Delest

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PRESENTEE A

L'UNIVERSITE DE BORDEAUX I

POUR OBTENIR LE GRADE DE
DOCTEUR D'ETAT ES SCIENCES

par

Marie-Pierre DELEST

UTILISATION DES LANGAGES ALGEBRIQUES ET DU CALCUL FORMEL POUR LE CODAGE ET L'ENUMERATION DES POLYOMINOS

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soutenue le 22 Mai 1987 devant la commission d'examen:

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Generating functions for column-convex polyominoes

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Abstract. Using bijections and language theory, we give the generating function for the number of column-convex polyominoes with perimeter $2n$. We prove also a result about their number with area m and k columns.

Introduction

Unit squares with vertices at integer points in the cartesian plane are called *cells*. A *polyomino* is a finite connected union of cells such that the interior is also connected (no cut point). The *area* of a polyomino is the number of cells, the *perimeter* is the length of the border.

Polyominoes are defined up to translation.

Polyominoes are very classical objects in combinatorics. Counting polyominoes according to their area or perimeter is a major unsolved problem in combinatorics.The first authors interested in this subject were Read [12] and Golomb [5]. Some authors Lunnon [9], Redelmeier [13] have given the first values for the number of polyominoes with a given area.

The physicists have given several asymptotic results. They call *animal* a set of points obtained by taking the centers of the cells of a polyomino. Giving a privileged direction for the growth of an animal allowes them to obtain generating functions (see Viennot [16] and references therein).

A *column* (resp. *row*) of a polyomino is the intersection between the polyomino and any infinite vertical (rep. horizontal) strip of unit squares.

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* This work was completed when the author was visiting San Diego in June 1984.

ERRATA

I-5,	ligne 11	: remplacer "of u" par "u"
I-5,	ligne 26	: remplacer "$F \rightarrow xxFB$" par "$FB \rightarrow xxFB$"
I-9,	figure 4	: mettre le "W" sur le point du dessus
I-14,	figure 8	: remplacer "Φ" par "β"
I-21,	ligne 8	: remplacer "$\alpha(\mathbf{R})$" par "$\alpha(\mathbf{V})$"
I-22,	ligne 1	: remplacer "$a(x,\bar{x},y,\bar{y})$" par "$h(x,\bar{x},y,\bar{y})$"
I-25,	ligne 2	: remplacer "ω_2 is West" par "ω_2 is East"
I-25,	ligne 7	: remplacer "$(W,\overline{W})$" par "$(\overline{W},W)$"
I-28,	ligne 2-	: remplacer "x^q" par "$\bar{x}^q$"
I-31,	ligne 25	: remplacer "$\alpha(\mathbf{L})$" par "$\alpha(\mathbf{G})$"
I-36,	ligne 3	: remplacer "W to N" par "W to N'"
II-8,	ligne 14	: remplacer "$x_n y_m$" par "$x^n y^m$"
II-15,	ligne 4	: remplacer "x (resp. $\bar{y}$)" par "x (resp. $\bar{x}$)"
II-15,	ligne 5	: remplacer "$\bar{x}$ (resp. y)" par "$\bar{y}$ (resp. y)"
II-15,	ligne 6	: remplacer "x (resp. $\bar{x}$)" par "$\bar{x}$ (resp. x)"
II-16,	ligne 7	: remplacer "- d(f)" par "(i) d(f)"
II-16,	ligne 24	: remplacer "$\underline{GD}$ + y" par "$\underline{\bar{G}D}$ + y"
II-16,	ligne 24	: remplacer "$\bar{y}$ $\underline{\bar{G}d}$" par "$\bar{y}$ $\underline{\bar{G}D}$"
III-2,	ligne 14	: remplacer "found" par "find"
III-3,	ligne 2-	: remplacer "by w" par "by $\tilde{w}$"
III-3,	ligne 1-	: remplacer "w" par "$\tilde{w}$"
III-15,	ligne 6	: remplacer "South" par "North"
III-15,	ligne 9	: remplacer "South" par "North"
III-15,	figure 9	: remplacer "9" par "8", l'axe vertical est orienté vers le bas de la feuille
III-20,	ligne 11	: remplacer "assymptotic" par "asymptotic"
IV-3,	ligne 3-	: remplacer "interesting" par "interested"
IV-7,	ligne 3	: remplacer "$\mathbb{C}[\langle x,y\rangle]$" par "$\mathbb{C}[\{x,y\}]$"
IV-7,	ligne 4	: remplacer "$\mathbb{C}[\langle u,v\rangle]$" par "$\mathbb{C}[\{u,v\}]$"
IV-10,	figure 5	: remplacer "$C^A=(5,4,4,7$" par "$C^A=(5,4,7$"
IV-11,	ligne 12	: remplacer deux fois "ddc-" par "dcc-"
IV-12,	ligne 3	: remplacer la ligne par "$\varphi(A)=x\varphi(A')\bar{y}$ with A' a dcc-polyomino defined by"
IV-25,	ligne 10	: remplacer "$-x^2z$" par "$-2x^2z$"
IV-34,	ligne 2	: remplacer "found" par "find"
V-1,	ligne 9	: remplacer "Nami" par "Naïmi"
V-2,	ligne 5	: remplacer "n processus" par "n+1 processus"
V-2,	ligne 13-	: remplacer "transformation "transformations"
V-2,	ligne 2-	: remplacer "raisonement" par "raisonnement"
V-3,	ligne 20	: remplacer "se travail" par "ce travail"
V-4,	ligne 3-	: remplacer "considéré" par "considérée"
V-5,	ligne 1	: remplacer "donné" par "donnée"
V-6,	ligne 6-	: remplacer "$\varphi_{.}\mathcal{A}$" par "$\varphi_i . \mathcal{A}$"
V-8,	ligne 1-	: remplacer "$\mathcal{A} \in \mathcal{T}_n$" par "$\mathcal{A} \in \mathcal{T}_{n+1}$"
V-12,	ligne 6-	: remplacer "vrai" par "vraie"
V-12,	ligne 5-	: remplacer "vrai" par "vraie"

