On Permutations and Weighted Complete Graphs

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Abstract

This paper is an extension of a comment submitted to OEIS, A006000. The original comment is here:

Consider a basis for the permutations of 1..n as the primitive permutations such that when rotated and/or read in reverse, P_n is created. E.g. for n=4 we have a basis in 1234, 1243 and 1324. The number of elements in a basis for P_n is given by A001710(n-1).

Consider a permutation as a Hamiltonian cycle on K_n (each element refers to a vertex, with consecutive entries implying an edge is present in the HC between the 2 vertices, with the last entry connected to the first entry), and assign each edge a weight from 1...A000217(n-1), the sum of which is A002817(n-1).

The weights can be assigned arbitrarily; this example uses 12->1, 13->2, 14->3, 23->4, 24->5 and 34->6.

We now want to know the total weight of the basis. For example using the basis given above, the cycles have weights 1+4+6+3 = 14, 1+5+6+2 = 14 and 2+4+5+3 = 14, with a total weight of 42. We can see that each edge appears (n-2)! times; for example, in the n=4 basis given above there are (4-2)!=2 copies of the edges 12, 13, etc. This means that the total weight for primitive Hamiltonian cycles on a simply weighted K_n is (n-2)!*A002817(n-1), which, in the case n=4, gives (4-2)!*21 = 42 as required.

It is not always the case that each cycle has a different weight; for n=5 for example, we have 12345 with a weight of 1+5+8+10+4=28, but the cycle 13524 has weight of 2+9+7+6+3=27.

To find the average weight of a cycle we take the total weight and divide by the number of cycles, i.e. the size of the basis. We have A002817(n-1)*(n-2)!/A001710(n-1) = A002817(n-1)*(n-2)!*2/(n-1)! = A002817(n-1)*2/(n-1).

This doesn't always give an integer, but multiplying by a further 2 does, and we arrive at sequence A006000.

For n>2, a(n) is 2 * (average cycle weight for n+1), e.g. the average cycle weight for n=4 is A002187(3)*2/3 = 21*2/3 = 14, and a(3) = 2*14 = 28.

Permutations

We define a permutation on n as an arrangement of the numbers 1,..,n. We write [n] for the collection of permutations on 1 to n. The magnitude of [n] is n!.

For example, there are 3! = 3*2*1 = 6 members of [3], namely 123, 132, 213, 231, 312 and 321.

Basis

We define a basis of [n] to be a collection of permutations such that by rotating the permutations and reading both forwards and backwards, we can generate [n], and a minimal basis as the smallest such basis.

For example, 123 is a (minimal) basis for [3]. 123 → 231 → 312 by rotation, and reading backwards gives 321, 132 and 213, which gives us the 6 elements of [3], so a (minimal) basis for [3] is any one of these permutations. {123,132} is a basis, but not minimal.

We will now use 'basis' to mean 'minimal basis' unless stated otherwise.

The order of a basis is (n-1)!/2, given by A001710 at OEIS. This is because a basis element of [n] generates n rotations and n reverses. |[n]| is n!, and if we divide by 2n we get the desired result.

A basis for [n] can be generated from a basis from [n-1] by considering the rotations of the basis elements of [n-1].

Consider a basis for [3], e.g. 123 for [3]. We create the set of rotations, i.e. 123, 231 and 312, and we say that the basis for [4] is 1 with (e+1), where e+1 represents 234, 342 and 423.

So a basis for [4] is 1234, 1342 and 1423.

Similarly, a basis for [5] is given by:

12345, 13452, 14523, 15234,
12453, 14532, 15324, 13245,
12534, 15342, 13425, 14253.

and so on.

Another method is to take a basis for [n] and insert n+1 at n of the n+1 insertion points, usually the last n.

