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A LATTICE PATH PROBLEM

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1. *Introduction.*

The following problem was brought to our attention a few years ago by Pavol Hell. A certain rectangular voting district, m miles by n miles, is to be divided into two connected constituencies, where the division is to take place only along lines determined by the mileage markers. In how many ways can this be done? We consider the following problem which offers a partial solution to the above problem. Consider the integer lattice $I_{mn} = \{(x,y): 0 \le x \le m,\ 0 \le y \le n,\ x,y \text{ integers}\}$. Let $f(m,n)$ denote the number of lattice paths on I_{mn} from $(0,0)$ to (m,n) that have the property that no vertex occurs more than once on any given path. We shall call such a path a non-intersecting path. It is trivial that $f(m,0) = 1$ and it is easy to check that $f(m,1) = 2^m$ for all $m \ge 1$.

In general, the problem of evaluating $f(m,n)$ appears to be difficult. In this paper we evaluate $f(m,2)$ and obtain bounds for $f(m,n)$ for $n \ge 3$.

2. *A Matrix Setting for the Problem.*

Consider any non-intersecting path from $(0,0)$ to (m,n). Label each unit square inside I_{mn} that lies to the "right" of the path by a 1 and those that lie to the "left" by a 0. This labelling then

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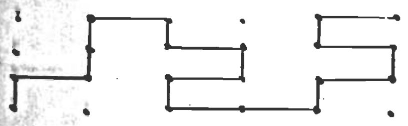
defines a one to one correspondence between the set of all non-intersecting paths on I_{mn} and a subset of the set of 0-1 $n\times m$ matrices. For example we have the following correspondence.

 $\longleftrightarrow \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 \end{bmatrix}$.

An $n\times m$ 0-1 matrix will be called <u>admissible</u> if it corresponds to a non-intersecting path on I_{mn} and inadmissible otherwise. Since the total number of $n\times m$ 0-1 matrices is 2^{mn} we have the following trivial upper bound

(1) $$f(m,n) \le 2^{mn}.$$

We say that two entries a_{ij} and $a_{k\ell}$ of an admissible matrix are 0-adjacent (respectively, 1-adjacent) if and only if $i = k$ and $|j - \ell| = 1$ or $j = \ell$ and $|i - k| = 1$ and $a_{ij} = a_{k\ell} = 0$ (respectively, $a_{ij} = a_{k\ell} = 1$). Further, we say that a_{ij} and $a_{k\ell}$ are 0-path (respectively, 1-path) adjacent if there is a sequence of entries $a_{ij},\dots,a_{k\ell}$ each of which is 0-adjacent (respectively, 1-adjacent) to its neighbours in the sequence. For convenience, we agree that a zero (one) entry is 0-path (1-path) adjacent to itself.

The following two observations are immediate:

(a) Any *zero* entry of an admissible matrix is 0-path adjacent to some entry in the first row or first column.

(b) Any *one* entry of an admissible matrix is 1-path adjacent to some entry in the last row or last column.

We now define for each $n \ge 2$, $2^n\times 2^n$ matrices B_n, $\underline{B}_n$ and $\bar{B}_n$. Associate with a column of 0-1 matrix, reading from top to bottom, a binary number in the natural way. The matrix $B_n = [b_{ij}]$, $0 \le i \le 2^n-1$, $0 \le j \le 2^n-1$ is now defined as follows:

$b_{ij} = 1$ if every admissible $n\times(m-1)$ matrix whose $(m-1)$st column is given by the binary representation of j can be extended to an admissible $n\times m$ matrix by adjoining an mth column given by the binary representation of i.

$b_{ij} = 0$ if every admissible $n\times(m-1)$ matrix whose $(m-1)$st column is given by the binary representation of j yields an inadmissible $n\times m$ matrix when we adjoin an mth column given by the binary representation of i.

