t^n}{B(n)}\right) \\
&= \sum_{n\geq 1} \mu(n)\frac{t^n}{B(n)}.
\end{aligned}$$

This is true, since we already know that $\overline{\mu}(n) = \mu(n)$ holds in this example.

For a poset P from Example **(b)** we obtain

$$\sum_{n\geq 1} \overline{\mu}(n)\frac{t^n}{B(n-1)} = -\left(\sum_{n\geq 1} \frac{t^n}{B(n-1)}\right) \cdot \left(\sum_{n\geq 0} \frac{t^n}{B(n)}\right)^{-1} = -t.$$

This holds since $\mu(\widehat{-1}, \hat{0}) = -1$ and $\mu(\widehat{-1}, y) = 0$ for $y > \hat{0}$.

In Example **(c)** we have

$$\begin{aligned}
\sum_{n\geq 1} \overline{\mu}(n)\frac{t^n}{e_1\cdots e_{n-1}} &= -\left(\sum_{n\geq 1} \frac{t^n}{e_1\cdots e_{n-1}}\right) \cdot \left(\sum_{n\geq 0} t^n\right)^{-1} \\
&= -\left(\sum_{n\geq 1} \frac{t^n}{e_1\cdots e_{n-1}}\right) \cdot (1-t) \\
&= \sum_{n\geq 2} e_{n-1} \cdot \frac{t^n}{e_1\cdots e_{n-1}} - \sum_{n\geq 1} \frac{t^n}{e_1\cdots e_{n-1}}.
\end{aligned}$$

Hence $\overline{\mu}(n) = e_{n-1} - 1$ for $n \geq 2$. This can also be proven by a direct combinatorial argument.

For the r-cubical lattice in Example **(d)** we have the computation

$$\begin{aligned}\sum_{n\geq 1} \overline{\mu}(n) \frac{t^n}{D(n)} &= -\left(\sum_{n\geq 1} \frac{t^n}{r^{n-1}\cdot(n-1)!}\right)\cdot\left(\sum_{n\geq 0}\frac{t^n}{n!}\right)^{-1} \\ &= -t\cdot e^{\frac{t}{r}}\cdot e^{-t} \\ &= -\sum_{n\geq 1}(1-r)^{n-1}\cdot\frac{t^n}{r^{n-1}\cdot(n-1)!}.\end{aligned}$$

Thus $\overline{\mu}(n) = -(1-r)^{n-1} = (-1)^n\cdot(r-1)^{n-1}$.

We may count the number of chains of length k in a Sheffer interval of length n by considering the element $\overline{\zeta}(\zeta-\delta)^{k-1}$ in the incidence algebra. The corresponding generating function is

$$\left(\sum_{n\geq 1}\frac{t^n}{D(n)}\right)\cdot\left(\sum_{n\geq 1}\frac{t^n}{B(n)}\right)^{k-1}.$$

Lemma 4 *Let $\overline{Z}_n(w)$ be the zeta polynomial in the variable w of a Sheffer interval of length n. Then the generating series for the zeta polynomials is given by*

$$\sum_{n\geq 1}\overline{Z}_n(w)\frac{t^n}{D(n)} = \left(\sum_{n\geq 1}\frac{t^n}{D(n)}\right)\cdot\left(\left(\sum_{n\geq 0}\frac{t^n}{B(n)}\right)^{w} - 1\right)\cdot\left(\sum_{n\geq 1}\frac{t^n}{B(n)}\right)^{-1}.$$

This formula is valid for any complex number w.

Proof: We only need to prove this identity for w being a positive integer, since the coefficients of $\frac{t^n}{D(n)}$ on both sides of the identity are polynomials in w. Recall that two polynomials are equal if they agree on an infinite number of values. Now observe that $\overline{\zeta}\zeta^{k-1}$ counts the number of multichains of length k, $\hat{0} = x_0 \leq x_1 \leq \cdots \leq x_k = y$, in a Sheffer interval $[\hat{0}, y]$ of length n such that $x_0 < x_1$. Hence, the number of multichains of length k without such a restriction is counted by

$$\overline{\zeta} + \overline{\zeta}\zeta + \cdots + \overline{\zeta}\zeta^{k-1} = \overline{\zeta}\frac{\zeta^k - \delta}{\zeta - \delta}.$$

The corresponding generating function is

$$\left(\sum_{n\geq 1}\frac{t^n}{D(n)}\right)\cdot\left(\left(\sum_{n\geq 0}\frac{t^n}{B(n)}\right)^{k} - 1\right)\cdot\left(\sum_{n\geq 1}\frac{t^n}{B(n)}\right)^{-1}.$$

□

This lemma implies Lemma 3 by setting $w = -1$.

For instance, the r-cubical lattice from Example **(d)** satisfies

$$\sum_{n\geq 1} \overline{Z}_n(w)\frac{t^n}{D(n)} = t \cdot e^{\frac{t}{r}} \cdot \left(e^{wt}-1\right) \cdot \left(e^{t}-1\right)^{-1}.$$

By setting $t = rx$, this may be written

$$\sum_{n\geq 0} \overline{Z}_{n+1}(w)\frac{x^n}{n!} = e^{x} \cdot \frac{e^{wrx}-1}{e^{rx}-1}.$$

5 Linear edge-labelings

In this section, we relax the usual R-labeling condition used to compute the Möbius function of a rank-selected poset to that of a linear edge-labeling. We define two functions η and ν which generalize the notion of the zeta function and the Möbius function.

