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# Highly Powerful Numbers.

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1. Introduction and Notation. Using the nomenclature of S. Golomb [5], a natural number $n > 1$ is said to be powerful if in its standard prime power decomposition, the exponents of the prime factors of $n$ are all greater than unity.

Let us write throughout for $n > 1$,

$$n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k} , \tag{1.1}$$

where $p_1, p_2, \ldots, p_k$ are distinct prime factor of $n$ in increasing order of magnitude. Thus $n$ is powerful if and only if

$$\alpha_i \geq 2 \; (i = 1,2,\ldots,k) .$$

In addition, we regard 1 as a powerful number. Every powerful number has the unique representation $a^2 b^3$ where $a,b$ are integers and $b$ is square-free. The number of such numbers not exceeding $x$ is $\{\zeta(3/2)/\zeta(3)\}\sqrt{x}$ (see [5]).

It follows that the asymptotic density of powerful numbers is zero.

In this paper, we introduce what we call "highly powerful numbers" and study some of their properties. We hope to continue our investigations on this subject in a later paper. We conclude this paper with a table of all highly powerful numbers not exceeding $10^{27}$. There are 288 such numbers upto two billion. Their asymptotic density is zero, of course, since they are a subset of the set of powerful numbers.

1.2. Definition. For every natural number $n \geq 1$ we define $\delta(1) = 1$ and for $n > 1$ given by (1.1),

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$$\delta(n) = \alpha_1\alpha_2 \cdot \cdot \cdot \alpha_k \ .$$

1.3 Definition A natural number $n > 1$ is called highly powerful whenever

$$\delta(n) > \delta(m)$$

for all natural numbers $m < n$.

It is immediate that a highly powerful number is itself powerful.

1.4 Defintion The core $\gamma(n)$ of $n$ given by (1.1) is given by

$$\gamma(n) = p_1p_2 \cdot \cdot \cdot p_k \ .$$

Also, $$\gamma(1) = 1 \ .$$

1.5 Definition A divisor $d$ of $n > 1$ is said to be a coreful divisor of $n$ if $d|n$ and $\gamma(n)|d$. Also $1$ is taken as a coreful divisor of $1$.

We see that the number of coreful divisors of $n$ equals $\delta(n)$. Hence we have

1.6. Theorem A number $n > 1$ is highly powerful if and only if it has more coreful divisors than any natural number $m$ less than itself.

In view of this theorem, highly powerful numbers may also be called "highly coreful numbers".

If $n$ is highly powerful, then clearly we have $\delta(n) < \delta(2n)$; hence it follows that there is at least one highly powerful number in the interval $(n,2n]$. Thus the number of highly powerful numbers is infinite.

In all that follows, $a_1, a_2, \ldots, a_\ell, \ldots$ denote the sequence of highly powerful (h.p. for brevity) numbers. We write

$$q_1 = 2 \ , \ q_2 = 3 \ , \ . \ . \ . \ . \ , \ q_k \ , \ . \ . \ .$$

for the sequence of primes. We also use the standard notation

$$(1.7) \qquad a_\ell = 2^{\alpha_1} 3^{\alpha_2} 5^{\alpha_3} \ . \ . \ . \ q_k^{\alpha_k}$$

for the general form of a highly powerful number. It is easy to see that

$$(1.8) \qquad \alpha_1 \geq \alpha_2 \geq \alpha_3 \geq . \ . \ . \geq \alpha_k > 1$$

Thoughout, $[x]$ denotes as usual the largest integer not exceeding $x$ .

1.9 Remark The definition of a h.p. number and its structure given by (1.8) are reminiscent of those for highly composite numbers introduced by Ramanujan [9] and studied by him, Erdös [2] and others. However there are several differences in the structural properties of h.p. numbers and highly composite numbers, revealed by the results to follow.

It may be recalled here that Pillai [7,8] generalized highly composite numbers in two directions, while Alaoglu and Erdös [1] introduced the concept of highly abundant and collosally abundant numbers and studied some of their properties.

2. Some properties of highly powerful numbers. The following results throw some light on the structure of h.p. numbers.

2.1 Theorem Every prime divides some h.p. number.

Proof. Let $q_{k+1}$ be the smallest prime that does not divide any h.p. number.

Then every h.p. number $a_\ell$ has the form

$$a_\ell = 2^{\alpha_1} 3^{\alpha_2} \ldots q_k^{\alpha_k} , \alpha_1 \geq \alpha_2 \geq \ldots .$$

Since the number of h.p. numbers is infinite, we choose $\ell$ so large that

$$2^{[\alpha_1/2]} > q_{k+1}^2 , \tag{2.2}$$

Then the number $n$ given by

$$n = 2^{[(\alpha_1+1)/2]} 3^{\alpha_2} \ldots q_k^{\alpha_k} q_{k+1}^2 \tag{2.3}$$

satisfies the relation

$$n = \frac{a_\ell}{2^{[\alpha_1/2]}} q_{k+1}^2 \tag{2.4}$$

on utilizing the relation $\alpha_1 = [\alpha_1/2] + [(\alpha_1+1)/2]$ for all positive integers $\alpha_1$. Hence, using (1.11) we get

$$n < a_\ell . \tag{2.5}$$

$$\delta(n) = \frac{2^{[(\alpha_1+1)/2]\delta(a_\ell)}}{\alpha_1} \geq \delta(a_\ell) . \tag{2.6}$$

This contradicts that $a_\ell$ is a h.p. number in view of (2.5), and the theorem follows.

2.7 <u>Theorem</u> For any h.p. numbers $a_\ell$ given by (1.7) we have

$$q_i^{[\alpha_i/2]} < q_{k+1}^2 \quad (1 \le i \le k) \tag{2.8}$$

Proof We proceed as in the proof of the last theorem. If for any $i$ $(1 \le i \le k)$, (2.8) is false, take the number $n$ given by

$$n = 2 \frac{a_\ell}{q_i^{[\alpha_i/2]}} q_{k+1}^2 \quad .$$

Then, $n < a_\ell$, since in (2.8) equality of both members is not possible, being [dist]inct prime powers. Also,

$$\delta(n) = 2^{[(\alpha_i+1)/2]} \delta(a_\ell)/\alpha_i \ge \delta(a_\ell) \quad ,$$

constituting a contradiction that $a_\ell$ is h.p.

2.9 Corollary $\alpha_k \le 5$, for $a_\ell > 2^7$.

Proof If $\alpha_i \ge 6$, then the inequality (2.8) would be false for $q_i^3 > q_{i+1}^2$ for $i \ge 2$.

2.10 Corollary: $\lim_{\ell\to\infty} k = \infty$.

2.11 Theorem In any h.p. number $a_\ell$, atmost one exponent in the prime power decomposition is 2, that is, $\alpha_i > 2$ for all $i < k$ in the notation (1.7).

Proof. If $p^2$ and $q^2$ occur as factors of a h.p. number $a$, where $p,q$ are primes and $p < q$, then take the number $\hat{a}$ obtained from $a$ by replacing the product $p^2q^2$ occuring in $a$ by $p^4$. Then $\hat{a} < a$, but $\delta(a) = \delta(\hat{a})$, a contradiction that $a$ is h.p.

2.12 Theorem $\lim_{\ell\to\infty} \alpha_i = \infty$ $(i = 1,2,\ldots)$.

Proof Suppose the theorem is false. Then for some particular $i$, there exists an infinite subsequence of h.p. numbers, say, $\bar{a}_1, \bar{a}_2, \ldots$, for which $\alpha_i'$s are bounded by $\beta$, say (here $\alpha_i$ denotes the power of the ith prime in the canonical expansion of $\bar{a}_1, \bar{a}_2, \ldots$; $\alpha_i$, of course varies with each member of this sequence).