$b_{ij} = *$ (unspecified) if there are admissible $n\times m$ matrices whose mth and $(m-1)$st columns are given by the binary representation of i and j respectively, but which are such that deleting the mth column does not result in an admissible $n\times(m-1)$ matrix.

The matrix B_n is thus well defined for every $n \ge 2$. $\underline{B}_n$ is the matrix obtained from B_n by replacing each $*$ entry by 0 and $\bar{B}_n$

is obtained by replacing each * entry by 1.

3. *Evaluation of* $f(m,2)$.

The following additional observations follow easily from observations (a) and (b) in Section 2. $A_{nm} = [a_{ij}]$ denotes an $n\times m$ matrix and we suppose $m \geq 2$ throughout.

(c) Any admissible $A_{2,m-1}$ can be extended to an admissible $A_{2,m}$ by adjoining an mth column $\binom{1}{1}$.

(d) Any admissible $A_{2,m-1}$ whose (m-1)st column is either $\binom{0}{0}$ or $\binom{1}{0}$ can be extended to an admissible $A_{2,m}$ by adjoining an mth column $\binom{1}{0}$.

(e) Any admissible $A_{2,m-1}$ whose (m-1)st column is not $\binom{1}{0}$ can be extended to an admissible $A_{2,m}$ by adjoining an mth column which is either $\binom{0}{0}$ or $\binom{0}{1}$.

(f) If $A_{2,m-1}$ is inadmissible but can be extended to an admissible $A_{2,m}$, then it is inadmissible because $a_{2,m-1} = 0$ and there is no 0-path adjacent entry $a_{1,p}$, $1 \leq p \leq n$ or a 0-path adjacent entry $a_{q,1}$, $1 \leq q \leq 2$. That is, the only inadmissible $A_{2,m-1}$ matrices that can be extended to an admissible $A_{2,m}$ are some of those whose (m-1)st column is $\binom{1}{0}$, and these can be extended by adjoining an mth column $\binom{0}{0}$. Further, all such matrices must have as their last k+1 columns, $\binom{1}{1}$ followed by k columns $\binom{1}{0}$ and the first m-k-2 columns must form an admissible $A_{2,m-k-2}$ for some k, $1 \leq k \leq m-2$.

From observations (c), (d), (e) and (f) we have that B_2, as defined in Section 2, is given by

$$B_2 = \begin{bmatrix} 1 & 1 & * & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

LEMMA 1. *For* $m \geq 4$,

$$f(m,2) = 4f(m-1,2)-3f(m-2,2)+2f(m-3,2)+f(m-4,2).$$

Proof. Let a_j^m be the number of admissible matrices $A_{2,m}$ whose last column is given by the binary representation of j. Then, since $f(m,2) = a_3^{m+1}$, we must show, for $m \geq 3$,

(2) $$a_3^{m+1} = 4a_3^m-3a_3^{m-1}+2a_3^{m-2}+a_3^{m-3}.$$

It follows from observations (c), (d), (e) and (f) above that

$$a_0^m = a_0^{m-1}+a_1^{m-1}+a_3^{m-1}+\sum_{i=1}^{m-2} a_3^i = a_0^{m-1}+a_1^{m-1}+\sum_{i=1}^{m-1} a_3^i$$

(3) $$a_1^m = a_0^{m-1}+a_1^{m-1}+a_3^{m-1}$$

$$a_2^m = a_0^{m-1}+a_2^{m-1}$$

$$a_3^m = a_0^{m-1}+a_1^{m-1}+a_2^{m-1}+a_3^{m-1}.$$

We have, by repeatedly appealing to the various equations given by (3),

$$a_3^{m+1}-4a_3^m+3a_3^{m-1}-2a_3^{m-2}-a_3^{m-3}$$

$$= a_0^m+a_1^m+a_2^m-3a_3^m+3a_3^{m-1}-2a_3^{m-2}-a_3^{m-3}$$

$$= (a_0^{m-1}+a_1^{m-1}+\sum_{i=1}^{m-1} a_3^i) + (a_0^{m-1}+a_1^{m-1}+a_3^{m-1}) + (a_0^{m-1}+a_2^{m-1})$$