A *linear edge-labeling* λ of a locally finite poset P is a map which assigns to each edge in the Hasse diagram of P an element from some linearly ordered poset Λ. If x and y is an edge in the poset, that is, y covers x in P, then we denote the label on this edge by $\lambda(x,y)$. A maximal chain $x = y_0 \prec y_1 \prec \cdots \prec y_k = z$ in an interval $[x,z]$ in P is called *rising* if the labels are weakly increasing with respect to the order of the poset Λ, that is, $\lambda(y_0,y_1) \leq_\Lambda \lambda(y_1,y_2) \leq_\Lambda \cdots \leq_\Lambda \lambda(y_{k-1},y_k)$. We will denote the number of rising maximal chains in the interval $[x,z]$ by $\eta(x,z)$. (Note that $\eta(x,x) = 1$.) A linear edge-labeling is called an *R-labeling* if $\eta(x,z) = 1$ for all intervals $[x,z]$. That is, $\eta = \zeta$, where ζ is the usual zeta-function in the incidence algebra of P.

Let P be a poset of rank n with a linear edge-labeling λ. We introduce the notation $[n] = \{1,2,\ldots,n\}$. For a maximal chain $c = \{\hat{0} = x_0 \prec x_1 \prec \ldots \prec x_n = \hat{1}\}$ in P, the *descent set* of the chain c is

$$D(c) = \{i \ : \ \lambda(x_{i-1},x_i) > \lambda(x_i,x_{i+1})\}.$$

Observe that $D(c)$ is a subset of the set $[n-1]$.

For a subset S of $[n-1]$, we define the *S-rank-selected subposet* P_S by

$$P_S = \{z \in P \ : \ z = \hat{0},\ z = \hat{1},\ \text{or } \rho(z) \in S\}.$$

We denote the Möbius function of P_S by $\mu_S(\hat{0},\hat{1})$. The *S-rank-selected Möbius invariant* $\beta(P,S) = \beta(S)$ is defined by

$$\beta(S) = \sum_{T\subseteq S} (-1)^{|S-T|}\alpha(T),$$

where $\alpha(T)$ is the number of maximal chains in the rank-selected subposet P_T. Recall that $\beta(S)$ is equal to Möbius function of P_S up to a sign, namely $\beta(S) = (-1)^{|S|-1}\mu_S(\hat{0},\hat{1})$. See [14]. Finally, we define $\beta^*(S)$ to be the number of maximal chains in the poset P having descent set S with respect to the linear edge-labeling λ.

The next proposition will explain the relation between $\beta^*(S)$ and η.

Proposition 2 *Let P be a graded poset of rank n, with a linear edge-labeling λ. Assume that the number of rising chains in the interval $[x, y]$ is $\eta(x, y)$. Let S be a subset of ranks, that is, $S \subseteq [n-1]$. Let η_S be the restriction of η to $I(P_S)$, the incidence algebra of P_S. Then the number of chains with descent set S, $\beta^*(S)$, is given by $(-1)^{|S|-1} \cdot \eta_S^{-1}(\hat{0}, \hat{1})$, where η_S^{-1} is the inverse of η_S in $I(P_S)$.*

When the linear edge-labeling is an R-labeling, Proposition 2 reduces to a well-known result of Björner and Stanley [3, Theorem 2.7]: if P is a graded poset of rank n, $S \subseteq [n-1]$, and P admits an R-labeling, then $\beta(S)$ equals the number of maximal chains in P having descent set S with respect to the given R-labeling λ. Notice that in this case $\beta^*(S) = \beta(S)$.

The following lemma is generalization of Philip Hall's formula for the Möbius function. It will be used in the proof of Proposition 2.

Lemma 5 *Let P be a locally finite poset. Let f be a function in in the incidence algebra of the poset P such that $f(x, x) = 1$ for all $x \in P$. Then f is invertible. Its inverse is given by $f^{-1}(x, x) = 1$ and for $x < z$:*

$$f^{-1}(x, z) = \sum_{x=y_0<y_1<\cdots<y_n=z} (-1)^n \cdot f(y_0, y_1) \cdot f(y_1, y_2) \cdots f(y_{n-1}, y_n).$$

We omit the proof of this lemma, since it is straightforward. Lemma 5 is also stated in a slightly more general form in [2, Proposition 4.11].

Proof of Proposition 2: Recall that $\beta^*(S)$ is the number of maximal chains in the poset P that have descent set S. Similarly, define $\alpha^*(S)$ to be the number of maximal chains in the poset P whose descent set is contained in the set S. Directly we have that

$$\alpha^*(S) = \sum_{T \subseteq S} \beta^*(T) \qquad \text{and} \qquad \beta^*(S) = \sum_{T \subseteq S} (-1)^{|S-T|} \cdot \alpha^*(T).$$

Define

$$\gamma(S) = \sum_{\hat{0}=z_0 \prec z_1 \prec \cdots \prec z_k = \hat{1}} \eta_S(z_0, z_1) \cdot \eta_S(z_1, z_2) \cdots \eta_S(z_{k-1}, z_k),$$

where the sum is over all maximal chains in P_S. This may be written as

$$\gamma(S) = \sum_{\hat{0}=y_0 < y_1 < \cdots < y_k = \hat{1}} \eta(y_0, y_1) \cdot \eta(y_1, y_2) \cdots \eta(y_{k-1}, y_k),$$

where the sum is over all chains in P whose ranks are exactly S. By Lemma 5, we may express $\eta_S^{-1}(\hat{0}, \hat{1})$ as

$$\begin{aligned} \eta_S^{-1}(\hat{0}, \hat{1}) &= \sum_{T \subseteq S} (-1)^{|T|+1} \cdot \gamma(T) \\ &= (-1)^{|S|-1} \cdot \sum_{T \subseteq S} (-1)^{|S|-|T|} \cdot \gamma(T). \end{aligned}$$

Thus to prove the proposition it is enough to prove that $\alpha^*(S) = \gamma(S)$, for all subsets S of $[n-1]$.

