OEIS A005642

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Abstract. We illustrate the 2-regular digraphs of sequence A005642 up to 6 nodes and comment on the number of unlabeled digraphs that are obtained by cutting one or two of their arcs to produce *fairly* 2-regular digraphs with two or four "external" edges.

1. Digraphs with 2-regular Degrees

The number of unlabeled digraphs of n nodes where each indegree is 2 and each outdegree is 2 and no multiarcs are admitted (but loops are) is designated here as $U^{(0)}(n)$, and the equivalent number of labeled graphs $L^{(0)}(n)$. The subset of connected graphs of this type is $U_c^{(0)}(n)$ and $L_c^{(0)}(n)$ written with a subscript c [4]. This information is gathered in Table 1, including the sequence numbers in the Online Encyclopedia of Integer Sequences.

One can also count the same types of graphs but discard those that contain loops, adding the subscript $\not L$ to indicate the absence of loops, which gives Table 2.

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n	0	1	2	3	4	5	6	7	8	9	10
$U^{(0)}$	1	0	1	3	8	27	131	711	5055		
$U_c^{(0)}$	1	0	1	3	7	24	117	663	4824		
$L^{(0)}$	1	0	1	6	90	2040	67950	3110940			
$L_c^{(0)}$	1	0	1	6	87	1980	66270	3050460			

Table 1. 2-regular unlabeled or labeled digraphs (not necessarily connected or connected). [5, A005641,A005642,A001499,A123544]

n	0	1	2	3	4	5	6	7	8	9	10
$U_{\not L}^{(0)}$	1	0	0	1	2	5	23	92	624		
$U_{c\not L}^{(0)}$	1	0	0	1	2	5	22	90	616		
$L_{\not L}^{(0)}$	1	0	0	1	9	216	7570	357435	22040361		
$L_{c\not L}^{(0)}$	1	0	0	1	9	216	7560	357120	22025430		

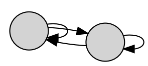
Table 2. 2-regular unlabeled or labeled digraphs (not necessarily connected or connected) without loops. [5, A219889,A307155,A007106,A007107]

Because sets of these 2-regular digraphs are still 2-regular digraphs, the standard Multiset/Euler-Transformation relates the connected graphs to the graphs with any number of components. (In this manuscript *connected* always means *weakly connected.*)

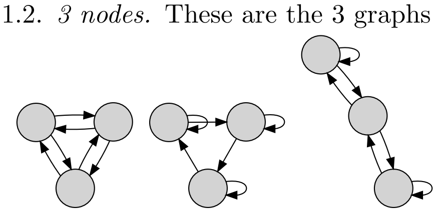
A superset of graphs were multiarcs are admitted has been enumerated elsewhere [3][2, A306892].

1.1. **Illustration of Connected Regular Graphs.**

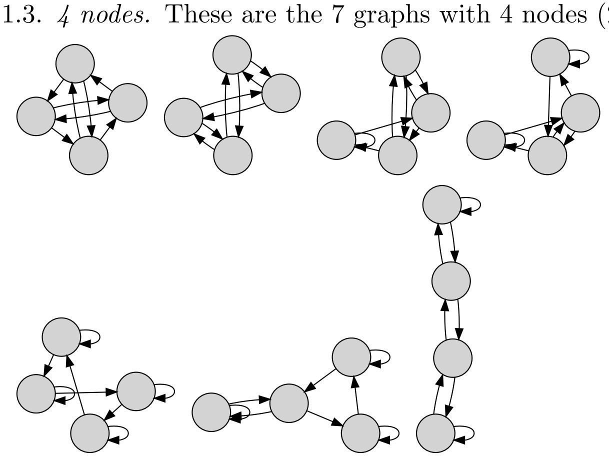
The following graphs illustrate the $U_c^{(0)}$ (including the $U_{c\not L}^{(0)}$) unlabeled regular digraphs up to $n = 6$. A graph classified in our manner by the symmetric statistcs of in- and outdegrees at the nodes remains in the same class if the directions of all arcs are reversed; if this is the same graph it is plotted only once, otherwise both (inequivalent) are plotted.