Choose $q_k$ so that $q_i^{\beta^2} < q_k^2$. By corollary 2.10, we can find a $\bar{a}_\ell$ so that $q_k | \bar{a}_\ell$. Consider the number $n$ given by

$$n = (\bar{a}_\ell q_i^{\beta^2})/(q_k^{\alpha_k} q_i^{\alpha_i}) \tag{2.13}$$

Then

$$\delta(n) = \delta(\bar{a}_\ell)\beta^2/(\alpha_i\alpha_k) \geq \delta(\bar{a}_\ell) ,$$

since $\beta^2 \geq \alpha_i\alpha_k$. But $n < \bar{a}_\ell$. Hence this contradicts that $\bar{a}_\ell$ is a h.p. number, and the theorem follows.

2.14 Theorem Let $a_\ell$ be a h.p. number with $\alpha_k = 2$, say

$$a_\ell = 2^{\alpha_1} 3^{\alpha_2} \ldots q_k^2 , \tag{2.15}$$

and $i$ $(1 \leq i \leq k-1)$ be fixed. Let $r = r(i)$ be the positive integer given by

$$r = \left[\frac{\log q_k}{\log q_i}\right] + 1 . \tag{2.16}$$

Then

$$q_i^{\alpha_i} > q_k^2 \tag{2.17}$$

and

(2.18) $$\alpha_i < 3r$$

Remark By (2.16) we have

(2.19) $$q_i^{r-1} < q_k < q_i^r \quad .$$

Proof of the theorem. (a) Assume

(2.20) $$q_i^{\alpha_i} < q_k^2 \quad .$$

Let $\hat{a}$ be given by

$$\hat{a} = a_\ell \, q_i^{\alpha_i}/q_k^2$$

Then $\hat{a} < a_\ell$ and also

$$\delta(\hat{a}) = \delta(a_\ell)\,\frac{2}{2} = \delta(a_\ell)$$

But this is a contradiction, since $a_\ell$ is a h.p. number. Hence (2.20) cannot hold.

(b) Assume

(2.21) $$\alpha_i \geq 3r$$

Take the number $\hat{a} = a_\ell q_k/q_i^r$ .

Then $\hat{a} < a_\ell$ on using (2.19) .

Also

$$\delta(\hat{a}) = (3/2)\left(\frac{\alpha_i - r}{\alpha_i}\right)\delta(a_\ell) \quad ,$$

we claim

$$\frac{3}{2}\left(\frac{\alpha_i - r}{\alpha_i}\right) \geq 1 \, ,$$

that is

$$3\alpha_i - 3r \geq 2\alpha_i \quad ,$$

that is

$$\alpha_i \geq 3r \quad ,$$

which is what we assume. Thus $\delta(\hat{a}) \geq \delta(a_\ell)$ , leading again to a contradiction.

Thus the theorem is established.

2.22 Remark Theorem 2.11 follows immediately from the above result, specifically from (2.17) , since for (2.17) to hold $\alpha_i$ should be at least 3 .

2.23 Theorem Let $a_\ell$ given by (1.7) be a h.p. number. For each pair of primes $q_i$ , $q_j$ $(1 \leq i < j \leq k)$ let $r = r(q_i, q_j)$ be defined by

$$r = \left[\frac{\log q_j}{\log q_i}\right] + 1 \quad .$$

Then we have

$$(2.24) \qquad (\alpha_j - 1)(r-1) < \alpha_i < (\alpha_j + 1)r \quad .$$

Remark. $r$ is equivalently defined by

$$(2.25) \qquad q_i^{r-1} < q_j < q_i^r \quad .$$

Proof (a). Assume

$$(2.26) \qquad \alpha_i \le (\alpha_j - 1)(r-1)$$

Take $\hat{a}$ given by

$$(2.27) \qquad \hat{a} = a_\ell \, q_i^{r-1}/q_j$$

Then

$$(2.28) \qquad \hat{a} < a_\ell$$

by (2.25) .

$$\delta(\hat{a}) = \delta(a) \left( \frac{\alpha_i + r - 1}{\alpha_i} \right) \left( \frac{\alpha_j - 1}{\alpha_j} \right)$$

we claim that

$$\left( \frac{\alpha_i + r - 1}{\alpha_i} \right) \left( \frac{\alpha_j - 1}{\alpha_j} \right) \ge 1 \quad .$$

This follows routinely from (2.26) . Thus $\delta(\hat{a}) \geq \delta(a)$ , which together with (2.28) lead to a contradiction.

(b). Assume

(2.29) $$\alpha_i \geq (\alpha_j+1)r \quad .$$

Let

$$\hat{a} = a_\ell q_j / q_i^r \quad .$$

Then, using (2.25) we have

(2.30) $$\hat{a} < a_\ell$$

Also

$$\delta(\hat{a}) = \delta(a_\ell) \left( \frac{\alpha_j+1}{\alpha_j} \right) \left( \frac{\alpha_i - r}{\alpha_i} \right)$$
$$\geq \delta(a_\ell)$$

by (2.30) . Hence we have a contradiction in view of (2.30) .

The theorem now follows.

2.31 Theorem If $a_\ell$ given by (1.7) is a h.p. number, and if

(2.32) $$\frac{q_{k+1}}{2^{[\alpha_1/9]/2}} < q_{j-1} < q_j < q_k \quad ,$$

then $$\alpha_j = 3 \quad .$$

Proof. Assume $\alpha_j > 3$ . Then $\alpha_i \geq \alpha_j \geq 4$ . Consider

$$\hat{a} = \frac{q^2_{k+1} a_\ell}{2^{[\alpha_1/9]} q_{j-1} q_j}$$

Then

$$\hat{a} < a_\ell \quad .$$

and

$$\begin{aligned}\delta(a) &= \delta(a_\ell)\left(\frac{\alpha_1 - [\alpha_1/9]}{\alpha_1}\right)\left(\frac{\alpha_{j-1}-1}{\alpha_{j-1}}\right)\left(\frac{\alpha_j - 1}{\alpha_j}\right)2 \\ &\geq \delta(a_\ell)\left(\frac{\alpha_1 - \alpha_1/9}{\alpha_1}\right)(3/4)^2\, 2 \\ &= \delta(a_\ell) \quad .\end{aligned}$$

We used here the fact that since $\alpha_{j-1}, \alpha_j$ are $\geq 4$ , $\frac{\alpha_{j-1}-1}{\alpha_{j-1}}$ , $\frac{\alpha_j - 1}{\alpha_j}$ are $\geq 3/4$ . Hence $\delta(\hat{a}) \geq \delta(a_\ell)$ while $\hat{a} < a_\ell$ , a contradiction. Hence $\alpha_j \leq 3$ .

But by theorem 2.11 , not nore than one exponent can be 2 . Hence $\alpha_j = 3$ .

2.33 Corollary For any fixed positive integer $r$ , $3^r$ divides $\delta(a_\ell)$ for sufficiently large $\ell$ ; all h.p. numbers after a stage have last exponents a string of 3's (possibly ending with a single 2) .

Proof: This follows from Theorem 2.31 , Theorem 2.12 and the prime number theorem.