$$- 3(a_0^{m-1}+a_1^{m-1}+a_2^{m-1}+a_3^{m-1}) + 3a_3^{m-1} - 2a_3^{m-2} - a_3^{m-3}$$

$$= -a_1^{m-1}-2a_2^{m-1}+2a_3^{m-1}-2a_3^{m-2}-a_3^{m-3}+\sum_{i=1}^{m-2} a_3^i$$

$$= -(a_0^{m-2}+a_1^{m-2}+a_3^{m-2}) - 2(a_0^{m-2}+a_2^{m-2})$$

$$+2(a_0^{m-2}+a_1^{m-2}+a_2^{m-2}+a_3^{m-2}) - 2a_3^{m-2} - a_3^{m-3} + \sum_{i=1}^{m-2} a_3^i$$

$$= -a_0^{m-2}+a_1^{m-2}-a_3^{m-3}+\sum_{i=1}^{m-3} a_3^i$$

$$= -(a_0^{m-3}+a_1^{m-3}+\sum_{i=1}^{m-3} a_3^i) + (a_0^{m-3}+a_1^{m-3}+a_3^{m-3})$$

$$-a_3^{m-3}+\sum_{i=1}^{m-3} a_3^i$$

$= 0$, as required.

We list the first few values of a_0^m, a_1^m, a_2^m and $a_3^m = f(m-1,2)$ in the table below:

m	a_0^m	a_1^m	a_2^m	a_3^m	
1	1	1	1	1	
2	3	3	2	4	+1
3	11	10	5	12	+1
4	38	33	16	38	0
5	126	109	54	125	−1
6	415	360	180	414	−1
7	1369	1189	595	1369	0
8	4521	3927	1964	4522	+1

6189 6190 6191 6192

It now follows immediately from the lemma that the following theorem holds.

THEOREM 1. *For* $m \geq 0$,

$$f(m,2) = \left(\frac{4+\sqrt{13}}{2\sqrt{13}}\right)\left(\frac{3+\sqrt{13}}{2}\right)^m + \left(\frac{\sqrt{13}-4}{2\sqrt{13}}\right)\left(\frac{3-\sqrt{13}}{2}\right)^m + \frac{i}{2\sqrt{3}}\left\{\left(\frac{1-i\sqrt{3}}{2}\right)^m - \left(\frac{1+i\sqrt{3}}{2}\right)^m\right\}.$$

Remark: Theorem 1 shows that $f(m,2) \sim c(1.817\ldots)^{2m}$ for some constant c. It also follows from the lemma that $f(m,2)$ has the following rational generating function

$$\frac{1-x^2}{1-4x+3x^2-2x-x^4} = \sum_{m=0}^{\infty} f(m,2)x^m.$$

A similar approach may be used to find the number of non-intersecting paths in $I_{m,2}$ from $(0,0)$ to a point of $I_{m,2}$ other than $(m,2)$. The complexity of the solution will vary with the choice of the terminal point. For example, we count the number of non-intersecting paths, say $g(m)$, from $(0,0)$ to $(0,2)$. The matrix analagous to B_2 is

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

Thus, corresponding to (3), defining the a_j^m in the natural way, we have

$$a_0^m = a_0^{m-1}+a_1^{m-1}+a_2^{m-1}$$

$$a_1^m = a_0^{m-1}+a_1^{m-1}$$

$$a_2^m = a_0^{m-1}+a_2^{m-1}$$

$$a_3^m = a_0^{m-1}+a_1^{m-1}+a_2^{m-1}+a_3^{m-1},$$

from which it follows that, for $m \geq 3$, keeping in mind that $a_3^{m+1} = g(m)$,

$$g(m) = 3g(m-1)-g(m-2)-g(m-3).$$

From this it follows that $g(m)$ has the rational generating function

$$\frac{1+x}{1-3x+x^2+x^3} = \sum_{m=0}^{\infty} g(m)x^m$$

and for $m \geq 0$

$$g(m) = -1 + \left(\frac{3+2\sqrt{2}}{2\sqrt{2}}\right)(1+\sqrt{2})^m - \left(\frac{3-2\sqrt{2}}{2\sqrt{2}}\right)(1-\sqrt{2})^m.$$