For simplicity let v, l_1, l_2, l_3, l_4 denote the respective images by ϕ of the series $v(x,b,/)$, $l_1(x,b,/)$, $l_2(x,b,/)$, $l_3(x,b,/)$, $l_4(x,b,/)$. These series are solutions of an commutative algebraic system coming from (11). Successively, we obtain :

$$l_3 = \frac{v}{1-x},$$

$$l_4 = \frac{v}{(1-x)^2},$$

$$l_1 = \frac{yv}{1-x},$$

$$l_2 = \frac{xyv}{1-x} + \frac{x^2y^2v}{(1-x)^3} + y.$$

noting that

$$v = x(v - l_1 - l_2)$$

we have the following result

Theorem 3.

The generating function for the number $g_{n,m}$ *of column-convex polyominoes with area* n *and* m *columns is*

$$\sum_{n\geqslant 1}\sum_{m\geqslant 1} g_{n,m}\, x_n\, y_m = \frac{xy(1-x)^3}{(1-x)^4 - xy(1-x)^2(1+x) - x^3y^2}.$$

Remark 4.

For $y = 1$, we obtain the same result as Klarner [6]

$$\sum_{n\geqslant 1} g_n\, x^n = \frac{x(1-x)^3}{1-5x+7x^2-4x^3}$$

where g_n is the number of column-convex polyominoes with area n.

Remark 5.

The parameter perimeter of a polyomino is not simply encoded in this bijection. Thus we introduce a new bijection in the following paragraph.

If a polyomino has a perimeter 2n and k corners it is easy to see that its site perimeter is 2n-k. Thus the generating function

$$(39)\quad p(z) = \sum_{m \geq 0} p_m z^m$$

where p_m is the number of polyomino with site perimeter m, is given by:

$$(40)\quad p(z) = c(z, 1/z).$$

Thus we have:

$$(41)\quad p(z) = \frac{1-x^2-2x^3+x^4+(x^2-x-1)\sqrt{1-2x-x^2-2x^3+x^4}}{2}.$$

Using technics of complex analysis described by P. Flajolet [4], we give an assymptotic evaluation for p_m.

The smallest singularity of p(z) is $\frac{3-\sqrt{5}}{2}$.

Around this singularity the function p(z) can be developped in:

$$(42)\quad p(z) = A(1 - x\frac{3+\sqrt{5}}{2}) + o(|1 - x\frac{3+\sqrt{5}}{2}|^{1/2}),$$

It is easy to prove that:

$$(43)\quad A = 2\, 5^{1/4}(\sqrt{5} - 2),$$

thus we have the following result

Theorem 9. *The number of parallelogram polyominoes with site perimeter m is asymptotically*

$$p_m = 5^{1/4}(\sqrt{5} - 2)\frac{1}{m\sqrt{\pi m}}\left(\frac{3+\sqrt{5}}{2}\right)^m + \left(\frac{3+\sqrt{5}}{2}\right)^m o(m^{-3/2}).$$

Thus the formula (18) is proved for p odd . In the same way it is easy to prove the same result in the case where p is even. Moreover we have

$$v(-1)=1 \text{ and } v'(-1)=-3 \qquad (19)$$

Therefore, using the equalities (15) to (19), we get the following

THEOREM 23 . *The number of directed column-convex polyominoes having a directed site perimeter n is*

$$s_n=\sum_{j=0}^{n-1}\sum_{i=0}^{j_2}(-1)^{n+i+1+j_+}\,2^{n-j-1}\,3^{2i+r}\,\frac{1}{n+j_++i}\binom{n+j_++i}{j+1;\,2i+r;\,j_2-i;\,n-j-1}$$

with $r=j \bmod 2$, $j_+=\left\lceil\frac{j+1}{2}\right\rceil$ *and* $j_2=\left\lfloor\frac{j}{2}\right\rfloor$.

We give in Figure 8, the table for $s_{n,k}$ and s_n.