1.1.1. *2 nodes.* This the graph with 2 nodes 

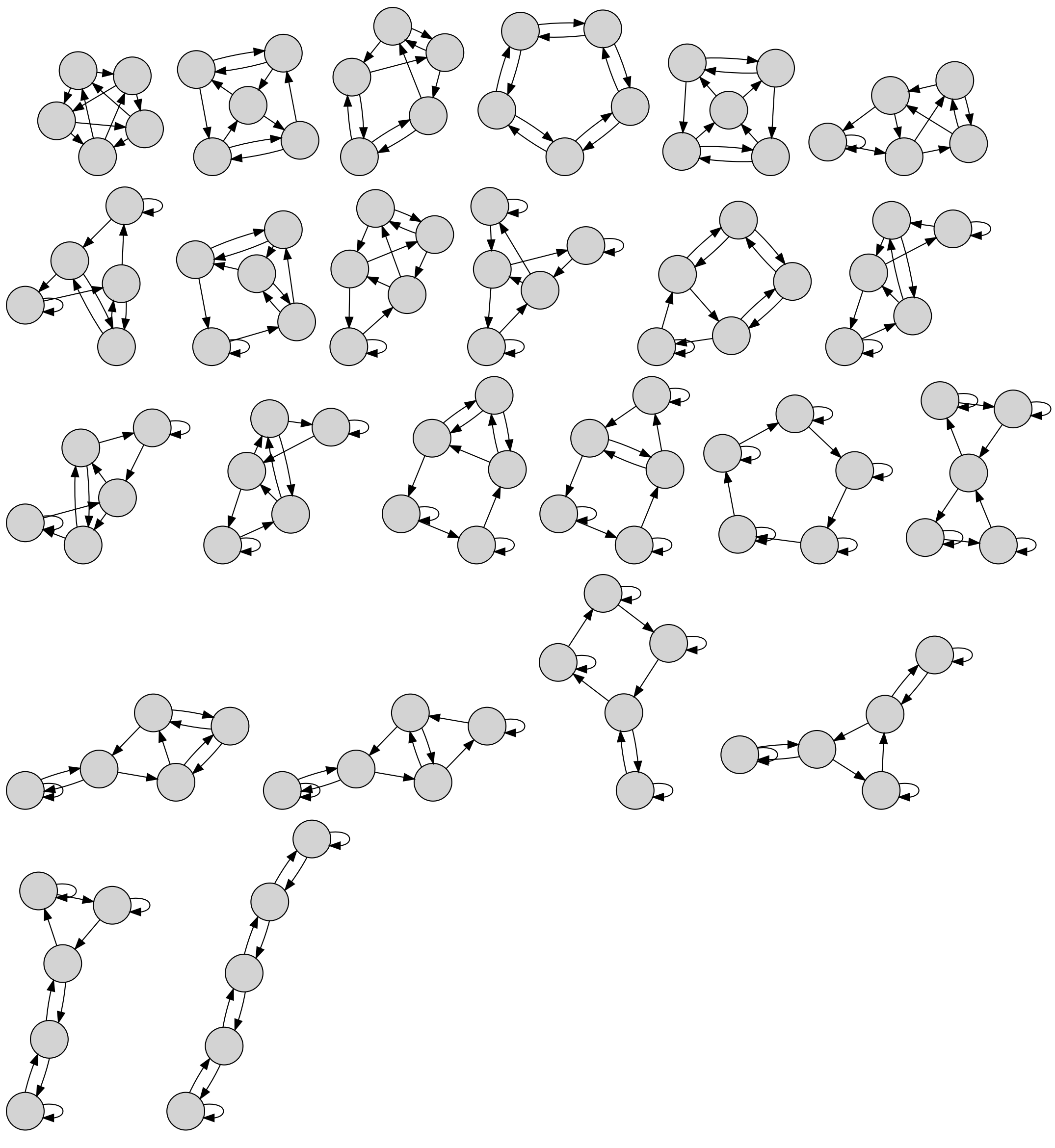
1.1.2. *3 nodes.* These are the 3 graphs with 3 nodes (1 without loops)