4. *The General Case - Upper and Lower Bounds.*

The two examples detailed in Section 3 make it clear that the unspecified entries, (*), of B_n complicate the problem of determining, for fixed n, a recurrence relation for $f(m,n)$. We therefore concentrate on the problem of obtaining upper and lower bounds.

Let $a(m,n,j)$ denote the number of admissible matrices A_{nm} whose last column is given by the binary representation of j, so that $a(m,2,j) = a_j^m$ and $a(m+1,n,2^n-1) = f(m,n)$. Let $\underline{B}_n = [\underline{b}_{ij}]$ and $\bar{B}_n = [\bar{b}_{ij}]$.

The matrices $\underline{B}_n$ and $\bar{B}_n$ may be used to define sequences $\underline{a}(m,n,j)$ and $\bar{a}(m,n,j)$ as follows:

$$(4)\qquad \begin{aligned} \underline{a}(m,n,j) &= \sum_{k=0}^{2^n-1} \underline{a}(m-1,n,k)\underline{b}_{jk} \\ \bar{a}(m,n,j) &= \sum_{k=0}^{2^n-1} \bar{a}(m-1,n,k)\bar{b}_{jk} \end{aligned}$$

where $\underline{a}(1,n,j) = \bar{a}(1,n,j) = 1$ for all n and all j, $0 \leq j \leq 2^n-1$. If we further define $\underline{f}(m,n) = \underline{a}(m-1,n,2^n-1)$ and $\bar{f}(m,n) = \bar{a}(m-1,n,2^n-1)$ we then have

$$(5)\qquad \underline{f}(m,n) \leq f(m,n) \leq \bar{f}(m,n),$$

and

THEOREM 2. *For* $m \geq 2$,

$$\underline{f}(m,n) = \text{sum of the entries in } \underline{B}_n^{m-1}$$
$$\bar{f}(m,n) = \text{sum of the entries in } \bar{B}_n^{m-1}.$$

Proof. It will suffice to prove the result for $\underline{f}(m,n)$. Let $e_m = [\underline{a}(m,n,0),\underline{a}(m,n,1),\ldots,\underline{a}(m,n,2^n-1)]$. It follows from (4) that for $m \geq 2$

$$\underline{a}(m,n,j) = e_{m-1}\cdot[\text{row } j \text{ of } \underline{B}_n]^T$$

and hence that

$$e_m = e_{m-1}\cdot \underline{B}_n^T = \cdots = e_1 \cdot (\underline{B}_n^T)^{m-1}.$$

Since $e_1 = [1,1,\ldots,1]$ we find that

$$\underline{f}(m,n) = \underline{a}(m+1,n,2^{n-1})$$

$$= \sum_{j=0}^{2^n-1} \underline{a}(m,n,j) = \text{sum of the entries of } e_m$$

$$= \text{sum of the entries of } \underline{B}_n^{m-1}$$

as asserted.

Letting $e = e_1 = [1,1,\ldots,1]$ we then have the following:

COROLLARY.

$$e \cdot \underline{B}_n^{m-1} \cdot e^T \leq f(m,n) \leq e \cdot \bar{B}_n^{m-1} \cdot e^T.$$

In order to estimate the sum of the entries of powers of a 0-1 matrix we appeal to the following classical theorem of Perron and Frobenius. (See for example [1], Chapter 1).

THEOREM. (Perron-Frobenius). *If A is a primitive 0-1 matrix there is a unique eigenvalue λ of A with the largest absolute value. λ is a positive real number and satisfies*

$$\lim_{k\to\infty} \left(\frac{1}{\lambda}A\right)^k \cdot e^T = y$$

where y is the eigenvector corresponding to λ.