## 3. Estimates for $(a_{\ell+1} - a_\ell)$ and $\delta(a_{\ell+1}) - \delta(a_\ell)$ .

3.1 Lemma. Let $a_\ell$ be the $\ell$th highly powerful number, given by (1.7) . For any fixed positive $s$ , we have

$$\frac{\alpha_1}{(\delta(a_\ell))^{1/s}} = 0(1) \quad (\ell \to \infty) \quad .$$

Proof. By (2.13) we have

$$\alpha_1 \to \infty \quad \text{as} \quad \ell \to \infty \quad .$$

Now by the prime number theorems (or Chebychev inequality) , $q_k = 0\ (k \log k)$ . Also, (2.16) and (2.18) give

$$\alpha_1 < 3[\log q_k/\log 2] + 1 = 0\ (\log k)$$

But since for each $i$ , $\alpha_i \geq 2$ , we have

$$\delta(a_\ell) > 2^k \quad .$$

Hence

$$\frac{\alpha_1}{(\delta(a_\ell))^{1/s}} < \frac{0(\log k)}{2^{k/s}} = 0(1) \quad \text{as} \quad \ell \to \infty \quad .$$

(3.2) <u>Lemma</u>. (Dickman [4] ; Norton [6] pp.8-10). Let $\psi(x,y)$ denote the number of positive integers $\leq x$ with no prime factor bigger than $y$ . Then for every $\alpha \geq 1$ ,

(3.3) $$\psi(x, x^{1/\alpha}) \sim \rho(\alpha)x$$

where $\rho(\alpha)$ is defined by:

$$\rho(\alpha) = 1 \quad , \quad 0 \le \alpha \le 1 \quad ; \quad \text{and}$$

$$\rho(\alpha) = \rho(N) - \int_N^{\alpha} V^{-1}\rho(V-1)dV \quad , \quad N < \alpha \le N+1$$
$$(N = 1,2,....)$$

Furthermore (Chowla and Vijayaraghavan [3] ; Norton [6] p.10)

(3.4) $$\lim_{\alpha\to\infty} \rho(\alpha) = 0 \quad .$$

3.5 Theorem $\overline{\lim_{\ell\to\infty}}\ \delta(a_{\ell+1})-\delta(a_\ell) = \infty$ .

Proof Choose any number $d > 0$ . We shall show that

$$\delta(a_{\ell+1})-\delta(a_\ell) > d$$

for some $\ell$ . By lemma 3.2 , there is an $\alpha > 0$ so that

$$\frac{\psi(x,x^{1/\alpha})}{x} \sim \frac{1}{d+1} \quad .$$

This follows from (3.3) , (3.4) and the continuity of $\rho(\alpha)$ . Hence we can find $N$ arbitrarily large so that for all $n$ satisfying $N \le n \le N + d - 1$ , $n$ has a prime factor $\ge (n)^{1/\alpha}$ .

Let $S(N)$ denote the sequence

$$S(N) = \{N, N+1,..., N+d-1\} .$$

Let $a_\ell$ be the smallest h.p. number with $\delta(a_\ell) \ge N$ . Then $\delta(a_{\ell-1}) < N$ . By lemma 3.1 , if $N$ is sufficiently large,

$$\alpha_1 < (\delta(a_\ell))^{1/\alpha} \quad .$$

Since $\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_k$, we see that $\alpha_1 \geq$ the largest prime factor of $\delta(a_\ell)$. Thus $\delta(a_\ell)$ is not a member of $S(N)$, and so

$$\delta(a_\ell) - \delta(a_{\ell-1}) > d \quad .$$

This completes the proof.

3.6 <u>Theorem</u> $\lim_{\ell\to\infty} (a_{\ell+1}-a_\ell) = \infty$

Proof. By Theorem 2.12, $\alpha_1$ grows without bound. Therefore, for any fixed $n$, for $\ell$ sufficiently large, $\alpha_1 > n$ for $a_\ell$, $a_{\ell+1}$, ..... Hence

$$2^n | a_\ell$$

$$2^n | a_{\ell+1}$$

Thus

$$2^n | (a_{\ell+1}-a_\ell) \quad .$$

It follows that $a_{\ell+1}-a_\ell \geq 2^n$ and this implies the theorem.

3.7 <u>Theorem</u> $\lim_{\ell\to\infty} \{\delta(a_{\ell+1})-\delta(a_\ell)\} = \infty$

<u>Proof</u>. By corollary 2.33, for any given $r$, for sufficiently large $\ell$, $3^r | \delta(a_\ell)$, and $3^r | \delta(a_{\ell+1})$. Thus

$$3^r | (\delta(a_{\ell+1})-\delta(a_\ell))$$

and the theorem follows.

3.8 Remark The above theorem is a stronger result than theorem 3.5.

3.9 Theorem $\lim_{\ell\to\infty} \frac{a_{\ell+1}}{a_\ell} = 1$ .

Proof. Suppose the theorem is false, then there exists an $r > 1$ and an infinite sequence $\bar{a}_1 , \bar{a}_2 , \ldots$ (which is a subsequence of the sequence of h.p. numbers $a_1 , a_2 , \ldots$) , satisfying

$$a_i^*/\bar{a}_i > r \quad (i = 1,2,3,\ldots) ,$$

$a_i^*$ is the element immediately following $\bar{a}_i$ in the sequence of all h.p. numbers.

Consider any element $\bar{a}$ in the sequence $\{\bar{a}_1 , \bar{a}_2 , \ldots\}$; say,

$$a = 2^{\alpha_1} 3^{\alpha_2} \ldots q_i^{\alpha_i} \ldots q_j^{\alpha_j} \ldots q_k^{\alpha_k} .$$

If $q_j/q_i < r$ , we claim

$$\alpha_i - \alpha_j \le 1 \tag{3.10}$$

If this is not the case, let

$$\hat{a} = \bar{a}$$

$$\hat{a} = \bar{a}\, q_j/q_i$$

Then

$$\hat{a} < r\,\bar{a}$$

$$\delta(\hat{a}) = \frac{\alpha_i - 1}{\alpha_i}\ \frac{\alpha_j - 1}{\alpha_j}\ \delta(\bar{a})$$

$$= (1 + \frac{\alpha_i - \alpha_j - 1}{\alpha_i \alpha_j})\,\delta(\bar{a})$$

This is $> \delta(\bar{a})$ on using $\alpha_i - \alpha_j > 1$ .

Hence

$$\frac{a^*}{\bar{a}} \le \frac{\hat{a}}{\bar{a}} < r \quad ,$$

contradicting the defining property of $\bar{a}$ .

Thus our claim in (3.10) is established. By the prime number theorem, there is an infinite sequence of primes

$$p_1\ ,\ p_2\ ,\ p_3\ ,\ .\ .\ .$$

so that

$$\text{(3.11)} \qquad \frac{1+r}{2} < \frac{p_{i+1}}{p_i} < r \qquad (i = 1,2,\ldots) \quad .$$

Clearly, (3.11) gives

$$p_{m+1} > (\frac{1+r}{2})^m \rho_1 \qquad (m = 1,2,3,\ldots)$$

Furthermore, if

$$p_{m+1} < q_k \quad ,$$

(where $q_k$ is the largest prime dividing $\bar{a}$ ) , and letting $\alpha_i$ be the exponent of

$p_i$ in the canonical expansion of $\bar{a}$ , $i \geq 1,2,\ldots, m+1$ ,

we get

$$\bar{\alpha}_1 - \bar{\alpha}_{m+1} = (\bar{\alpha}_1-\bar{\alpha}_2)+(\bar{\alpha}_2-\bar{\alpha}_3)+\ldots+(\bar{\alpha}_m-\bar{\alpha}_{m+1})$$

By applying (3.10) to each term in the right-hand side of the above equation, we find

$$(3.12) \qquad \bar{\alpha}_1 - \bar{\alpha}_{m+1} \leq m \quad .$$

Choose $m$ so large that

$$\left( \frac{1+r}{2} \right)^m > p_1 \quad .$$

Thus

$$(3.13) \qquad p_{m+1} > p_1^2$$

Choose $\bar{a}$ so large that

$$(3.14) \qquad \bar{\alpha}_{m+1} \geq 3 \quad \text{while} \quad \bar{\alpha}_1 > 2(m+1) \quad .$$

By theorem 2.23 ,

$$\bar{\alpha}_1 > ( \bar{\alpha}_{m+1}-1 )\left[ \frac{\log p_{m+1}}{\log p_1} \right]$$

$$> (\bar{\alpha}_{m+1}-1)2$$

on using (3.13) . But applying (3.12) to the above, we have

$$\bar{\alpha}_1 > 2(\bar{\alpha}_1 - m) - 2$$

This gives

$$\bar{\alpha}_1 < 2(m+1)$$

This contradicts our choice of $\bar{a}$ as given by (3.14), and thus the theorem is established.