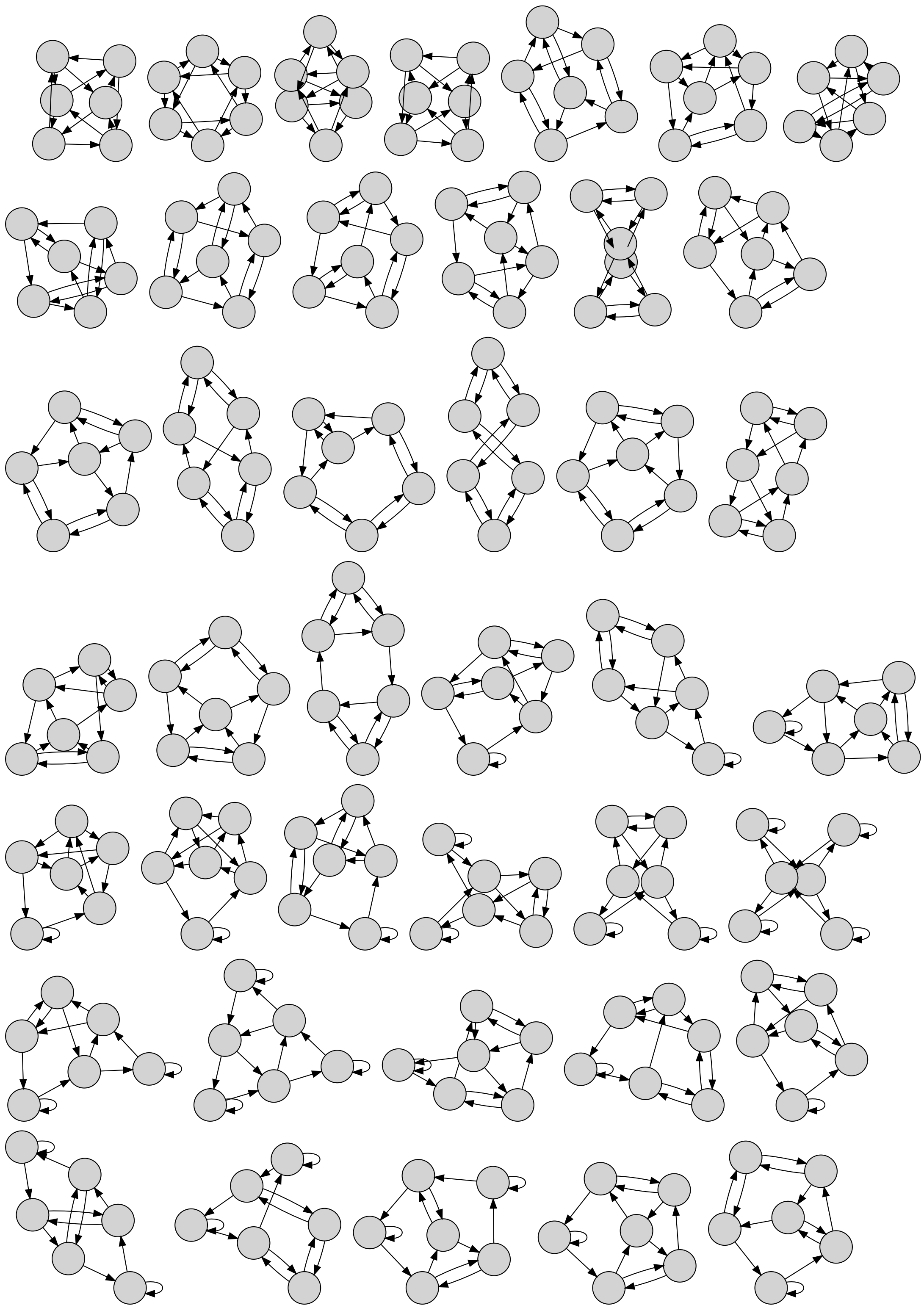
1.1.3. *4 nodes.* These are the 7 graphs with 4 nodes (2 without loops)