It follows from this theorem that

(6) $$e \cdot A^k \cdot e^T \sim e \cdot y\lambda^k \quad \text{as } k\to\infty.$$

Thus, if the matrices $\underline{B}_n$ and $\bar{B}_n$ are both primitive we obtain, via corollary 1, the following bounds:

$$\underline{c}_n \underline{\lambda}_n^m \leq f(m,n) \leq \bar{c}_n \bar{\lambda}_n^m$$

where $\underline{\lambda}_n$ and $\bar{\lambda}_n$ are, respectively, the largest eigenvalu[illegible] $\underline{B}_n$ and $\bar{B}_n$ and $\underline{c}_n$ and $\bar{c}_n$ are constants depending solely on n. (Note that $\underline{b}_{i,1}$, $i = 0,1,\ldots,2^n-1$, $\underline{b}_{2^n-1,j}$, $j = 0,1,\ldots,2^n-1$, and $\underline{b}_{1,2^n-1}$ are all 1's and therefore, for all $n \geq 1$, $\underline{B}_n$ and $\bar{B}_n$ are indeed primitive matrices.)

In the case n = 3 one may verify that

$$B_3 = \begin{bmatrix} 1 & 1 & * & 1 & * & * & * & 1 \\ 1 & 1 & 0 & 1 & * & * & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

Thus $\underline{B}_3$ and $\bar{B}_3$ have the characteristic polynomials

$$\underline{P}_3(\lambda) = \lambda^2(\lambda^6-8\lambda^5+19\lambda^4-18\lambda^3+2\lambda^2+8\lambda-4)$$

and

$$\bar{P}_3(\lambda) = \lambda(\lambda^7-8\lambda^6+13\lambda^5+6\lambda^4-15\lambda^3-6\lambda^2+2\lambda+1)$$

respectively. One then finds that the largest eigenvalues of $\underline{B}_3$ and $\bar{B}_3$ are $\underline{\lambda}_3 = 4.78183\ldots$ and $\bar{\lambda}_3 = 5.56165\ldots$ respectively and therefore

$$\underline{c}_3(1.684\ldots)^{3m} < f(m,3) < \bar{c}_3(1.772\ldots)^{3m}.$$

It is clear that the above method may be used in principle to estimate f(m,n) for n > 3. However, calculation of the associated characteristic polynomials becomes an extremely tedious task. One may, however, use the following alternate approach: It follows from (6) that if A i[illegible] primitive 0-1 matrix then

$$\lim_{k\to\infty} \frac{e\cdot A^{k}\cdot e^{T}}{e\cdot A^{k-1}\cdot e^{T}} = \lambda. \tag{7}$$

Thus λ may be approximated by calculating successive ratios of sums of entries of the powers of A. Of course it is difficult to measure the rate of convergence to λ. However our calculations suggest that in the case $n = 4$

$$\underline{c}_4(1.661\ldots)^{4m} < f(m,4) < \bar{c}_4(1.757\ldots)^{4m}.$$

We do not give all of the details here but we record the matrix B_4 so that the interested reader may verify matters for himself.

$$\begin{bmatrix}
1 & 1 & * & 1 & * & * & * & 1 & * & * & * & * & * & * & * & 1\\
1 & 1 & 0 & 1 & * & * & 0 & 1 & * & * & 0 & * & * & * & 0 & 1\\
1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & * & 0 & * & 0 & 0 & 0 & 1 & 0\\
1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & * & * & 0 & 0 & 0 & 0 & 1 & 1\\
1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0\\
1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0\\
1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0\\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1\\
1 & 1 & * & 1 & 0 & 0 & 0 & 0 & 1 & 1 & * & 1 & 1 & 1 & 1 & 0\\
1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0\\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0\\
1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0\\
1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0\\
1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0\\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{bmatrix}$$

We now prove a theorem which enables us to obtain estimates of $f(m,n)$ in terms of $f(m,k)$, $k < n$.