We shall next establish

3.15 <u>Theorem</u> $\lim_{\ell\to\infty} \delta(a_{\ell+1})/\delta(a_\ell) = 1$.

we first need the following

3.16 <u>Lemma</u>. Let

(3.17) $$a_\ell = 2^{\alpha_1} 3^{\alpha_2} \dots q_k^{\alpha_k}$$

be a h.p. number; if

(3.18) $$\alpha_1 \geq (\alpha_i + 1)^2 \quad (i = 2,3,\dots,k) \quad ,$$

then for all $q_j (j = 1,2,\dots,i-1)$ satisfying

(3.19) $$q_j > \frac{1}{2} q_i \quad (j = 1,2,\dots i-1)$$

we have

(3.20) $$\alpha_j - \alpha_i \leq 1 \quad .$$

*Proof*. If $\alpha_j - \alpha_i > 1$ , then define $\hat{a}$ by

$$\hat{a} = (a_\ell q_i)/2q_j$$

Then

$$\hat{a} < a_\ell \quad ;$$

$$\delta(\hat{a}) = \delta(a)\left(\frac{\alpha_1 - 1}{\alpha_1}\right)\left(\frac{\alpha_j - 1}{\alpha_j}\right)\left(\frac{\alpha_i + 1}{\alpha_i}\right)$$

$$= \delta(a_\ell)\left(\frac{A-1}{A}\right)\left(\frac{B-1}{B}\right)\left(\frac{C+1}{C}\right) \quad , \quad \text{say}$$

where $A = \alpha_1$, $B = \alpha_{i-1}$ , $C = \alpha_i$ , all $\geq 2$ . We claim that

$$\left(\frac{A-1}{A}\right)\left(\frac{B-1}{B}\right)\left(\frac{C+1}{C}\right) \geq 1 \quad . \tag{3.21}$$

This is the same as

$$(A-1)(B-1)(C+1) \geq ABC \; ;$$

equivalently,

$$A(B-C-1) \geq BC + B - C - 1 \quad , \tag{3.22}$$

or

$$A \geq \frac{BC}{B-C-1} + 1$$

(Note that $B-C-1 > 1$) .

Let

$$f(B) = \frac{BC}{B-C-1} \quad .$$

Then

$$f'(B) < 0 \text{ and so } f \text{ is decreasing.}$$

$f$ is maximum when $B = C+2$, since $B-C \geq 2$. A simple calculation shows that

$$\text{Max } f(B) = C^2 + 2C$$

It is therefore sufficient to show that

$$A \geq C^2 + 2C + 1 \quad ,$$

that is

$$\alpha_1 \geq (\alpha_i + 1)^2 \quad ,$$

which establishes our claim (3.21).

Hence

$$\delta(\hat{a}) \geq \delta(a_\ell) \quad ;$$

this is a contradiction of the definition of $a_\ell$. The lemma follows.

3.23 Corollary. For any fixed $m \geq 3$ for $a_\ell$ sufficiently large, there exist $\alpha_i = m$ and $\alpha_{i+i} = m - 1$.

Proof. Choose $a_\ell$ so large that

$$2^{(4m+1)^2} < p_{k-1}$$

and further that there is a prime $p_i$ satisfying

$$p_i^{m-1} < p_{k-1} < p_i^m \quad ,$$

By theorem (2.23), since $\alpha_{k-1} = 3$

$$m < 2(m-1) < \alpha_i < 4m \quad .$$

Applying lemma 3.16, we see that $\alpha_i$, $\alpha_{i+1}$, ..., $\alpha_{k-1}$ is a non-increasing sequence of integers with $(\alpha_j - \alpha_{j+1}) \le 1$, $\alpha_i > m$, $\alpha_{k-1} = 3$. Thus for some $j$, $\alpha_j = m$ and $\alpha_{j+1} = m-1$.

3.24 <u>Theorem</u> $\lim_{\ell\to\infty} \delta(a_{\ell+1})/\delta(a_\ell) = 1$

<u>Proof</u>. If the theorem is false, there exists a $d$ with $0 < d < 1$ and an [in]finite sequence $a_{\ell_1}$, $a_{\ell_2}$, .... so that

(3.25)
$$\frac{\delta(a_{\ell_i+1})}{\delta(a_{\ell_i})} > 1 + d.$$

Choose $m$ so large that

(3.26)
$$4/(m^2+m) < d \quad .$$

Choose $a_{\ell_i+1}$ so large that there is an $\alpha_j = m + 1$ and $\alpha_{j+1} = m$.

Let

$$\hat{a} = a_{\ell_i+1}(\,p_j/p_{j+1}\,)$$

Then

$$\hat{a} < a_{\ell_i+1}$$

Also

$$\delta(\hat{a}) = \delta(a_{\ell_i+1})\left(\frac{m+2}{m+1}\right)\left(\frac{m-1}{m}\right)$$

$$= \left(1 - \frac{2}{(m+1)m}\right)\delta(a_{\ell_i+1})$$

$$> \left(1 - \frac{2}{m(m+1)}\right)(1+d)\delta(a_{\ell_i}) \quad ,$$

on using (3.25) . Now using $1 + d < 2$ , we get

$$\delta(\hat{a}) > \left(1 + d - \frac{4}{m^2+m}\right)\delta(a_{\ell_i})$$

$$> \delta(a_{\ell_i}) \quad ,$$

recalling our choice of $m$ given by (3.26) . Thus if $\hat{a} > a_{\ell_i}$ , then

$$a_{\ell_i+1} \le \hat{a} \quad ,$$

contradicting (3.27) . If $\hat{a} < a_{\ell_i}$ , we contradict the definition of $a_{\ell_i}$ . (Note that $\hat{a} \neq a_{\ell_i}$ because $\delta(\hat{a}) \neq \delta(a_{\ell_i})$) .

The theorem is thus established.

Remarks. Professors Carole Lacampagne and John Selfridge have informed the author that they have proved that there are only nineteen highly powerful numbers which end in the square of a prime. These are given below.

