6336
−6341
6999
4897 3
1030
5206
5374
5378
−5379
5185
5228
5243

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IN THE SOCIAL SCIENCES
Ventura Hall

July 15, 1977

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Dear "N.J.A.",

I have quite a few sequences to offer you, for your new edition. A lot of them are based on the following notion, which I call "Eta-sequences":

$$\text{eta}_k\,(r;s) = [(k+1)r + s] - [kr + s]$$

with "[x]" being the greatest-integer function, and r and s arbitrary real numbers.

The default option is for s to be 0, r positive and irrational, and k running from 1 to infinity. The prototype example is with r = sqrt(2). Then the sequence runs

```
seq:   1212112121121212112121121212112121121211212112...
deriv:   1  2 1  2 1 1  2 1  2 1 1  2 1  2 1  2 1 1  2...
```

new A6337

Its structure is that it equals its own "derivative", which is formed by counting 1's between 2's (thus the first 1 is skipped, since it is not preceded by a 2). If you put s=1/2 instead, you get

```
seq:        21211212121121211212121121211212112121212112...
deriv:        1  2 1 1  2 1  2 1 1  2 1  2 1  2 1 1  2
2nd deriv:          2     1      2     1     1      2
```

new A6338

which equals its own 2nd derivative.

Not all eta-sequences equal their own derivatives or nth derivatives. Such things only happen for r=quadratic. Eta-sequences are intimately related to continued fractions, particularly to "nearest-integer continued fractions", described in Perron's out-of-print classic, "Die Lehre von den Kettenbruechen" under the title "Kettenbrueche nach naechsten Ganzen".

You have already published one eta-type sequence (which belongs to r=sqrt(2)) -- number 21 (appropriately enough!), from my friend Alexander Nagel, who in his article thanks me for showing him the first examples of such sequences.

N21
= A1030

Obviously there are an uncountable number of such sequences, so one should pick carefully the interesting values of r and s. I suggest particularly the following choices, and their reasons:

(1) r=sqrt(2), s=0....equals own deriv ✓ A6337

(2) r=phi [Golden ratio], s=0....equals own deriv A6339

(3) r=sqrt(2), s=1/2....equals own 2nd deriv, also happens to be given by counting how many triangular #'s occur between successive squares ✓ A6338

(4) r=phi, s=1/2....equals own 2nd deriv A6340

Seems to have been detached from letter!

I am enclosing an unfinished write-up on Eta-sequences which I believe ought to be of interest not only to you, but also to R.L. Graham and perhaps others at Bell Labs. There are many fascinating features of Eta-sequences which I have not included in this write-up. I have been too busy writing a book called "Goedel, Escher, Bach: an Eternal Golden Braid" to finish "Eta-Lore", but I shall soon have finished "GEB, EGB", and will then be able to devote my time to other matters.

I will send along "Pi-Mu Sequences" also, which will come at the end of "Eta-Lore", not being strictly of the same family but being similar in spirit. The first pi-mu sequence is quite fascinating.

Now for some other kinds of sequences. How about the recursive functions

✓ G(0) = 0
G(n) = n - G(G(n-1)) A5206

✓ H(0) = 0
H(n) = n - H(H(H(n-1))) A5374

G(1) = 1
G(2) = 2 - 1 = 1
G(3) = 3 - 1 = 2
G(4) = 4 - 1 = 3
G(5) = 5 - 2 = 3

etc.. You can plot them as "trees" (i.e. put n above G(n)) and you get an interestingly recursive Fibonacci-tree for G:

```
14    15   16      17        18       19        20       21
 |     |    |       |         |        |         |        |
 |- 9 -|   10       |-- 11 --|         |-- 12 --|        13
      |     |             |                  |            |
      |-- 6 --|           7                  |----- 8 -----|
          |               |                         |
          |------ 4 ------|                         5
                  |                                 |
                  |------------- 3 -----------------|
                                 |
                                 2
                                 |
                                 1
```

✓

For H, you get an analogous third-order tree, and so on for higher orders. The sequence of their values is monotone non-decreasing. In the case of G, it equals the sequence [n phi]. But not for H or higher orders. The "fork-functions" for such trees are defined as the # of branches at n. It is always either 1 or 2, and in the case of G, it is eta(phi), but it is not any eta-sequence for H or higher orders. However, these fork-sequences "give themselves back" under suitable generalizations of the notion of "derivative" just as do eta-sequences.