To do this, consider a maximal chain $\hat{0} = z_0 \prec z_1 \prec \cdots \prec z_k = \hat{1}$ in P_S. We would like to extend this chain to a maximal chain in P, say $\hat{0} = x_0 \prec x_1 \prec \cdots \prec x_n = \hat{1}$, such that the subchain in each interval $[z_{i-1}, z_i]$ is rising. Clearly this may be done in $\eta(z_0, z_1) \cdot \eta(z_1, z_2) \cdots \eta(z_{k-1}, z_k)$ possible ways. Observe that the chains constructed have their descent set in S. Also, each maximal chain with descent set contained in S may be constructed in this way. Thus the equality $\alpha^*(S) = \gamma(S)$ holds. □

The r-cubical lattice from Example **(d)** has a very nice linear edge-labeling described as follows: for the cover relation $A \prec B$, label the corresponding edge in the Hasse diagram by (i, a), where i is the unique index such that $A_i \neq B_i$, and let a be the singleton element in $A_i - B_i$. Also, for the relation $\hat{0} \prec B$ let the label be the special element G. Hence, the set of labels Λ are $([r] \times I) \cup \{G\}$.

So far we have not given a linear order on the set of labels Λ. We now do this. Let p be an integer such that $0 \leq p \leq r$. Choose any linear order of Λ which satisfies the following condition

$$(i,j) <_\Lambda G \Longrightarrow i \leq r-p, \qquad \text{and} \qquad (i,j) >_\Lambda G \Longrightarrow i > r-p. \tag{1}$$

That is, the labels above the element G in the ordering of Λ are those whose first coordinate is from the set $\{r-p+1, r-p+2, \ldots, r\}$. There are p possible first coordinates there. The labels below the element G have their first coordinate from the set $\{1, 2, \ldots, r-p\}$.

Lemma 6 *Let Λ be a linear order on the set $([r] \times I) \cup \{G\}$ satisfying condition (1). Then the above described linear edge-labeling for the r-cubical lattice has the following η function*

$$\eta(x,y) = \begin{cases} 1 & \textit{if } \hat{0} = x = y, \\ p^{\rho(y)-1} & \textit{if } \hat{0} = x < y, \\ 1 & \textit{if } \hat{0} < x \leq y. \end{cases}$$

Proof: We first show that there is a unique rising chain for a binomial interval $[A, B]$. This interval is a boolean poset. Observe that every chain in the interval $[A, B]$ consists of the same set of labels. Since the interval is boolean, every permutation of the labels corresponds to a maximal chain in $[A, B]$. Hence there exists a unique chain where the labels are increasing.

For a Sheffer interval $[\hat{0}, B]$, all the chains begin with the label G. Recall that the labels in Λ greater than G are of the form (i, a), where $i \in \{r-p+1, r-p+2, \ldots, r\}$. Hence all the other labels of a rising chain must have their first entry in this set. Thus we conclude that an atom $A = (A_1, \ldots, A_r)$ in a rising chain in $[\hat{0}, B]$ satisfies $A_j = B_j$ for $j \leq r-p$. Let H be the set of elements missing from $B = (B_1, \ldots, B_r)$, that is, $H = I - \bigcup_{i=1}^{r} B_i$. For each element in H there p possibilities in which set $A_{r-p+1}, \ldots, A_r$ it may belong. There are thus $p^{|H|}$ possible atoms for a rising chain in $[\hat{0}, B]$.

From such an atom A to the element B there exists only one unique rising chain, since $[A, B]$ is a binomial interval. Thus there are $p^{|H|} = p^{\rho(B)-1}$ rising chains in the interval $[\hat{0}, B]$. □

6 The ν-function of rank selections

Let P be a Sheffer poset. Proposition 2 and Lemma 6 suggests that we study the following function, η, in the incidence algebra of the Sheffer poset P:

$$\eta(x,y) = \begin{cases} 1 & \text{if } \hat{0} = x = y, \\ \overline{\eta}(n) & \text{if } \hat{0} = x < y \text{ and } \rho(y) = n, \\ 1 & \text{if } \hat{0} < x \leq y, \end{cases}$$

where $\overline{\eta}$ is a function from the positive integers $\mathbb{P}$ to the complex numbers $\mathbb{C}$, since the incidence algebra we consider is over the complex numbers.

Let S be a subset of the positive integers $\mathbb{P}$. For an interval $[x,y]$ in the Sheffer poset P define the rank-selected interval $[x,y]_S$ by

$$[x,y]_S = \{z \in [x,y] \ : \ z = x, \ z = y, \text{ or } \rho(z) \in S\}.$$

Let ν_S be the inverse of η with respect to this rank selection, that is,

$$\delta(x,y) = \sum_{z \in [x,y]_S} \nu_S(x,z) \cdot \eta(z,y).$$

For a binomial interval $[x,y]$, that is $x > \hat{0}$, we have that $\nu_S(x,y)$ is the Möbius function of the rank-selected interval $[x,y]_S$, also denoted by $\mu_S(x,y)$. Thus our interest will focus on the values of $\nu_S(\hat{0},y)$. By considering Lemma 5, we may conclude that $\nu_S(\hat{0},y)$ only depends on the rank of y. Thus we can view $\overline{\nu}_S$ as an element of $M(P)$ and we are allowed to write $\overline{\nu}_S(n)$. Also define $\beta^*(n,S) = (-1)^{|S|-1}\overline{\nu}_S(n)$.