$2^2$
$2^4 \cdot 3^2$
$2^5 \cdot 3^2$
$2^7 \cdot 3^3 \cdot 5^2$
$2^6 \cdot 3^4 \cdot 5^2$
$2^5 \cdot 3^5 \cdot 5^2$
$2^7 \cdot 3^4 \cdot 5^2$
$2^8 \cdot 3^4 \cdot 5^2$
$2^7 \cdot 3^5 \cdot 5^3 \cdot 7^2$
$2^7 \cdot 3^4 \cdot 5^4 \cdot 7^2$
$2^8 \cdot 3^5 \cdot 5^3 \cdot 7^2$
$2^8 \cdot 3^4 \cdot 5^4 \cdot 7^2$

$2^9 \cdot 3^6 \cdot 5^4 \cdot 7^3 \cdot 11^2$
$2^{11} \cdot 3^7 \cdot 5^4 \cdot 7^3 \cdot 11^3 \cdot 13^2$
$2^{11} \cdot 3^6 \cdot 5^5 \cdot 7^4 \cdot 11^3 \cdot 13^3 \cdot 17^2$
$2^{10} \cdot 3^7 \cdot 5^5 \cdot 7^4 \cdot 11^3 \cdot 13^3 \cdot 17^2$
$2^{11} \cdot 3^7 \cdot 5^5 \cdot 7^4 \cdot 11^3 \cdot 13^3 \cdot 17^2$
$2^{11} \cdot 3^7 \cdot 5^5 \cdot 7^4 \cdot 11^3 \cdot 13^3 \cdot 17^3 \cdot 19^2$
$2^{11} \cdot 3^8 \cdot 5^5 \cdot 7^4 \cdot 11^3 \cdot 13^3 \cdot 17^3 \cdot 19^2$

Results concerning an estimate for the number of highly powerful numbers not exceeding $x$, and related matters will be dealt with in a separate paper.

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Powerful Numbers $a_\ell = 2^{\alpha_1} 3^{\alpha_2} 5^{\alpha_3} 7^{\alpha_4} 11^{\alpha_5} \cdots$ up to $10^{27}$

| $a_\ell$ | $\delta(a_\ell)$ | $\alpha_1$ | $\alpha_2$ | $\alpha_3$ | $\alpha_4$ | $\alpha_5$ | ... |
|---|---|---|---|---|---|---|---|
| 4 | 2 | 2 | | | | | |
| 8 | 3 | 3 | | | | | |
| 16 | 4 | 4 | | | | | |
| 32 | 5 | 5 | | | | | |
| 64 | 6 | 6 | | | | | |
| 128 | 7 | 7 | | | | | |
| 144 | 8 | 4 | 2 | | | | |
| 216 | 9 | 3 | 3 | | | | |
| 288 | 10 | 5 | 2 | | | | |
| 432 | 12 | 4 | 3 | | | | |
| 864 | 15 | 5 | 3 | | | | |
| 1296 | 16 | 4 | 4 | | | | |
| 1728 | 18 | 6 | 3 | | | | |
| 2592 | 20 | 5 | 4 | | | | |
| 3456 | 21 | 7 | 3 | | | | |
| 5184 | 24 | 6 | 4 | | | | |
| 7776 | 25 | 5 | 5 | | | | |
| 10368 | 28 | 7 | 4 | | | | |
| 15552 | 30 | 6 | 5 | | | | |
| 20736 | 32 | 8 | 4 | | | | |
| 31104 | 35 | 7 | 5 | | | | |
| 41472 | 36 | 9 | 4 | | | | |
| 62208 | 40 | 2 | 5 | | | | |
| 86400 | 42 | 7 | 3 | 2 | | | |
| 108000 | 45 | 5 | 3 | 3 | | | |
| 129600 | 48 | 6 | 4 | 2 | | | |
| 194400 | 50 | 5 | 5 | 2 | | | |
| 216000 | 54 | 6 | 3 | 3 | | | |
| 259200 | 56 | 7 | 4 | 2 | | | |
| 324000 | 60 | 5 | 4 | 3 | | | |
| 432000 | 63 | 7 | 3 | 3 | | | |
| 518400 | 64 | 8 | 4 | 2 | | | |
| 648000 | 72 | 6 | 4 | 3 | | | |
| 972000 | 75 | 5 | 5 | 3 | | | |
| 1296000 | 84 | 7 | 4 | 3 | | | |
| 1944000 | 90 | 6 | 5 | 3 | | | |
| 2592000 | 96 | 8 | 4 | 3 | | | |
| 3888000 | 105 | 7 | 5 | 3 | | | |
| 5184000 | 108 | 9 | 4 | 3 | | | |
| 6480000 | 112 | 7 | 4 | 4 | | | |
| 7776000 | 120 | 8 | 5 | 3 | | | |
| 11664000 | 126 | 7 | 6 | 3 | | | |
| 12960000 | 128 | 8 | 4 | 4 | | | |
| 15552000 | 135 | 9 | 5 | 3 | | | |
| 19440000 | 140 | 7 | 5 | 4 | | | |
| 23328000 | 144 | 8 | 6 | 3 | | | |
| 31104000 | 150 | 10 | 5 | 3 | | | |
| 38880000 | 160 | 8 | 5 | 4 | | | |

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|---|---|---|---|---|---|---|
| | 46656000 | 162 | 9 | 6 | 3 | |
| | 58320000 | 168 | 7 | 6 | 4 | |
| | 77760000 | 180 | 9 | 5 | 4 | |
| | 116640000 | 192 | 8 | 6 | 4 | |
| | 155520000 | 200 | 10 | 5 | 4 | |
| | 190512000 | 210 | 7 | 5 | 3 | 2 |
| | 222264000 | 216 | 6 | 4 | 3 | 3 |
| | 311040000 | 220 | 11 | 5 | 4 | |
| | 317520000 | 224 | 7 | 4 | 4 | 2 |
| | 333396000 | 225 | 5 | 5 | 3 | 3 |
| | 381024000 | 240 | 8 | 5 | 3 | 2 |
| | 444528000 | 252 | 7 | 4 | 3 | 3 |
| | 635040000 | 256 | 8 | 4 | 4 | 2 |
| | 666792000 | 270 | 6 | 5 | 3 | 3 |
| | 889056000 | 288 | 8 | 4 | 3 | 3 |
| 1 | 333584000 | 315 | 7 | 5 | 3 | 3 |
| 1 | 778112000 | 324 | 9 | 4 | 3 | 3 |
| 2 | 222640000 | 336 | 7 | 4 | 4 | 3 |
| 2 | 667168000 | 360 | 8 | 5 | 3 | 3 |
| 4 | 000752000 | 378 | 7 | 6 | 3 | 3 |
| 4 | 445280000 | 384 | 8 | 4 | 4 | 3 |
| 5 | 334336000 | 405 | 9 | 5 | 3 | 3 |
| 6 | 667920000 | 420 | 7 | 5 | 4 | 3 |
| 8 | 001504000 | 432 | 8 | 6 | 3 | 3 |
| 10 | 668672000 | 450 | 10 | 5 | 3 | 3 |
| 13 | 335840000 | 480 | 8 | 5 | 4 | 3 |
| 16 | 003008000 | 486 | 9 | 6 | 3 | 3 |
| 20 | 003760000 | 504 | 7 | 6 | 4 | 3 |
| 26 | 671680000 | 540 | 9 | 5 | 4 | 3 |
| 40 | 007520000 | 576 | 8 | 6 | 4 | 3 |
| 53 | 343360000 | 600 | 10 | 5 | 4 | 3 |
| 80 | 015040000 | 648 | 9 | 6 | 4 | 3 |
| 106 | 686720000 | 660 | 11 | 5 | 4 | 3 |
| 120 | 022560000 | 672 | 8 | 7 | 4 | 3 |
| 133 | 358400000 | 675 | 9 | 5 | 5 | 3 |
| 160 | 030080000 | 720 | 10 | 6 | 4 | 3 |
| 240 | 045120000 | 756 | 9 | 7 | 4 | 3 |
| 280 | 052640000 | 768 | 8 | 6 | 4 | 4 |
| 320 | 060160000 | 792 | 11 | 6 | 4 | 3 |
| 373 | 403520000 | 800 | 10 | 5 | 4 | 4 |
| 400 | 075200000 | 810 | 9 | 6 | 5 | 3 |
| 480 | 090240000 | 840 | 10 | 7 | 4 | 3 |
| 560 | 105280000 | 864 | 9 | 6 | 4 | 4 |
| 746 | 807040000 | 880 | 11 | 5 | 4 | 4 |
| 800 | 150400000 | 900 | 10 | 6 | 5 | 3 |
| 960 | 180480000 | 924 | 11 | 7 | 4 | 3 |
| 1120 | 210560000 | 960 | 10 | 6 | 4 | 4 |
| 1600 | 300800000 | 990 | 11 | 6 | 5 | 3 |
| 1680 | 315840000 | 1008 | 9 | 7 | 4 | 4 |
| 2240 | 421120000 | 1056 | 11 | 6 | 4 | 4 |
| 2800 | 526400000 | 1080 | 9 | 6 | 5 | 4 |
| 3360 | 631680000 | 1120 | 10 | 7 | 4 | 4 |