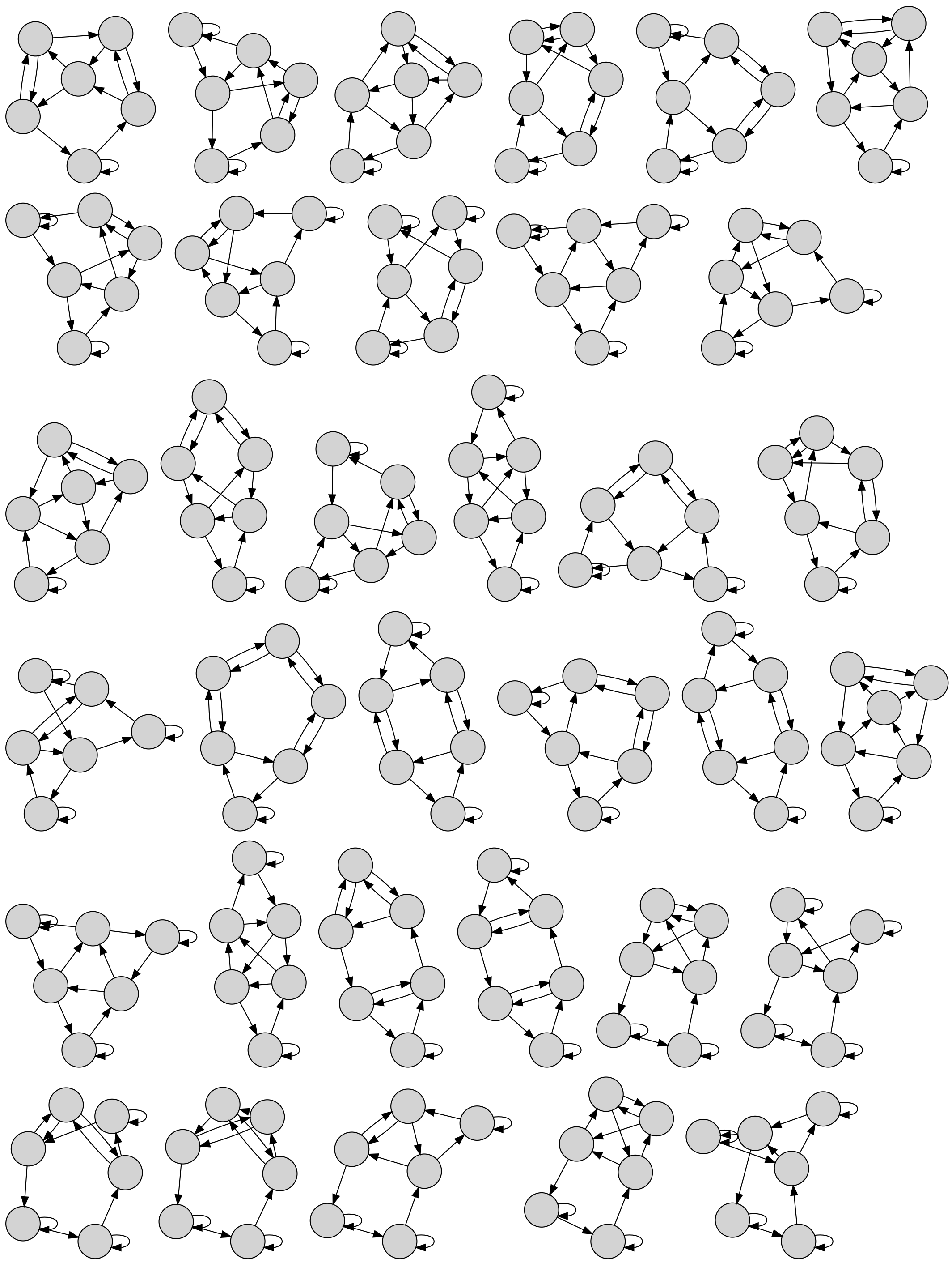


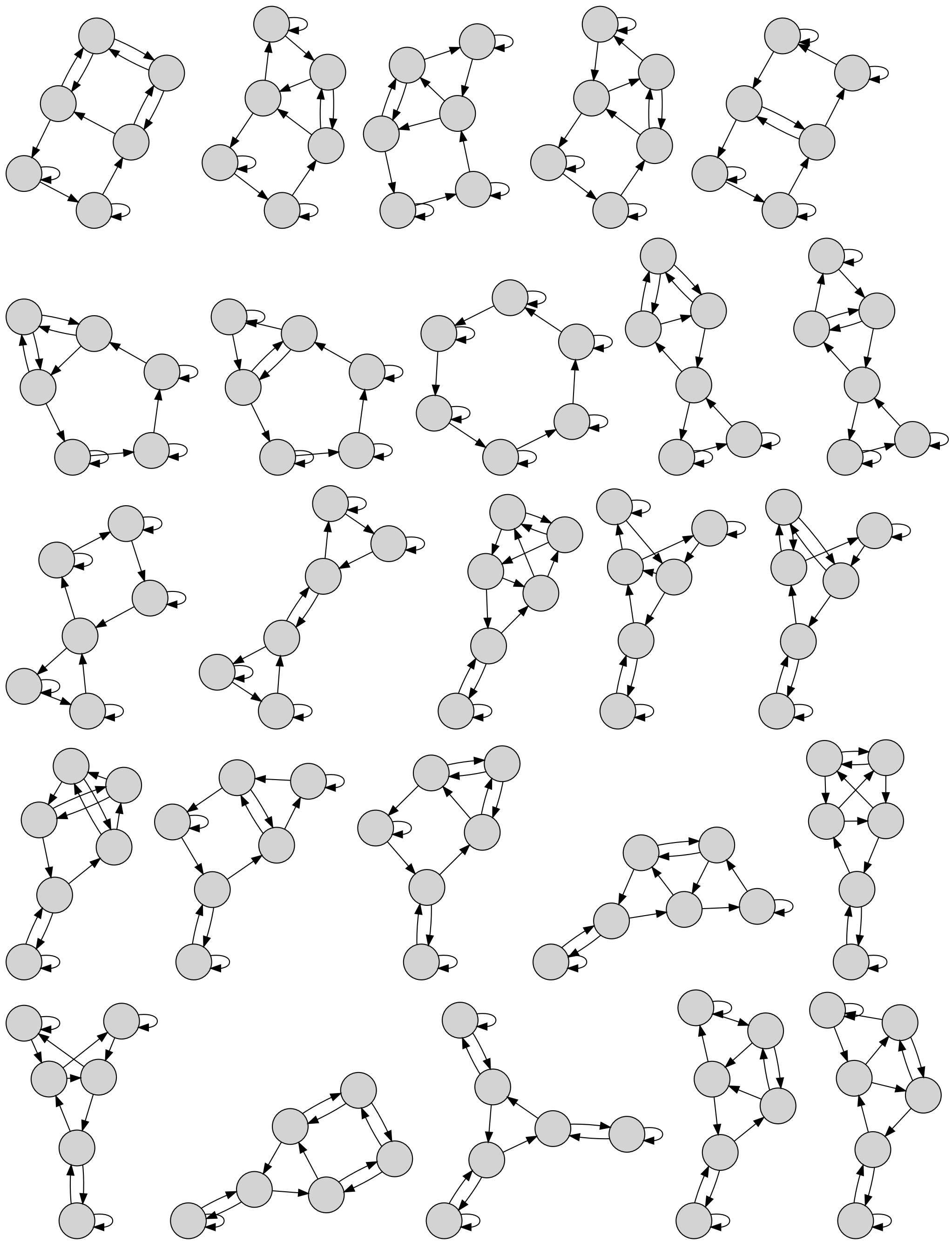
1.1.4. *5 nodes.* These are the 24 graphs with 5 nodes (5 without loops):