Another thing to do is to flip the "trees" in a mirror, relabel from left to right, and then come up with a recursive formula for the "flip-tree". Left for the reader.

Then you can intertwine recursively, as in ✓5378

$$F(0) = 1, \quad M(0) = 0$$
$$F(n) = n - M(F(n-1))$$
$$G(n) = n - F(M(n-1))$$

✓5379

Make the trees!

Now the following sequence is a horse of an entirely nother color.

$$Q(1) = Q(2) = 1$$
$$Q(n) = Q(n-Q(n-1)) + O(n-Q(n-2))$$

✓5185

This sequence is absolutely CRAZY. I showed it to Paul Erdős, and he found it quite intriguing. It is like Fibonacci-numbers because you're always adding previous terms, but you are told how far back to count by the last two terms (instead of just adding them). There is NO rhyme nor rule to the behavior of this sequence (despite the fact that if you try the first several values at n = 3 times a power of 2, you will think you see a pattern. But it flops out there somewhere. In fact, it is not obvious by any means that Q(n) even EXISTS for all n -- yet it "clearly" does, from computing it out to 50,000 terms.

Now how about my "weird numbers" --

seq:	1	3	7	12	18	26	35	45	56	69 ....
1st diff's:	2	4	5	6	8	9	10	11	13...	

✓5228

The idea here is that the top sequence is the complement of its first-difference sequence. It behaves like triangular numbers asymptotically, but if you subtract off the nth triangular number, what is left? A funny sequence.

An idea inspired by a combination of the above "trees" and eta-sequences was to make the tree whose fork-function is eta(r). Choose r other than phi and you get a mess. Even if r is RATIONAL! For example, choose r=3/2. Then your tree looks crazy globally. Up its right hand side runs the following:

6999

1 2 4 7 11 17 26 40 61 92 139 ....

6341

This sequence is recursively defined by a(n) = [3a(n-1)/2]+1. I believe (but have never shown) that it is intimately related to the 3n+1 problem. In any case, the evenness-oddness pattern here is what is of interest: oeeoooeeoeo... Tell me the rule non-recursively, and in

Now here is a funny sequence:

4, 7, 9, 12, 13, 15, 20, 23, 26, 27, 28, 31, 36, 38, 39, ... 5243

I call them the "C-bar numbers" and here's why. They are the numbers which are NOT "C-numbers", and these latter are defined in a recursive way. You begin with 1 and 2 and then you make all the numbers you can as SUMS OF SUCCESSIVE PREVIOUS ELEMENTS. Thus, 3 enters after 1 and 2. This allows 5 (=2+3) and 6 (=1+2+3). Then you have 1,2,3,5,6. Now you can throw in 8 (=3+5), 10 (=2+3+5), and so on. It turns out that the farther out you go, the more numbers seem to be included among the C's. Therefore, the C-bar's are more "special" and it is intriguing to ask if they are even infinite in number, or if eventually every number is a c-number. Erdös liked this question quite a lot, too.

4897?

Another Erdös liked was this one:

2 3 5 9 14 17 26 27 33 41 44 50 51 53 65 5244

The rule is to take all products of any two previous elements, subtract 1, and throw them all in (e.g. 17 got thrown in because it equals 2x9-1). How dense is this asymptotically?

Now we have another Fibonacci-like sequence, which happens to connect back up to eta-sequences... Amazingly tight little world of recursion. The idea is that

$$v(1) = 1, \quad v(2) = 2$$

$$v(n) = v(n-1) + v(n-1-a(n))$$

Should be a(n-1) not a(n)

where a(n) is: the number of EVEN v's so far. Here's how we begin:

1 2 3 5 8 11 16 21 29 40 51 67 88 109 138 6336

Now take 1+v(n) modulo 2 and you get back eta(phi)! Quite a surprise. I hate to admit it, but I have never tried to prove this so I don't even know if this is trivial or deep. I have made up so many problems in my life that I haven't had the time to think about most of them, unfortunately.

There are some more but I hope these will entertain you for now.

Sincerely,

Doug Hofstadter

Doug Hofstadter

P.S. — From Aug. 10 on, at Computer Science Dept, Indiana Univ., Bloomington 47401.

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June 25, 1991

Professor D. R. Hofstadter
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Dear Doug:

I enclose a copy of your letter. It is probably the most interesting I received in eighteen years of correspondence! If you can find copies of "Eta-Lore" and "Pi-Mu Sequences" I would love to see them. They may still turn up in my files, in which case I'll send them to you.

Best regards,

N. J. A. Sloane

Encl.