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|---|---|---|---|---|---|---|---|
| 4480 | 842240000 | 1152 | 12 | 6 | 4 | 4 | |
| 4800 | 902400000 | 1155 | 11 | 7 | 5 | 3 | |
| 5601 | 052800000 | 1200 | 10 | 6 | 5 | 4 | |
| 6721 | 263360000 | 1232 | 11 | 7 | 4 | 4 | |
| 8401 | 579200000 | 1260 | 9 | 7 | 5 | 4 | |
| 9681 | 819840000 | 1296 | 9 | 6 | 4 | 3 | 2 |
| 11202 | 105600000 | 1320 | 11 | 6 | 5 | 4 | |
| 13442 | 526720000 | 1344 | 12 | 7 | 4 | 4 | |
| 14200 | 002432000 | 1350 | 10 | 5 | 3 | 3 | 3 |
| 16803 | 158400000 | 1400 | 10 | 7 | 5 | 4 | |
| 17750 | 003040000 | 1440 | 8 | 5 | 4 | 3 | 3 |
| 21300 | 003648000 | 1458 | 9 | 6 | 3 | 3 | 3 |
| 26625 | 004560000 | 1512 | 7 | 6 | 4 | 3 | 3 |
| 33606 | 316800000 | 1540 | 11 | 7 | 5 | 4 | |
| 35500 | 006080000 | 1620 | 9 | 5 | 4 | 3 | 3 |
| 53250 | 009120000 | 1728 | 8 | 6 | 4 | 3 | 3 |
| 71000 | 012160000 | 1800 | 10 | 5 | 4 | 3 | 3 |
| 106500 | 018240000 | 1944 | 9 | 6 | 4 | 3 | 3 |
| 142000 | 024320000 | 1980 | 11 | 5 | 4 | 3 | 3 |
| 159750 | 027360000 | 2016 | 8 | 7 | 4 | 3 | 3 |
| 177500 | 030400000 | 2025 | 9 | 5 | 5 | 3 | 3 |
| 213000 | 036480000 | 2160 | 10 | 6 | 4 | 3 | 3 |
| 319500 | 054720000 | 2268 | 9 | 7 | 4 | 3 | 3 |
| 372750 | 063840000 | 2304 | 8 | 6 | 4 | 4 | 3 |
| 426000 | 072960000 | 2376 | 11 | 6 | 4 | 3 | 3 |
| 497000 | 085120000 | 2400 | 10 | 5 | 4 | 4 | 3 |
| 532500 | 091200000 | 2430 | 9 | 6 | 5 | 3 | 3 |
| 639000 | 109440000 | 2520 | 10 | 7 | 4 | 3 | 3 |
| 745500 | 127680000 | 2592 | 9 | 6 | 4 | 4 | 3 |
| 994000 | 170240000 | 2640 | 11 | 5 | 4 | 4 | 3 |
| 1065000 | 182400000 | 2700 | 10 | 6 | 5 | 3 | 3 |
| 1278000 | 218880000 | 2772 | 11 | 7 | 4 | 3 | 3 |
| 1491000 | 255360000 | 2880 | 10 | 6 | 4 | 4 | 3 |
| 2130000 | 364800000 | 2970 | 11 | 6 | 5 | 3 | 3 |
| 2236500 | 383040000 | 3024 | 9 | 7 | 4 | 4 | 3 |
| 2982000 | 510720000 | 3168 | 11 | 6 | 4 | 4 | 3 |
| 3727500 | 638400000 | 3240 | 9 | 6 | 5 | 4 | 3 |
| 4473000 | 766080000 | 3360 | 10 | 7 | 4 | 4 | 3 |
| 5964001 | 021440000 | 3456 | 12 | 6 | 4 | 4 | 3 |
| 6390001 | 094400000 | 3465 | 11 | 7 | 5 | 3 | 3 |
| 7455001 | 276800000 | 3600 | 10 | 6 | 5 | 4 | 3 |
| 8946001 | 532160000 | 3696 | 11 | 7 | 4 | 4 | 3 |
| 11182501 | 915200000 | 3780 | 9 | 7 | 5 | 4 | 3 |
| 13419002 | 298240000 | 3840 | 10 | 8 | 4 | 4 | 3 |
| 14910002 | 553600000 | 3960 | 11 | 6 | 5 | 4 | 3 |
| 17892003 | 064320000 | 4032 | 12 | 7 | 4 | 4 | 3 |
| 22365003 | 830400000 | 4200 | 10 | 7 | 5 | 4 | 3 |
| 26838004 | 596480000 | 4224 | 11 | 8 | 4 | 4 | 3 |
| 29820005 | 107200000 | 4320 | 12 | 6 | 5 | 4 | 3 |
| 35784006 | 128640000 | 4368 | 13 | 7 | 4 | 4 | 3 |
| 44730007 | 660800000 | 4620 | 11 | 7 | 5 | 4 | 3 |
| 59640010 | 214400000 | 4680 | 13 | 6 | 5 | 4 | 3 |