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$s_{n,k}$	1	2	3	4	5	6	7	8	s_n
1	1								1
2	1	2							3
3	1	6	5						12
4	1	12	27	14					54
5	1	20	85	112	42				260
6	1	30	205	492	450	132			1310
7	1	42	420	1582	2565	1782	429		6821
8	1	56	770	4172	10415	12562	7007	1430	36413

Figure 8. The table for the numbers $s_{n,k}$ and s_n.

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Triangle: no of column-convex polyominoes with ~~area~~ perimeter n and m columns

7 - NUMBER OF DIRECTED COLUMN-CONVEX POLYOMINOES ACCORDING TO THE BOND PERIMETER

Using the same results than in the previous one, we give in this section a formula for the number $p_{n,k}$ of dcc-polyominoes with bond perimeter 2n+2 and k columns. we have the following equation for P(x,y)

$$x^4P^3-2x^2(1-x^2)P^2+(1-x^2)(1-x^2-x^2y)P-yx^2(1-x^2)=0. \qquad (20)$$

Dividing by $(1-x^2)^2$, and let s be $x^2/(1-x^2)$, we get

$$s^2P^3-2sP^2+(1-ys)P-sy=0.$$

Let g be sP, we have

$$P(sP-1)^2-sy(P+1)=0.$$

We get the following system of equations

$$\begin{cases} P = ys \dfrac{(P+1)}{(g-1)^2}, \\ g = s P. \end{cases}$$

This system has a form which allow us to use Good's formula, with t=ys, u=s,

$$\varphi(P,g)=\frac{(P+1)}{(g-1)^2} \quad \text{and} \quad \psi(P,g)=P.$$

Thus we have

$$(P,t^mu^r)=\frac{1}{mr}\sum_{i=1}^{r}\sum_{j=0}^{m-1} i(\varphi^m,P^jg^i)(\psi^r,P^{m-1-j}g^{r-i}),$$

and consequently

$$(P,t^mu^r)=\frac{1}{m}\binom{m}{r+1}\binom{2m+r-1}{r}.$$

Afterwards, we substitute the values of t and u

$$P(x,y)=\sum_{m\geq 1}\sum_{i\geq 0}\sum_{r=0}^{m-1}\frac{1}{m}\binom{m}{r+1}\binom{2m+r-1}{r}\binom{m+r+i-1}{i}y^{m}x^{2(m+r+i)}.$$

At length, we get the following

THEOREM 24. *The number of directed column-convex polyominoes with bond perimeter* 2n+2 *and* k *columns is*

$$p_{n,k}=\frac{1}{k}\sum_{r=0}^{M_p}\binom{k}{r+1}\binom{2k+r-1}{r}\binom{n-1}{k+r-1},$$

with M_p *the minimum of* k-1 *and* n-k.

Summing over k, we get the

CORROLARY 25. *The number of directed column-convex polyominoes having a bond perimeter* 2n+2 *is*

$$p_n=\sum_{k=1}^{n}\frac{1}{k}\sum_{r=0}^{M_p}\binom{k}{r+1}\binom{2k+r-1}{r}\binom{n-1}{k+r-1},$$

with M_p *the minimum of* k-1 *and* n-k.

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$p_{n,k}$	1	2	3	4	5	6	7	p_n
1	1							1
2	1	1						2
3	1	4	1					6
4	1	9	9	1				20
5	1	16	37	16	1			71
6	1	25	105	106	25	1		263
7	1	36	240	446	245	36	1	1005

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Figure 9. Table for the numbers $p_{n,k}$ and p_n .

We give in Figure 9 the table for the values of p_n and $p_{n,k}$.

REMARK 26. Using the equation (20) with y=1, it is easy to found an asymptotic value for p_n and we get

$$p_n \approx \left(\frac{3+2\sqrt[3]{100+5\sqrt{10}}}{6}\right)^n n^{-3/2} .$$

8 - NUMBER OF DIRECTED COLUMN-CONVEX POLYOMINOES WITH GIVEN AREA

In this section, we give an exact formula for the number r_n of dcc-polyominoes having an area n.

Let X be the alphabet {a,x}.

DEFINITION 27. R is *the language of the words* w *of* X^* *satisfying*

(i) w *is in* $(xx+a)^*xx$,

(ii) $|w|$ *is even.*

For each dcc-polyomino A having k columns, we define the word $w=\rho(A)$ in X^* using the following construction:

- if A has one column thus $C^A=(c_1^A)$ and then $\rho(A)=x^{2c_1^A}$,
- if A has k columns then $\rho(A)=w_1w_2\ldots w_k$ with

  for every $i\in[1,k-1]$, $w_i=x^{2g_i^A}ax^{2(c_i^A-g_i^A-1)}a$

  and $w_k=x^{2c_k^A}$.

Clearly, w is a word of R. The number of columns of the dcc-polyomino A is

$$\frac{|\rho(A)|_a}{2} + 1,$$

and the area of A is

$$\frac{|\rho(A)|}{2} .$$

An example of this coding is given in figure 10.