When we set $\overline{\eta}(n) \equiv 1$ for all $n \in \mathbb{P}$ then η is equal to ζ, the classical zeta function. In this case we obtain that for all intervals $[x,y]$, $\nu_S(x,y)$ is equal to $\mu_S(x,y)$, the Möbius function of $[x,y]_S$, and that $\beta^*(n,S)$ is equal to the S-rank-selected Möbius invariant for a Sheffer n-interval $[\hat{0},y]$.

Lemma 7 *For a Sheffer poset P, with factorial functions $B(n)$ and $D(n)$ and $S \subseteq \mathbb{P}$, we have*

$$-\sum_{n \geq 1} (\overline{\eta}(n) + \overline{\nu}_S(n)) \frac{t^n}{D(n)} = \left(\sum_{n \in S} \overline{\nu}_S(n) \frac{t^n}{D(n)} \right) \cdot \left(\sum_{n \geq 1} \frac{t^n}{B(n)} \right).$$

Proof: Let χ to be the characteristic function of the set $S \cup \{0\}$. That is, $\chi(n) = 1$ if $n \in S \cup \{0\}$, and $\chi(n) = 0$ if $n \notin S \cup \{0\}$. We may view $g(\hat{0},y) = \chi(\rho(y)) \cdot \overline{\nu}_S(\hat{0},y)$ as an element of $M(P)$.

Let $y \in P$ be an element of rank $n \geq 1$. Then we have that

$$\begin{aligned} 0 &= \sum_{z \in [\hat{0},y]_S} \nu_S(\hat{0},z) \cdot \eta(z,y) \\ &= \sum_{\hat{0} < z < y, \ \rho(z) \in S} \nu_S(\hat{0},z) \cdot \eta(z,y) + \eta(\hat{0},y) + \nu_S(\hat{0},y). \end{aligned}$$

In other words,

$$\begin{aligned}
-(\overline{\eta}+\overline{\nu}_S)(n) &= \sum_{\hat{0}<z<y} \chi(\rho(z))\overline{\nu}_S(\hat{0},z) \\
&= \sum_{\hat{0}<z<y} g(\hat{0},z) \\
&= \sum_{\hat{0}<z<y} g(\hat{0},z)\cdot(\zeta-\delta)(z,y) \\
&= \sum_{\hat{0}<z\leq y} g(\hat{0},z)\cdot(\zeta-\delta)(z,y) \\
&= (g(\zeta-\delta))(\hat{0},y).
\end{aligned}$$

Hence $-(\overline{\eta}+\overline{\nu}_S) = g(\zeta-\delta)$. Apply the isomorphism ψ:

$$\begin{aligned}
-\sum_{n\geq 1}(\overline{\eta}(n)+\overline{\nu}_S(n))\frac{t^n}{D(n)} &= \left(\sum_{n\geq 1} g(n)\frac{t^n}{D(n)}\right)\cdot\left(\sum_{n\geq 1}\frac{t^n}{B(n)}\right) \\
&= \left(\sum_{n\geq 1} \chi(n)\overline{\nu}_S(n)\frac{t^n}{D(n)}\right)\cdot\left(\sum_{n\geq 1}\frac{t^n}{B(n)}\right) \\
&= \left(\sum_{n\in S} \overline{\nu}_S(n)\frac{t^n}{D(n)}\right)\cdot\left(\sum_{n\geq 1}\frac{t^n}{B(n)}\right).
\end{aligned}$$

□

When $S=\mathbb{P}$ then we write $\overline{\nu}(n)=\overline{\nu}_{\mathbb{P}}(n)$. Lemma 7 then implies that

$$-\sum_{n\geq 1}\overline{\eta}(n)\frac{t^n}{D(n)} = \left(\sum_{n\geq 1}\overline{\nu}(n)\frac{t^n}{D(n)}\right)\cdot\left(\sum_{n\geq 0}\frac{t^n}{B(n)}\right). \tag{2}$$

By setting $\overline{\eta}(n)\equiv 1$, we have $\overline{\mu}(n)=\overline{\nu}(n)$ for all $n\in\mathbb{P}$. In this case, equation (2) implies Lemma 3

Lemma 8 *Let $S=k\cdot\mathbb{P}=\{k,2k,3k,\ldots\}$. Then*

$$-\sum_{n\in S}\overline{\nu}_S(n)\frac{t^n}{D(n)} = \left(\sum_{n\geq 1}\overline{\eta}(k\cdot n)\frac{t^{k\cdot n}}{D(k\cdot n)}\right)\cdot\left(\sum_{n\geq 0}\frac{t^{k\cdot n}}{B(k\cdot n)}\right)^{-1}.$$

Proof: As in Example **(e)**, let $P^{(k)}$ denote the subposet of the poset P, where we select the ranks which are divisible by k. Recall that $B_k(n)$ and $D_k(n)$ are the factorial functions of $P^{(k)}$, and they satisfy $B(n\cdot k)=B_k(n)\cdot B(k)^n$ and $D(n\cdot k)=D_k(n)\cdot D(k)\cdot B(k)^{n-1}$.

Let $\eta^{(k)}$ be the restriction of η to $P^{(k)}$, and let $\nu^{(k)}$ be the inverse of $\eta^{(k)}$ in the incidence algebra of $P^{(k)}$. Then we have $\overline{\eta}^{(k)}(n)=\overline{\eta}(k\cdot n)$, and $\overline{\nu}^{(k)}(n)=\overline{\nu}_S(k\cdot n)$.