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|---|---|---|---|---|---|---|---|---|
| 67095011 | 491200000 | 4800 | 10 | 8 | 5 | 4 | 3 | |
| 77993513 | 357760000 | 4860 | 9 | 5 | 4 | 3 | 3 | 3 |
| 89460015 | 321600000 | 5040 | 12 | 7 | 5 | 4 | 3 | |
| 116990270 | 036640000 | 5184 | 8 | 6 | 4 | 3 | 3 | 3 |
| 134190022 | 982400000 | 5280 | 11 | 8 | 5 | 4 | 3 | |
| 155987026 | 715520000 | 5400 | 10 | 5 | 4 | 3 | 3 | 3 |
| 178920030 | 643200000 | 5460 | 13 | 7 | 5 | 4 | 3 | |
| 215982036 | 990720000 | 5544 | 11 | 7 | 4 | 3 | 3 | 2 |
| 233980540 | 073280000 | 5832 | 9 | 6 | 4 | 3 | 3 | 3 |
| 311974053 | 431040000 | 5940 | 11 | 5 | 4 | 3 | 3 | 3 |
| 350970810 | 109920000 | 6048 | 8 | 7 | 4 | 3 | 3 | 3 |
| 389967566 | 788800000 | 6075 | 9 | 5 | 5 | 3 | 3 | 3 |
| 467961080 | 146560000 | 6480 | 10 | 6 | 4 | 3 | 3 | 3 |
| 701941620 | 219840000 | 6804 | 9 | 7 | 4 | 3 | 3 | 3 |
| 818931890 | 256480000 | 2912 | 8 | 6 | 4 | 4 | 3 | 3 |
| 935922160 | 293120000 | 7128 | 11 | 6 | 4 | 3 | 3 | 3 |
| 1 091909187 | 008640000 | 7200 | 10 | 5 | 4 | 4 | 3 | 3 |
| 1 169902700 | 366400000 | 7290 | 9 | 6 | 5 | 3 | 3 | 3 |
| 1 403883240 | 439680000 | 7560 | 10 | 7 | 4 | 3 | 3 | 3 |
| 1 637863780 | 512960000 | 7776 | 9 | 6 | 4 | 4 | 3 | 3 |
| 2 183818374 | 017280000 | 7920 | 11 | 5 | 4 | 4 | 3 | 3 |
| 2 339805400 | 732800000 | 8100 | 10 | 6 | 5 | 3 | 3 | 3 |
| 2 807766480 | 879360000 | 8316 | 11 | 7 | 4 | 3 | 3 | 3 |
| 3 275727561 | 025920000 | 8640 | 10 | 6 | 4 | 4 | 3 | 3 |
| 4 679610801 | 465600000 | 8910 | 11 | 6 | 5 | 3 | 3 | 3 |
| 4 913591341 | 538880000 | 9072 | 9 | 7 | 4 | 4 | 3 | 3 |
| 6 551455122 | 051840000 | 9504 | 11 | 6 | 4 | 4 | 3 | 3 |
| 8 189318902 | 564800000 | 9720 | 9 | 6 | 5 | 4 | 3 | 3 |
| 9 827182683 | 077760000 | 10080 | 10 | 7 | 4 | 4 | 3 | 3 |
| 13 102910244 | 103680000 | 10368 | 12 | 6 | 4 | 4 | 3 | 3 |
| 14 038832404 | 396800000 | 10395 | 11 | 7 | 5 | 3 | 3 | 3 |
| 16 378637805 | 129600000 | 10800 | 10 | 6 | 5 | 4 | 3 | 3 |
| 19 654365366 | 155520000 | 11088 | 11 | 7 | 4 | 4 | 3 | 3 |
| 24 567956707 | 694400000 | 11340 | 9 | 7 | 5 | 4 | 3 | 3 |
| 29 481548049 | 233280000 | 11520 | 10 | 8 | 4 | 4 | 3 | 3 |
| 32 757275610 | 259200000 | 11880 | 11 | 6 | 5 | 4 | 3 | 3 |
| 39 308730732 | 311040000 | 12096 | 12 | 7 | 4 | 4 | 3 | 3 |
| 49 135913415 | 388800000 | 12600 | 10 | 7 | 5 | 4 | 3 | 3 |
| 58 963096098 | 466560000 | 12672 | 11 | 8 | 4 | 4 | 3 | 3 |
| 65 514551220 | 518400000 | 12960 | 12 | 6 | 5 | 4 | 3 | 3 |
| 78 617461464 | 622080000 | 13104 | 13 | 7 | 4 | 4 | 3 | 3 |
| 98 271826830 | 777600000 | 13860 | 11 | 7 | 5 | 4 | 3 | 3 |
| 131 029102441 | 036800000 | 14040 | 13 | 6 | 5 | 4 | 3 | 3 |
| 147 407740246 | 166400000 | 14400 | 10 | 8 | 5 | 4 | 3 | 3 |
| 196 543653661 | 555200000 | 15120 | 12 | 7 | 5 | 4 | 3 | 3 |
| 294 815480492 | 332800000 | 15840 | 11 | 8 | 5 | 4 | 3 | 3 |
| 393 087307323 | 110400000 | 16380 | 13 | 7 | 5 | 4 | 3 | 3 |
| 491 359134153 | 888000000 | 16632 | 11 | 7 | 6 | 4 | 3 | 3 |
| 540 495047569 | 276800000 | 16800 | 10 | 7 | 5 | 4 | 4 | 3 |
| 589 630960984 | 665600000 | 17280 | 12 | 8 | 5 | 4 | 3 | 3 |
| 687 902787815 | 443200000 | 17325 | 11 | 7 | 5 | 5 | 3 | 3 |
| 786 174614646 | 220800000 | 17640 | 14 | 7 | 5 | 4 | 3 | 3 |

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|---|---|---|---|---|---|---|---|---|---|---|
| 884 | 446441476 | 998400000 | 17820 | 11 | 9 | 5 | 4 | 3 | 3 | |
| 982 | 718268307 | 776000000 | 18144 | 12 | 7 | 6 | 4 | 3 | 3 | |
| 1080 | 990095138 | 553600000 | 18480 | 11 | 7 | 5 | 4 | 4 | 3 | |
| 1179 | 261921969 | 331200000 | 18720 | 13 | 8 | 5 | 4 | 3 | 3 | |
| 1375 | 805575630 | 886400000 | 18900 | 12 | 7 | 5 | 5 | 3 | 3 | |
| 1474 | 077402461 | 664000000 | 19008 | 11 | 8 | 6 | 4 | 3 | 3 | |
| 1621 | 485142707 | 830400000 | 19200 | 10 | 8 | 5 | 4 | 4 | 3 | |
| 1768 | 892882953 | 996800000 | 19440 | 12 | 9 | 5 | 4 | 3 | 3 | |
| 1965 | 436536615 | 552000000 | 19656 | 13 | 7 | 6 | 4 | 3 | 3 | |
| 2063 | 708363446 | 329600000 | 19800 | 11 | 8 | 5 | 5 | 3 | 3 | |
| 2161 | 980190277 | 107200000 | 20160 | 12 | 7 | 5 | 4 | 4 | 3 | |
| 2751 | 611151261 | 772800000 | 20475 | 13 | 7 | 5 | 5 | 3 | 3 | |
| 2948 | 154804923 | 328000000 | 20736 | 12 | 8 | 6 | 4 | 3 | 3 | |
| 3242 | 970285415 | 660800000 | 21120 | 11 | 8 | 5 | 4 | 4 | 3 | |
| 3930 | 873073231 | 104000000 | 21168 | 14 | 7 | 6 | 4 | 3 | 3 | |
| 4127 | 416726892 | 659200000 | 21600 | 12 | 8 | 5 | 5 | 3 | 3 | |
| 4323 | 960380554 | 214400000 | 21840 | 13 | 7 | 5 | 4 | 4 | 3 | |
| 5404 | 950475692 | 768000000 | 22176 | 11 | 7 | 6 | 4 | 4 | 3 | |
| 5896 | 309609846 | 656000000 | 22464 | 13 | 8 | 6 | 4 | 3 | 3 | |
| 6485 | 940570831 | 321600000 | 23040 | 12 | 8 | 5 | 4 | 4 | 3 | |
| 7566 | 930665969 | 875200000 | 23100 | 11 | 7 | 5 | 5 | 4 | 3 | |
| 8046 | 824753660 | 172480000 | 23328 | 9 | 6 | 4 | 4 | 3 | 3 | 3 |
| 8254 | 833453785 | 318400000 | 23400 | 13 | 8 | 5 | 5 | 3 | 3 | |
| 8647 | 920761108 | 428800000 | 23520 | 14 | 7 | 5 | 4 | 4 | 3 | |
| 9466 | 852651364 | 908800000 | 23760 | 11 | 6 | 5 | 4 | 3 | 3 | 2 |
| 10809 | 900951385 | 536000000 | 24192 | 12 | 7 | 6 | 4 | 4 | 3 | |
| 11495 | 463933800 | 246400000 | 24300 | 10 | 6 | 5 | 3 | 3 | 3 | 3 |
| 12971 | 881141662 | 643200000 | 24960 | 13 | 8 | 5 | 4 | 4 | 3 | |
| 14200 | 278977047 | 363200000 | 25200 | 10 | 7 | 5 | 4 | 3 | 3 | 2 |
| 16093 | 649507320 | 344960000 | 25920 | 10 | 6 | 4 | 4 | 3 | 3 | 3 |
| 21619 | 801902771 | 072000000 | 26208 | 13 | 7 | 6 | 4 | 4 | 3 | |
| 22700 | 791997909 | 625600000 | 26400 | 11 | 8 | 5 | 5 | 4 | 3 | |
| 22990 | 927867600 | 492800000 | 26730 | 11 | 6 | 5 | 3 | 3 | 3 | 3 |
| 24140 | 474260980 | 517440000 | 27216 | 9 | 7 | 4 | 4 | 3 | 3 | 3 |
| 28400 | 557954094 | 726400000 | 27720 | 11 | 7 | 5 | 4 | 3 | 3 | 2 |
| 32187 | 299014640 | 689920000 | 28512 | 11 | 6 | 4 | 4 | 3 | 3 | 3 |
| 40234 | 123768300 | 862400000 | 29160 | 9 | 6 | 5 | 4 | 3 | 3 | 3 |
| 48280 | 948521961 | 034880000 | 30240 | 10 | 7 | 4 | 4 | 3 | 3 | 3 |
| 64374 | 598029281 | 379840000 | 31104 | 12 | 6 | 4 | 4 | 3 | 3 | 3 |
| 68972 | 783602801 | 478400000 | 31185 | 11 | 7 | 5 | 3 | 3 | 3 | 3 |
| 80468 | 247536601 | 724800000 | 32400 | 10 | 6 | 5 | 4 | 3 | 3 | 3 |
| 96561 | 897043922 | 069760000 | 33264 | 11 | 7 | 4 | 4 | 3 | 3 | 3 |
| 120702 | 371304902 | 587200000 | 34020 | 9 | 7 | 5 | 4 | 3 | 3 | 3 |
| 144842 | 845565883 | 104640000 | 34560 | 10 | 8 | 4 | 4 | 3 | 3 | 3 |
| 160936 | 495073203 | 449600000 | 35640 | 11 | 6 | 5 | 4 | 3 | 3 | 3 |
| 193123 | 794087844 | 139520000 | 36288 | 12 | 7 | 4 | 4 | 3 | 3 | 3 |
| 241404 | 742609805 | 174400000 | 37800 | 10 | 7 | 5 | 4 | 3 | 3 | 3 |
| 289685 | 691131766 | 209280000 | 38016 | 11 | 8 | 4 | 4 | 3 | 3 | 3 |
| 321872 | 990146406 | 899200000 | 38880 | 12 | 6 | 5 | 4 | 3 | 3 | 3 |
| 386247 | 588175688 | 279040000 | 39312 | 13 | 7 | 4 | 4 | 3 | 3 | 3 |
| 482809 | 485219610 | 348800000 | 41580 | 11 | 7 | 5 | 4 | 3 | 3 | 3 |
| 643745 | 980292813 | 798400000 | 42120 | 13 | 6 | 5 | 4 | 3 | 3 | 3 |