THEOREM 3. *Let* $t \ge 0$, $k \ge 1$ *and let* $n = 2t(k+1)+k$. *Then*

$$f(m,n) \ge f(m,k)^{2t+1}.$$

Proof. Divide the integer lattice I_{mn} into $2t+1$ horizontal rectangles $I_1, I_2, \ldots, I_{2t+1}$ of width $k+1$. That this is possible follows from the definition of n in terms of t and k. In fact we have, for $j = 1,2,\ldots,2t+1$

$$I_j = \{(x,y): 0 \le x \le m,\ (k+1)(j-1) \le y \le (k+1)(j-1)+k\}.$$

For j odd let P_j denote any of the $f(m,k)$ non-intersecting paths in I_j from $(0,(k+1)(j-1))$ to $(m,(k+1)(j-1)+k)$ and for j even let P_j denote any one of the $f(m,k)$ non-intersecting paths in I_j from $(m,(k+1)(j-1))$ to $(0,(k+1)(j-1)+k)$. These paths may then be juxtaposed to give a non-intersecting path in I_{mn}. The number of non-intersecting paths in I_{mn} is thus at least $f(m,k)^{2t+1}$.

COROLLARY. *For each fixed* n,

$$\lim_{m\to\infty} f(m,n)^{\frac{1}{mn}} = \beta_n \quad \textit{exists.}$$

Proof. Let $\alpha = \liminf_{m\to\infty} f(m,n)^{\frac{1}{mn}} \le \limsup_{m\to\infty} f(m,n)^{\frac{1}{mn}} = \gamma$. Let $\varepsilon > 0$ and let k be such that

$$f(k,n)^{\frac{1}{kn}} > \gamma-\varepsilon \quad \text{and} \quad (\gamma-\varepsilon)^{1-\frac{1}{k}} > \gamma-2\varepsilon.$$

Henceforth k is fixed. Let m be given and let t be defined by $2t(k+1)+k \le m < 2(t+1)(k+1)+k$. We may suppose that m is so large that

$t>k^2$. Then, by Theorem 3 we have

$$f(m,n) = f(n,m) \ge f(n,2t(k+1)+k) \ge f(n,k)^{2t+1}$$
$$= f(k,n)^{2t+1} > (\gamma-\varepsilon)^{kn(2t+1)}.$$

Thus

$$\alpha = \liminf_{m\to\infty} f(m,n)^{\frac{1}{mn}} \ge (\gamma-\varepsilon)^{\frac{k(2t+1)}{m}} > (\gamma-\varepsilon)^{1-\frac{1}{k}} > \gamma-2\varepsilon.$$

from which it follows that $\alpha = \gamma$ as required.

In a similar manner, it follows that $\beta = \lim_{m\to\infty} f(m,m)^{\frac{1}{m^2}}$ exists. In fact, it follows easily from Theorem 3 that

$$\beta > \beta_k^{\frac{k}{k+1}},$$

for all k. Our calculations, based on (7), indicate that $\beta_6 > 1.653...$ and hence that

$$f(m,m) > c(1.583...)^{m^2}.$$

Finally, we indicate how upper bounds for $f(m,n)$ for $n \le m$ may be obtained. Note that of the 16 ways of partitioning the unit squares of $I_{2,2}$ into two sets, at most 14 can correspond to portions of non-intersecting lattice paths. Specifically, the patterns $\begin{bmatrix}0&1\\1&0\end{bmatrix}$ and $\begin{bmatrix}1&0\\0&1\end{bmatrix}$ are not possible. Thus, if m and n are both even, say $m = 2r$, $n = 2s$,

$$f(m,n) \le 14^{rs} \sim (1.935...)^{mn}$$

and a slightly weaker bound follows if m and n are not both even. Similarly, one may show that of the $2^{16} = 65536$ ways of partitioning the unit squares of $I_{4,4}$ into two sets, at most 22662 can correspond to portions of non-intersecting paths. Then, if m and n are both divisible by 4 one has

$$f(m,n) \le (22662)^{\frac{mn}{16}} \sim (1.872...)^{mn}$$

with slightly weaker results in the remaining cases. No doubt further such improvements are possible.