Now apply equation (2) to the Sheffer poset $P^{(k)}$.

$$-\sum_{n\geq 1} \overline{\eta}^{(k)}(n)\frac{t^n}{D_k(n)} = \left(\sum_{n\geq 1} \overline{\nu}^{(k)}(n)\frac{t^n}{D_k(n)}\right)\cdot\left(\sum_{n\geq 0}\frac{t^n}{B_k(n)}\right).$$

Substitute $t \longrightarrow \frac{t^k}{B(k)}$. We obtain

$$-\sum_{n\geq 1} \overline{\eta}(k\cdot n)\frac{t^{k\cdot n}}{D_k(n)\cdot B(k)^n} = \left(\sum_{n\geq 1} \overline{\nu}_S(k\cdot n)\frac{t^{k\cdot n}}{D_k(n)\cdot B(k)^n}\right)\cdot\left(\sum_{n\geq 0}\frac{t^{k\cdot n}}{B_k(n)\cdot B(k)^n}\right).$$

Multiply by $\frac{B(k)}{D(k)}$ and apply $B(n\cdot k) = B_k(n)\cdot B(k)^n$ and $D(n\cdot k) = D_k(n)\cdot D(k)\cdot B(k)^{n-1}$.

$$\begin{aligned} -\sum_{n\geq 1} \overline{\eta}(k\cdot n)\frac{t^{k\cdot n}}{D(k\cdot n)} &= \left(\sum_{n\geq 1} \overline{\nu}_S(k\cdot n)\frac{t^{k\cdot n}}{D(k\cdot n)}\right)\cdot\left(\sum_{n\geq 0}\frac{t^{k\cdot n}}{B(k\cdot n)}\right) \\ &= \left(\sum_{n\in S} \overline{\nu}_S(n)\frac{t^{n}}{D(n)}\right)\cdot\left(\sum_{n\geq 0}\frac{t^{k\cdot n}}{B(k\cdot n)}\right) \end{aligned}$$

This completes the proof of the lemma. □

By combining Lemmas 7 and 8 we conclude:

Lemma 9 *Let $S = k\cdot\mathbb{P} = \{k, 2k, 3k, \ldots\}$. Then*

$$\sum_{n\geq 1}(\overline{\eta}(n)+\overline{\nu}_S(n))\frac{t^n}{D(n)} = \left(\sum_{n\geq 1}\frac{\overline{\eta}(k\cdot n)\cdot t^{k\cdot n}}{D(k\cdot n)}\right)\cdot\left(\sum_{n\geq 0}\frac{t^{k\cdot n}}{B(k\cdot n)}\right)^{-1}\cdot\left(\sum_{n\geq 1}\frac{t^n}{B(n)}\right). \quad (3)$$

□

As an example of Lemma 9, consider the r-cubical lattice of Example **(d)** when $k = 2$, that is, $S = 2\cdot\mathbb{P} = \{2, 4, 6, \ldots\}$. Recall that the factorial functions are $B(n) = n!$ and $D(n) = r^{n-1}\cdot(n-1)!$. Let $\overline{\eta}(n) = p^{n-1}$, as suggested in Lemma 6. We then obtain

$$\sum_{n\geq 1}(p^{n-1}+\overline{\nu}_S(n))\frac{t^n}{D(n)} = t\cdot\sinh\left(\frac{pt}{r}\right)\cdot(\cosh(t))^{-1}\cdot\left(e^t - 1\right).$$

This may also be written as

$$\sum_{n\geq 0}\overline{\nu}_S(n+1)\frac{x^n}{n!} = \sinh(px)\cdot\operatorname{sech}(rx)\cdot(e^{rx}-1) - e^{px}.$$

Let $1\leq j\leq k$, and consider those exponents which are $\equiv j \bmod k$ in equation (3).

$$\sum_{\substack{m\geq 0\\ n=mk+j}}(\overline{\eta}(n)+\overline{\nu}_S(n))\frac{t^n}{D(n)} = \left(\sum_{n\geq 1}\frac{\overline{\eta}(k\cdot n)\cdot t^{k\cdot n}}{D(k\cdot n)}\right)\cdot\left(\sum_{n\geq 0}\frac{t^{k\cdot n}}{B(k\cdot n)}\right)^{-1}\cdot\left(\sum_{\substack{m\geq 0\\ n=mk+j}}\frac{t^n}{B(n)}\right).$$

Divide this equation by t^j, substitute $t^k \to -t^k$, and then multiply by t^j. This gives

$$\sum_{\substack{m\geq 0\\ n=mk+j}} (-1)^m \cdot (\overline{\eta}(n) + \overline{\nu}_S(n)) \frac{t^n}{D(n)} =$$

$$= \left(\sum_{n\geq 1} \frac{(-1)^n \cdot \overline{\eta}(k\cdot n)\cdot t^{k\cdot n}}{D(k\cdot n)}\right) \cdot \left(\sum_{n\geq 0} \frac{(-1)^n \cdot t^{k\cdot n}}{B(k\cdot n)}\right)^{-1} \cdot \left(\sum_{\substack{m\geq 0\\ n=mk+j}} \frac{(-1)^m \cdot t^n}{B(n)}\right).$$

We are interested in the values of $\beta^*(n,S)$. Since $|S \cap [n-1]| = \left\lfloor \frac{n-1}{k} \right\rfloor = m$, clearly $\beta^*(n,S) = (-1)^{m-1} \cdot \overline{\nu}_S(n)$ holds. Thus we have shown:

Proposition 3 *For $S = k \cdot \mathbb{P} = \{k, 2k, 3k, \ldots\}$, we have*

$$\sum_{\substack{m\geq 0\\ n=mk+j}} \beta^*(n,S) \frac{t^n}{D(n)} =$$

$$= -\left(\sum_{n\geq 1} \frac{(-1)^n \cdot \overline{\eta}(k\cdot n)\cdot t^{k\cdot n}}{D(k\cdot n)}\right) \cdot \left(\sum_{n\geq 0} \frac{(-1)^n \cdot t^{k\cdot n}}{B(k\cdot n)}\right)^{-1} \cdot \left(\sum_{\substack{m\geq 0\\ n=mk+j}} \frac{(-1)^m \cdot t^n}{B(n)}\right) +$$

$$+ \sum_{\substack{m\geq 0\\ n=mk+j}} \frac{(-1)^m \cdot \overline{\eta}(n)\cdot t^n}{D(n)}.$$

Observe that Lemmas 7, 8, and 9 and Proposition 3 reduce to results of Stanley [13, 14] when we apply them to a binomial poset and set $\overline{\nu}(n) \equiv 1$.