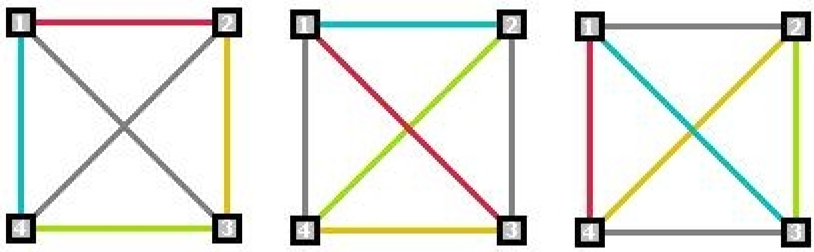
For example, with 1234, 1243 and 1324 we get:

15234, 12534, 12354, 12345,
15243, 12543, 12453, 12435,
15324, 13524, 13254, 13245.

Permutations as Hamiltonian Cycles

If we label the n vertices of a complete graph K(n) 1 through to n, then each basis element represents a primitive Hamiltonian cycle - the other Hamiltonian cycles are the rotations and reverses of the primitive cycles.

For example with [4], using the basis 1234, 1342 and 1423, we get the following primitive Hamiltonian cycles, using red as edge 1, orange, green and blue as edges 2, 3 and 4 respectively.



Simply Weighted Hamiltonian Cycles

Given K(n), we can assign each edge a weight. If every element from {1,...,n(n-1)/2} is used, then the weighting is said to be simple.

n(n+1)/2 is a triangular number, and is the sum 1+2+....+n, given by A000217 at OEIS. It is often written T(n). The total sum of the weights of the edges is T(T(n-1)), where T(T(n)) is:

$= [(n^2 + n) / 2 * (n^2 + n)/2 +1] / 2$

$= [(n^2 + n) * (n^2 + n + 2)] / 8$

$= [n * (n + 1) * (n^2 + n + 2)] / 8.$

= A002817.

T(T(n)) is even for 0, 2, 5, and 7 mod 8.

Weight of a Permutation/Hamiltonian Cycle

Probably the simplest weighting system is to assign the weights to edges as follows:

	1	2	3	4	5
1	x	1	2	3	4
2	1	x	5	6	7
3	2	5	x	8	9
4	3	6	8	x	10
5	4	7	9	10	x

Given a permutation of [n], we consider each pair of consecutive entries as an edge, an read the weight of from a table of weights. The last and first entries are also considered a pair as we are considering cycles not paths.

For example, the permutation 42351 consists of edges 42, 23, 35, 51 and 14, with respective weights 6, 5, 9, 4 and 3. This gives the total weight of permutation 42351 as:

6 + 5 + 9 + 4 + 3 = 27.

The cycle weights are not always equal, for example 12345 has a weight of:

1 + 5 + 8 + 10 + 4 = 28.

Total Weight of a Basis and Average Weight of a Cycle

The total weight of a basis can be calculated by observing that is a basis every edge appears exactly (n-2)! times.

For example, the edge 12 appears (in bold italics) 6 times in this basis for [5]:

15234, ***12534***, ***12354***, ***12345***,
15243, ***12543***, ***12453***, ***12435***,
15324, 13524, 13254, 13245.

and 35 also appears 6 times:

15234, ***12534***, ***12354***, 12345,
15243, 12543, ***12453***, ***12435***,
15324, ***13524***, 13254, 13245.

This means that each edge weight appears (n-2)! times and so the total weight for a basis is:

(n-2)! * T(T(n-1)) =

= [(N+1) * (N + 2) * (N^2 + N + 2)] / 8, where N = n – 1.

As there are (n-1)!/2 basis elements, we need to divide this by (n-1)!/2 to get the average weight of a cycle:

= (N – 1)! * [(N + 1)! * (N^2 + N + 2)] / 8 * 2 / N!

= [(N + 1) * (N^2 + N + 2)] / 4.

which is A006000(N) / 2.

For example, with n=4, N=3, so the average weight of a cycle is 4 * 14 / 4 = 14.

JavaScript Code

The following code calculates the formula for n:

```
<script>
function tri(n) {return n*(n+1)/2;}
for (i=2;i<40;i++) document.write(tri(tri(i))*2/i+", ");
</script>
```

and produces the following output:

6, 14, 27.5, 48, 77, 116, 166.5, 230, 308, 402, 513.5, 644, 795, 968, 1164.5, 1386, 1634, 1910, 2215.5

As A006000(n) is odd for n == 0 mod 4, multiplying by 4 instead of 2 gives an integer sequence.