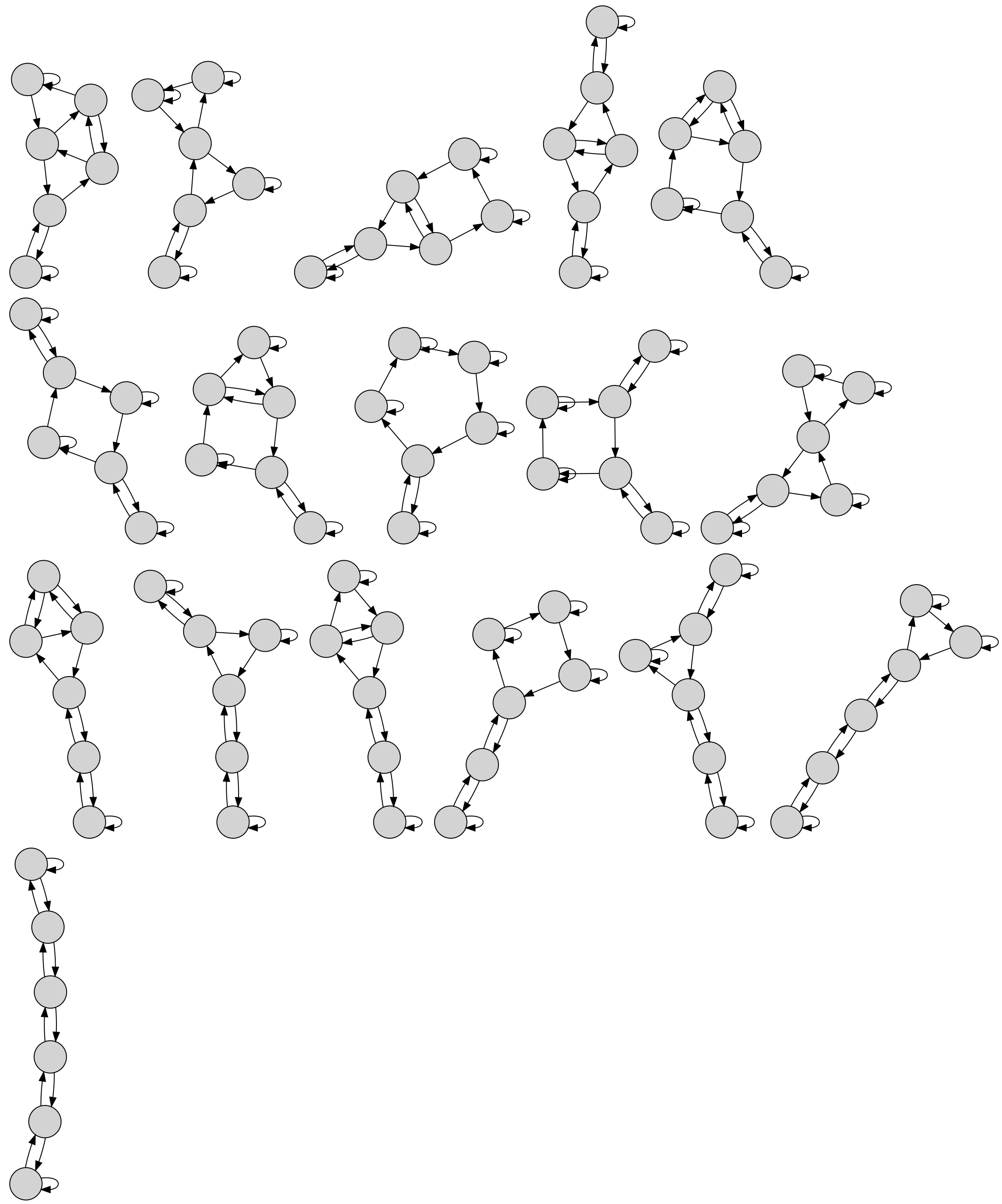


1.1.5. *6 nodes.* These are the 117 graphs with 6 nodes (22 without loops)