7 Augmented r-signed permutations

Definition 3 *An augmented r-signed permutation is a list of the form*

$$(G, (i_1, j_1), (i_2, j_2), \ldots, (i_n, j_n)),$$

where $i_1, i_2, \ldots, i_n \in [r]$ and $(j_1, j_2, \ldots, j_n)$ form a permutation on n elements.

We view the elements $i_1, \ldots, i_n$ as signs; hence the name r-signed permutation. Since we list the special element G first, we say that the permutation is augmented.

The descent set of an augmented r-signed permutation $\pi = (G = s_0, s_1, \ldots, s_n)$ is the set

$$D(\pi) = \{i \ : \ s_{i-1} >_\Lambda s_i\},$$

where Λ is a linear order on the set $([r] \times [n]) \cup \{G\}$. We say that the augmented r-signed permutation π is Λ-*alternating* if $D(\pi) = \{2, 4, 6, \ldots\}$.

Recall the r-cubical lattice of Example **(d)**. Let B be an element of rank $n+1$ in this lattice. Without loss of generality, we may assume that $I - \bigcup_{i=1}^{r} B_i = [n]$. Observe that the set of maximal chains in the Sheffer interval $[\hat{0}, B]$ corresponds to augmented r-signed permutations. The number of augmented r-signed permutations having a certain descent set is equal to the number of maximal chains with this same descent set.

We will now apply this connection with maximal chains to count alternating augmented r-signed permutations which are Λ-alternating. We assume that the linear order Λ satisfies condition (1). Thus by Lemma 6, we will assume $\overline{\eta}(n) = p^{n-1}$. Apply now Proposition 3 to the r-cubical lattice with $k = 2$ and $j = 2$. Then Proposition 3 yields the following generating functions:

$$\sum_{n \geq 1} \frac{(-1)^n \cdot \overline{\eta}(2n) \cdot t^{2n}}{D(2n)} = -t \cdot \sin\left(\frac{pt}{r}\right),$$

$$\sum_{n \geq 0} \frac{(-1)^n \cdot t^{2n}}{B(2n)} = \cos(t),$$

$$\sum_{\substack{m \geq 0 \\ n = 2m+2}} \frac{(-1)^m \cdot t^n}{B(n)} = 1 - \cos(t),$$

$$\sum_{\substack{m \geq 0 \\ n = 2m+2}} \frac{(-1)^m \cdot \overline{\eta}(n) \cdot t^n}{D(n)} = t \cdot \sin\left(\frac{pt}{r}\right).$$

Thus in this case, Proposition 3 specializes to the identity

$$\sum_{\substack{m \geq 0 \\ n = 2m+2}} \beta^*(n, S) \frac{t^n}{D(n)} = t \cdot \sin\left(\frac{pt}{r}\right) \cdot \sec(t) \cdot (1 - \cos(t)) + t \cdot \sin\left(\frac{pt}{r}\right)$$

$$= t \cdot \sin\left(\frac{pt}{r}\right) \cdot \sec(t).$$

When $j = 1$ we obtain

$$\sum_{\substack{m \geq 0 \\ n = 2m+1}} \frac{(-1)^m \cdot t^n}{B(n)} = \sin(t),$$

$$\sum_{\substack{m \geq 0 \\ n = 2m+1}} \frac{(-1)^m \cdot \overline{\eta}(n) \cdot t^n}{D(n)} = t \cdot \cos\left(\frac{pt}{r}\right).$$

Thus we also have the identity

$$\sum_{\substack{m \geq 0 \\ n = 2m+1}} \beta^*(n, S) \frac{t^n}{D(n)} = t \cdot \sin\left(\frac{pt}{r}\right) \cdot \sec(t) \cdot \sin(t) + t \cdot \cos\left(\frac{pt}{r}\right)$$

$$= t \cdot \sec(t) \cdot \left(\sin\left(\frac{pt}{r}\right) \cdot \sin(t) + \cos\left(\frac{pt}{r}\right) \cdot \cos(t)\right)$$

$$= t \cdot \sec(t) \cdot \cos\left(t - \frac{pt}{r}\right).$$

Adding these two equations yields

$$\sum_{n\geq 1} \beta^*(n,S)\frac{t^n}{D(n)} = t \cdot \sec(t) \cdot \left(\sin\left(\frac{pt}{r}\right) + \cos\left(\frac{(r-p)t}{r}\right)\right).$$

This can be made into an exponential generating function by dividing by t, and letting $t = rx$. Thus we have found the exponential generating function for $\beta^*(n+1,S)$, where $S = 2\cdot\mathbb{P} = \{2,4,6,\ldots\}$. Equivalently:

Proposition 4 *Let Λ be a linear order on the set $([r]\times[n])\cup\{G\}$ satisfying condition (1). Then the exponential generating function of the number of Λ-alternating augmented r-signed permutations is*

$$\frac{\sin(px) + \cos((r-p)x)}{\cos(rx)}. \tag{4}$$

When $r = p = 1$, the right hand side of equation (4) becomes the classical expression $\tan(x) + \sec(x)$. This is the exponential generating function of the Euler numbers, which counts the number of alternating permutations in the symmetric group. When $r = 2$ and $p = 1$, the right hand side becomes $\sec(2x)\cdot(\sin(x)+\cos(x))$. This generating function has been obtained by Purtill [11], when enumerating alternating augmented signed permutations. Also, two very similar generating functions have been obtained by Steingrímsson [16] which are equivalent to the two cases $p = 1$ and $p = r-1$.