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|---|---|---|---|---|---|---|---|---|---|---|---|
| 724214 | 227829415 | 523200000 | 43200 | 10 | 8 | 5 | 4 | 3 | 3 | 3 | |
| 965618 | 970439220 | 697600000 | 45360 | 12 | 7 | 5 | 4 | 3 | 3 | 3 | |
| 1448428 | 455658831 | 046400000 | 47520 | 11 | 8 | 5 | 4 | 3 | 3 | 3 | |
| 1931237 | 940878441 | 395200000 | 49140 | 13 | 7 | 5 | 4 | 3 | 3 | 3 | |
| 2414047 | 426098051 | 744000000 | 49896 | 11 | 7 | 6 | 4 | 3 | 3 | 3 | |
| 2655452 | 168707856 | 918400000 | 50400 | 10 | 7 | 5 | 4 | 4 | 3 | 3 | |
| 2896856 | 911317662 | 092800000 | 51840 | 12 | 8 | 5 | 4 | 3 | 3 | 3 | |
| 3379666 | 396537272 | 441600000 | 51975 | 11 | 7 | 5 | 5 | 3 | 3 | 3 | |
| 3862475 | 881756882 | 790400000 | 52920 | 14 | 7 | 5 | 4 | 3 | 3 | 3 | |
| 4345285 | 366976493 | 139200000 | 53460 | 11 | 9 | 5 | 4 | 3 | 3 | 3 | |
| 4828094 | 852196103 | 488000000 | 54432 | 12 | 7 | 6 | 4 | 3 | 3 | 3 | |
| 5310904 | 337415713 | 836800000 | 55440 | 11 | 7 | 5 | 4 | 4 | 3 | 3 | |
| 5793713 | 822635324 | 185600000 | 56160 | 13 | 8 | 5 | 4 | 3 | 3 | 3 | |
| 6759332 | 793074544 | 883200000 | 56700 | 12 | 7 | 5 | 5 | 3 | 3 | 3 | |
| 7242142 | 278294155 | 232000000 | 57024 | 11 | 8 | 6 | 4 | 3 | 3 | 3 | |
| 7966356 | 506123570 | 755200000 | 57600 | 10 | 8 | 5 | 4 | 4 | 3 | 3 | |
| 8690570 | 733952986 | 278400000 | 58320 | 12 | 9 | 5 | 4 | 3 | 3 | 3 | |
| 9656189 | 704392206 | 976000000 | 58968 | 13 | 7 | 6 | 4 | 3 | 3 | 3 | |
| 10138999 | 189611817 | 324800000 | 59400 | 11 | 8 | 5 | 5 | 3 | 3 | 3 | |
| 10621808 | 674831427 | 673600000 | 60480 | 12 | 7 | 5 | 4 | 4 | 3 | 3 | |
| 13518665 | 586149089 | 766400000 | 61425 | 13 | 7 | 5 | 5 | 3 | 3 | 3 | |
| 14484284 | 556588310 | 464000000 | 62208 | 12 | 8 | 6 | 4 | 3 | 3 | 3 | |
| 15932713 | 012247141 | 510400000 | 63360 | 11 | 8 | 5 | 4 | 4 | 3 | 3 | |
| 19312379 | 408784413 | 952000000 | 63504 | 14 | 7 | 6 | 4 | 3 | 3 | 3 | |
| 20277998 | 379223634 | 649600000 | 64800 | 12 | 8 | 5 | 5 | 3 | 3 | 3 | |
| 21243617 | 349662855 | 347200000 | 65520 | 13 | 7 | 5 | 4 | 4 | 3 | 3 | |
| 26554521 | 687078569 | 184000000 | 66528 | 11 | 7 | 6 | 4 | 4 | 3 | 3 | |
| 28968569 | 113176620 | 928000000 | 67392 | 13 | 8 | 6 | 4 | 3 | 3 | 3 | |
| 31865426 | 024494283 | 020800000 | 69120 | 12 | 8 | 5 | 4 | 4 | 3 | 3 | |
| 37176330 | 361909996 | 857600000 | 69300 | 11 | 7 | 5 | 5 | 4 | 3 | 3 | |
| 40555996 | 758447269 | 299200000 | 70200 | 13 | 8 | 5 | 5 | 3 | 3 | 3 | |
| 42487234 | 699325710 | 694400000 | 70560 | 14 | 7 | 5 | 4 | 4 | 3 | 3 | |
| 47798139 | 036741424 | 531200000 | 71280 | 11 | 9 | 5 | 4 | 4 | 3 | 3 | |
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| 174294224 | 164279335 | 916800000 | 83160 | 11 | 7 | 5 | 4 | 3 | 3 | 3 | 2 |
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| 297410642 | 895279974 | 860800000 | 88200 | 14 | 7 | 5 | 5 | 4 | 3 | 3 | |

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| 993477077 | 736392214 | 725760000 | 103680 | 10 | 8 | 4 | 4 | 3 | 3 | 3 | 3 |

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