The graphs with more than one component are created from these graphs with a standard Multiset/Euler Transformation.

n	1	2	3	4	5	6	7	8	9	10
$U^{(2)}$	0	1	1	3	11	46	232	1443		
$U_c^{(2)}$	0	1	1	2	7	33	184	1215		
$L^{(2)}$	0	2	6	60	1020	27180	1036980	53580240		
$L_c^{(2)}$	0	2	6	48	840	23040	904680	47859840		

TABLE 3. 2-regular unlabeled or labeled digraphs (not necessarily connected or connected) where 1 node is indegree=0 and outdegree=1, 1 node indegree=1 and outdegree=0, the others indegree=outdegree=2

n	1	2	3	4	5	6	7	8	9	10
$U_{\not L}^{(2)}$	0	1	0	0	2	7	31	194		
$U_{c\not L}^{(2)}$	0	1	0	0	1	5	26	170		
$L_{\not L}^{(2)}$	0	2	0	0	140	3510	135072	7063280		
$L_{c\not L}^{(2)}$	0	2	0	0	120	3240	126000	6632640		

TABLE 4. 2-regular unlabeled or labeled digraphs (not necessarily connected or connected) where 1 node is indegree=0 and outdegree=1, 1 node indegree=1 and outdegree=0, the others indegree=outdegree=2

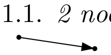
2. Digraphs with One Arc Cut: Bipole Graphs

Chopping an arc in one of the graphs of Section 1 creates 2 new nodes at the cut, one of these hat indegree 1 and outdegree 0, the other indegree 0 and outdegree 1 if the orientation of the arc is kept in the new sections. We call these *fairly* regular digraphs bipole digraphs — nomenclature adopted from Chae et al. [1]. Depending on the symmetry of the 2-regular original digraph, cuts may yield up to as many bipole digraphs as there were arcs in the original graph. A cut may also turn a connected digraph into a bipole digraph with 2 components. So in the bipole graphs with n nodes there is one node with indegree 1 and outdegree 0, one node with indegree 0 and outdegree 1, and the remaining $n-2$ nodes have indegree and outdegree 2. As in the original arcs there are no multiarcs, and loops may reside. We put an upper label (2) at the notations to indicate that there are 2 *outer* nodes in these graphs and arrive at the counts of tables 3 for the bipole graphs and 4 for the bipole graphs without loops.

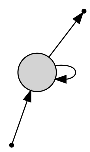
2.1. Illustrations of Connected Bipole Graphs.

The section illustrates the $U_c^{(2)}$ (including the $U_{c\not L}^{(2)}$) bipole graphs. In the following pictures, the 2 "external" input output nodes are shown as small dots, and the $n-2$ remaining nodes as grey circles.

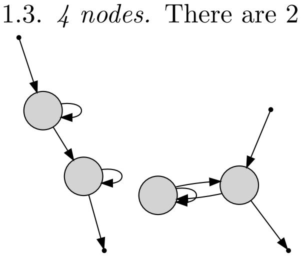
2.1.1. *2 nodes.* There is one graph on 2 nodes (1 without loops):



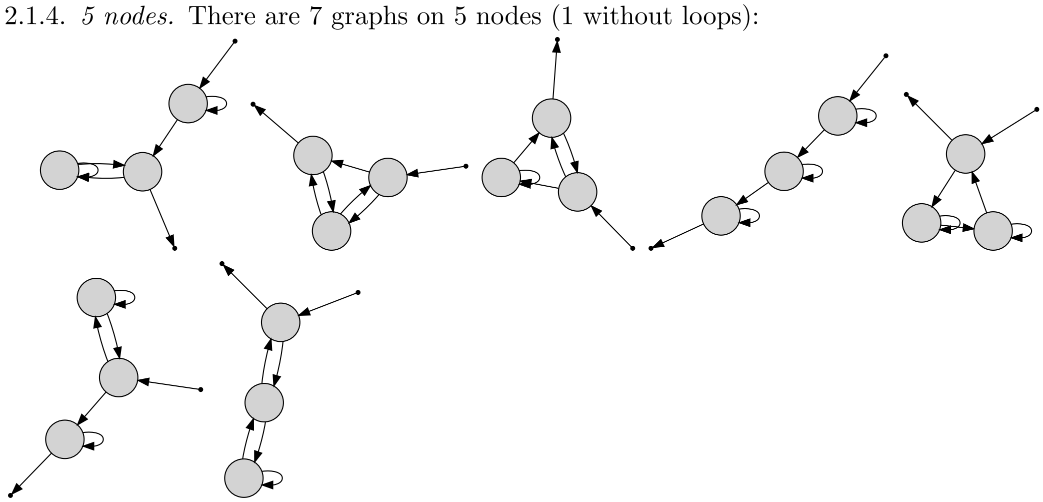
2.1.2. *3 nodes.* There is one graph on 3 nodes (0 without loops):