Note that Proposition 4 may be proved directly with a combinatorial argument. That is, by conditioning on where the largest element is in the permutation, one obtains recursion formulas. From these recursion formulas one may easily set up linear differential equations for the generating function. However, this straightforward argument does not explain why the generating functions that occur have the form that they indeed have.

By the Hardy-Littlewood-Karamata Tauberian theorem [4] we can easily compute the asymptotics for $\beta(n+1,S)$. Recall that a function $L(t)$ *varies slowly at infinity* if for every $s > 0$, $L(st) \sim L(t)$ as $t \to \infty$. The theorem says that if $A(z) = \sum_{n\geq 0} a_n z^n$ has radius of convergence R, a_n are nonnegative and monotonic increasing, and if for some $\rho > 0$ and some L which varies slowly at infinity

$$A(x) \sim (R-x)^{-\rho} L\left(\frac{1}{R-x}\right) \qquad \text{as} \qquad x \longrightarrow R^-,$$

then

$$a_n \sim \frac{n^{\rho-1}L(n)}{\Gamma(\rho)} R^{-n-\rho}.$$

Lemma 10 *For $S = 2\cdot\mathbb{P} = \{2,4,6,\ldots\}$, we have when $n \longrightarrow \infty$*

$$\beta^*(n+1,S) \sim \frac{4}{\pi}\cdot\sin\left(\frac{p\cdot\pi}{2r}\right)\cdot\left(\frac{2r}{\pi}\right)^n \cdot n!.$$

Proof: Apply the Hardy-Littlewood-Karamata Tauberian theorem to the generating function in equation (4), with $R = \frac{\pi}{2r}$, $\rho = 1$, and $L = 2\cdot r^{-1}\cdot\sin\left(\frac{p\pi}{2r}\right)$. □

8 A linear generalization of the 4-cubical lattice

Recall that the isotropic subspace lattice in Example (j) may be viewed as a linear version of the cubical lattice. In this section we will obtain a lattice that is a linear generalization of the 4-cubical lattice. This construction is closely related to the quaternions as the isotropic subspace lattice is related to the complex numbers. It is interesting to note that the construction presented here only works over an infinite field.

Consider the matrices A_1, A_2, A_3, and A_4 defined below.

$$A_1 = I_4, \quad A_2 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad A_4 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

These matrices fulfill the quaternion relations, that is A_1 is the identity, and we have that $A_2^2 = A_2^2 = A_3^2 = -A_1$, $A_2A_3 = A_4$, $A_3A_4 = A_2$, $A_4A_2 = A_3$, $A_3A_2 = -A_4$, $A_4A_3 = -A_2$, and $A_2A_4 = -A_3$. Thus we can represent the the quaternions $\mathbb{H}$ as 4 by 4 matrices, by the following map:

$$\Phi(a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}) = aA_1 + bA_2 + cA_3 + dA_4.$$

That is,

$$\Phi(a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}) = \begin{pmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{pmatrix}.$$

It is easy to check that this map is indeed an algebra map. Moreover $\det(\Phi(a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k})) = (a^2 + b^2 + c^2 + d^2)^2$.

Lemma 11 *For a nonzero vector $\mathbf{x} \in \mathbb{R}^4$, the set $\{A_1\mathbf{x}, A_2\mathbf{x}, A_3\mathbf{x}, A_4\mathbf{x}\}$ forms an orthogonal basis for $\mathbb{R}^4$.*

Proof: Observe that A_2, A_3, and A_4 are skew symmetric. For $i \neq j$, we have that $A_i^* \cdot A_j = \pm A_i \cdot A_j = \pm A_k$ for some $k = 2, 3, 4$. Now $(A_i\mathbf{x})^* \cdot A_j\mathbf{x} = \pm\mathbf{x}^* A_k \mathbf{x} = 0$, since A_k is skew-symmetric. □

Observe that this lemma fails in a vector space over a field of a finite characteristic.

Let B_i be the $4n$ by $4n$ matrix given by $B_i = A_i \otimes I_n$. Observe that the we have the quaternion relations for B_1, B_2, B_3, and B_4. That is, $B_2 \cdot B_3 = B_4$, and so on. Observe that from the previous lemma we obtain

Lemma 12 *For a nonzero vector $\mathbf{x} \in \mathbb{R}^{4n}$, the set $\{B_1\mathbf{x}, B_2\mathbf{x}, B_3\mathbf{x}, B_4\mathbf{x}\}$ is orthogonal.*

We can now define three skew-symmetric bilinear forms on $\mathbb{R}^{4n}$ by

$$\langle \mathbf{x} \mid \mathbf{y} \rangle_i = \mathbf{x}^* B_i \mathbf{y} \qquad \text{for } i = 2, 3, 4.$$

For V a subspace of $\mathbb{R}^{4n}$, define V' as

$$V' = \left\{\mathbf{x} \in \mathbb{R}^{4n} \ : \ \forall \mathbf{y} \in V, \ \langle \mathbf{x} \,|\, \mathbf{y}\rangle_2 = \langle \mathbf{x} \,|\, \mathbf{y}\rangle_3 = \langle \mathbf{x} \,|\, \mathbf{y}\rangle_4 = 0\right\}.$$

We call a subspace V *isotropic* if $V \subseteq V'$. That V is isotropic is equivalent to for all $\mathbf{x}, \mathbf{y} \in V$ we have $\langle \mathbf{x} \,|\, \mathbf{y}\rangle_2 = \langle \mathbf{x} \,|\, \mathbf{y}\rangle_3 = \langle \mathbf{x} \,|\, \mathbf{y}\rangle_4 = 0$.