5. *Some Open Questions.*

Clearly there are a number of unanswered questions which remain. We summarize some of these.

1. Evaluate β_n for $n \ge 3$ and evaluate β.

2. Is the sequence $f(m,n)^{\frac{1}{mn}}$ decreasing in both m and n? The first few values of $f(m,n)$ and $f(m,n)^{\frac{1}{mn}}$ are given in the following tables. If $f(m,n)^{\frac{1}{mn}}$ is decreasing, it would follow that $f(m,n) < c(1.758...)^{mn}$ for $m \ge 5$, $n \ge 4$.

n \ m	1	2	3	4	5	6	7	8
1	2	4	8	16	32	64	128	256
2		12	38	125	414	1369	4522	14934
3			184	976	5382	29739		
4				8512	79384			

n \ m		2	3	4	5	6	7	8
1	2	2	2	2	2	2	2	2
2		1.861	1.834	1.829	1.827	1.825	1.824	1.823
3			1.785	1.775	1.773	1.772		
4				1.760	1.758			

3. For even n, let $h(n)$ denote the number of non-intersecting paths in I_{nn} which are Hamiltonian. We can show that $\alpha = \lim_{n\to\infty} h(n)^{\frac{1}{n^2}}$ exists and that $2^{\frac{1}{3}} \le \alpha \le 12^{\frac{1}{4}}$. We believe $\alpha < \beta$.

4. For k even, $2n \le k \le n^2+2n$ let $f_k(n)$ be the number of non-intersecting paths from (0,0) to (n,n) which have exactly k edges, so that

$$f(n,n) = \sum_{\substack{k=2n \\ k \text{ even}}}^{n^2+2n} f_k(n).$$

$f_{2n}(n) = \binom{2n}{n}$ is the classical ballot problem result. The evaluation of $f_k(n)$ for $k > 2n$ appears difficult.

REFERENCE

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A BALANCED HYPERGRAPH DEFINED BY CERTAIN SUBTREES OF A TREE

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ABSTRACT

Let T be a tree with vertex set $V = \{v_1, v_2, \ldots, v_p\}$. For each i, $1 \le i \le p$, let a_i be a nonnegative integer and define $E_i \equiv \{v \in V : d(v_i,v) \le a_i\}$. We show that the hypergraph (V,E), where $E = \{E_1, E_2, \ldots, E_p\}$, is balanced. This result generalizes two previously known min-max relations for this hypergraph.

Let T be a tree with vertex set $V = \{v_1, v_2, \ldots, v_p\}$. For each i, $1 \le i \le p$, let a_i be a nonnegative integer and define $E_i \equiv \{v \in V: d(v_i,v) \le a_i\}$. Thus each E_i is the vertex set of a subtree of T "centred" at v_i. Let $R \equiv \{(v_1,a_1),(v_2,a_2),\ldots,(v_p,a_p)\}$ and associate a hypergraph $H_{T,R} = (V,E)$ with T and R, where $E \equiv \{E_1,E_2,\ldots,E_p\}$. The hypergraph terms we will use can be found in Berge [2].

Min-max theorems for $H_{T,R}$ have been obtained in [3] and [5]. One result from [3] implies that the maximum cardinality of matching in $H_{T,R}$ is equal to the minimum size of a transversal of $H_{T,R}$. The main result of [5] is that the maximum cardinality of a strongly stable set of vertices of $H_{T,R}$ is equal to the minimum size of a cover in $H_{T,R}$. These min-max results are also consequences of a more general property of $H_{T,R}$.

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