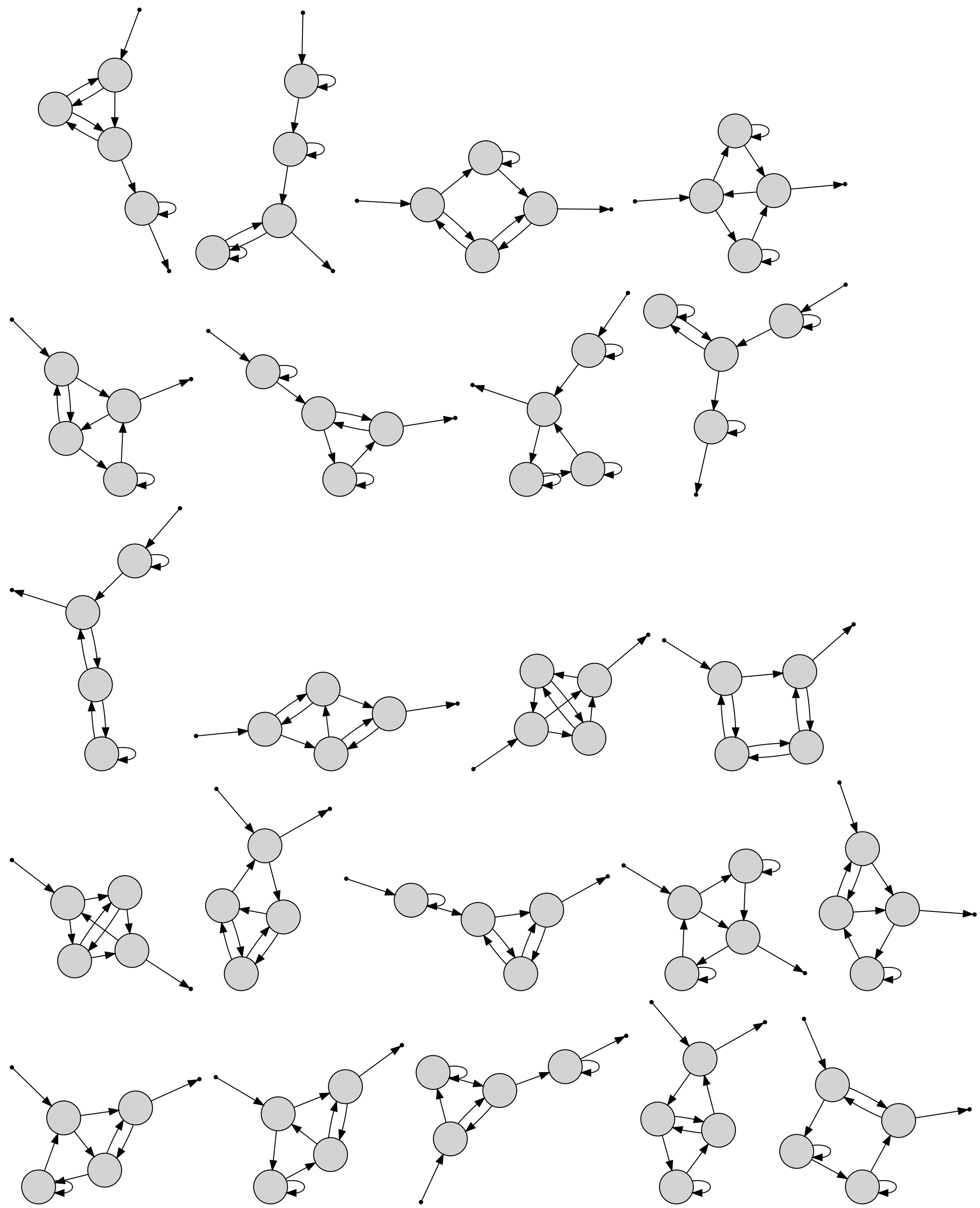
2.1.3. *4 nodes.* There are 2 graphs on 4 nodes (0 without loops):