Lemma 13 *If V is an isotropic subspace of $\mathbb{R}^{4n}$ then the four subspaces $B_1V = V$, B_2V, B_3V, and B_4V are pairwise orthogonal.*

Proof: To show that B_iV and B_jV are orthogonal to each other, consider $(B_i\mathbf{x})^*(B_j\mathbf{y})$ where $\mathbf{x}, \mathbf{y} \in V$. Let k be the index such that $B_iB_j = \pm B_k$. Note that $k \neq 1$. Thus we have $(B_i\mathbf{x})^*(B_j\mathbf{y}) = \mathbf{x}^* B_i^* B_j \mathbf{y} = \pm \mathbf{x}^* B_i B_j \mathbf{y} = \pm \mathbf{x}^* B_k \mathbf{y} = \langle \mathbf{x} \,|\, \mathbf{y}\rangle_k = 0$, since V is isotropic. □

Proposition 5 *If V is an isotropic subspace of $\mathbb{R}^{4n}$ then* $\dim(V') = 4n - 3 \cdot \dim(V)$.

Proof: Since B_2V, B_3V, and B_4V are pairwise orthogonal, we obtain

$$\dim\left(\bigoplus_{i=2}^{4} B_iV\right) = \sum_{i=2}^{4} \dim\left(B_iV\right) = 3 \cdot \dim\left(V\right).$$

We also have

$$\begin{aligned}
V' &= \bigcap_{i=2}^{4} \left\{\mathbf{x} \in \mathbb{R}^{4n} \ : \ \forall \mathbf{y} \in V \ \ \langle \mathbf{x} \,|\, \mathbf{y}\rangle_i = 0\right\} \\
&= \bigcap_{i=2}^{4} \left\{\mathbf{x} \in \mathbb{R}^{4n} \ : \ \forall \mathbf{y} \in V \ \ \mathbf{x}^* B_i \mathbf{y} = 0\right\} \\
&= \bigcap_{i=2}^{4} \left\{\mathbf{x} \in \mathbb{R}^{4n} \ : \ \forall \mathbf{y} \in B_iV \ \ \mathbf{x}^* \mathbf{y} = 0\right\} \\
&= \bigcap_{i=2}^{4} (B_iV)^{\perp} \\
&= \left(\bigoplus_{i=2}^{4} B_iV\right)^{\perp}.
\end{aligned}$$

Hence we conclude that

$$\begin{aligned}
\dim\left(V'\right) &= \dim\left(\left(\bigoplus_{i=2}^{4} B_iV\right)^{\perp}\right) \\
&= 4n - \dim\left(\bigoplus_{i=2}^{4} B_iV\right) \\
&= 4n - 3 \cdot \dim\left(V\right).
\end{aligned}$$

□

Corollary 1 *An isotropic subspace of* $\mathbb{R}^{4n}$ *has dimension at most* n.

Proof: Since $V \subseteq V^{\perp}$ we have that $\dim(V) \leq \dim(V') = 4n - 3 \cdot \dim(V)$. This inequality implies $\dim(V) \leq n$.

□

Define $I_n^4(\mathbb{R})$ to be the poset of all isotropic subspaces of $\mathbb{R}^{4n}$, with the order relation $V \leq W$ if $V \supseteq W$, and adjoin a minimal element $\hat{0}$. This poset is graded and has rank $n+1$.

Proposition 6 *The set of maximal chains in* $I_n^4(\mathbb{R})$ *is parametrized by the set*

$$\left(1 + \mathbb{R} + \cdots + \mathbb{R}^{4n-1}\right) \cdot \left(1 + \mathbb{R} + \cdots + \mathbb{R}^{4n-5}\right) \cdots \left(1 + \mathbb{R} + \mathbb{R}^2 + \mathbb{R}^3\right).$$

The construction of the isotropic subspace lattice in Example (j), is done in an analogous manner, by starting with the two matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

These matrices describe how to embed the complex numbers as 2 by 2 real matrices. Observe that we do not need versions of Lemmas 11, 12, and 13, and hence the construction may be done over a finite field.

9 Concluding remarks

In this paper we have generalized the theory of binomial posets and their rank selections to Sheffer posets. We are presently considering a number of related questions. For example, the ν-function associated with the r-cubical lattice enabled us to enumerate augmented r-signed permutations. We would like to find other examples of posets whose associated ν-function leads to elegant enumerative results.

It would be interesting to find other useful examples of linear edge labelings, such as the classical R-labelings. For example, see Stanley [15] for an extension of R-labelings, called relative ER-labelings. In our language a *relative ER-labeling* is a linear edge labeling such that $\eta(x, y) \leq 1$ and if $\eta(x, y) = 1$ then $\eta(x', y') = 1$ for all $x \leq x' \leq y' \leq y$.

So far we know three linear analogues of the r-cubical lattice, in the cases $r = 1$, 2, and 4. They correspond to the real numbers, the complex numbers and the quaternions. It is natural to ask if there is a similar construction based on the Cayley numbers, i.e., when $r = 8$. Also, it would be nice to find a construction that works for all possible r.

Recently the authors have discovered a deeper theory about the r-cubical lattice and augmented r-signed permutations. This will appear in a forthcoming paper on the r-cubical lattice [7].

Finally, the generating function $\sec(rx) \cdot (\sin(px) + \cos((r - p)x))$ suggests the following question: Do other such classes of generating functions naturally occur? More generally, is there a theory of trigonometric generating functions in enumerative combinatorics?

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