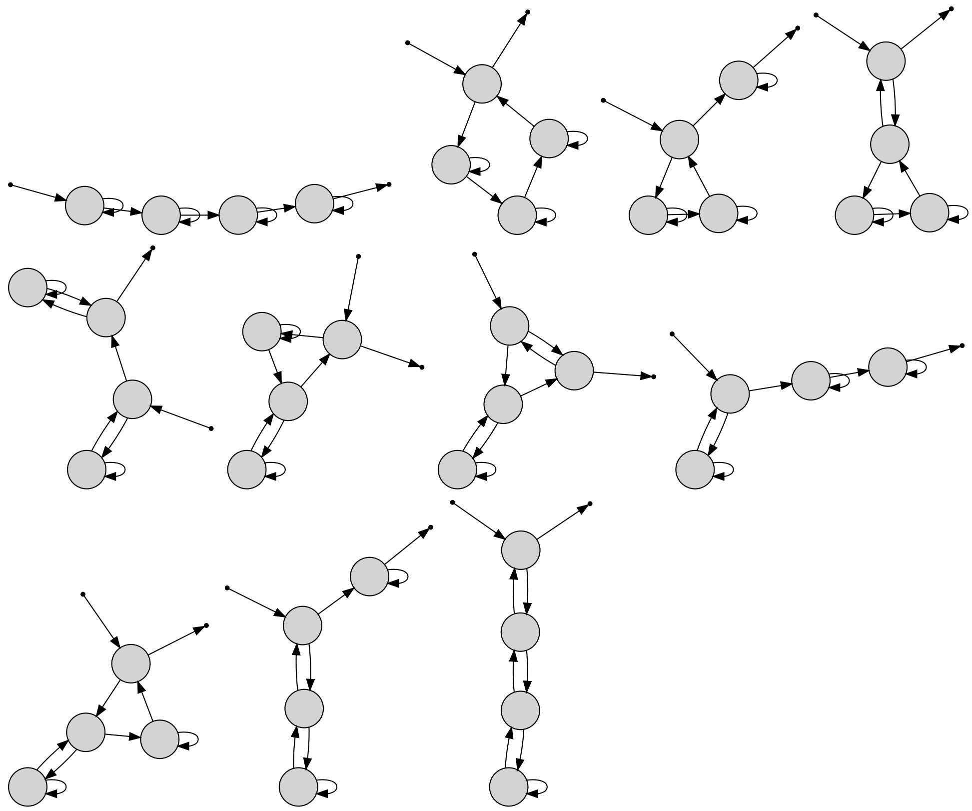


2.1.4. *5 nodes.* There are 7 graphs on 5 nodes (1 without loops):

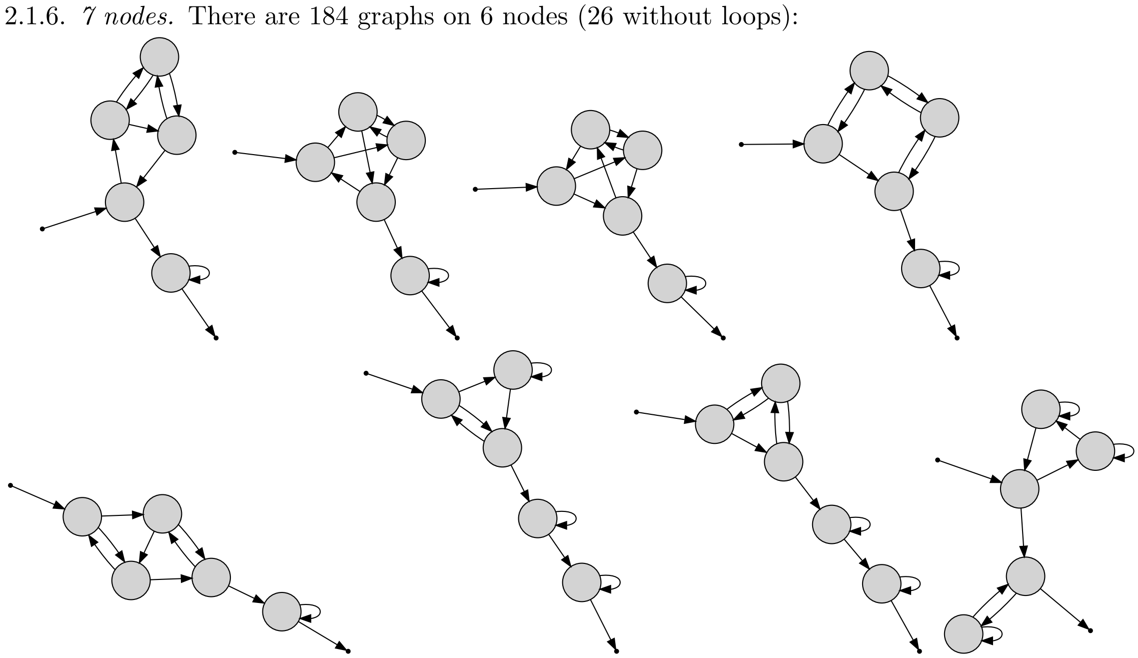


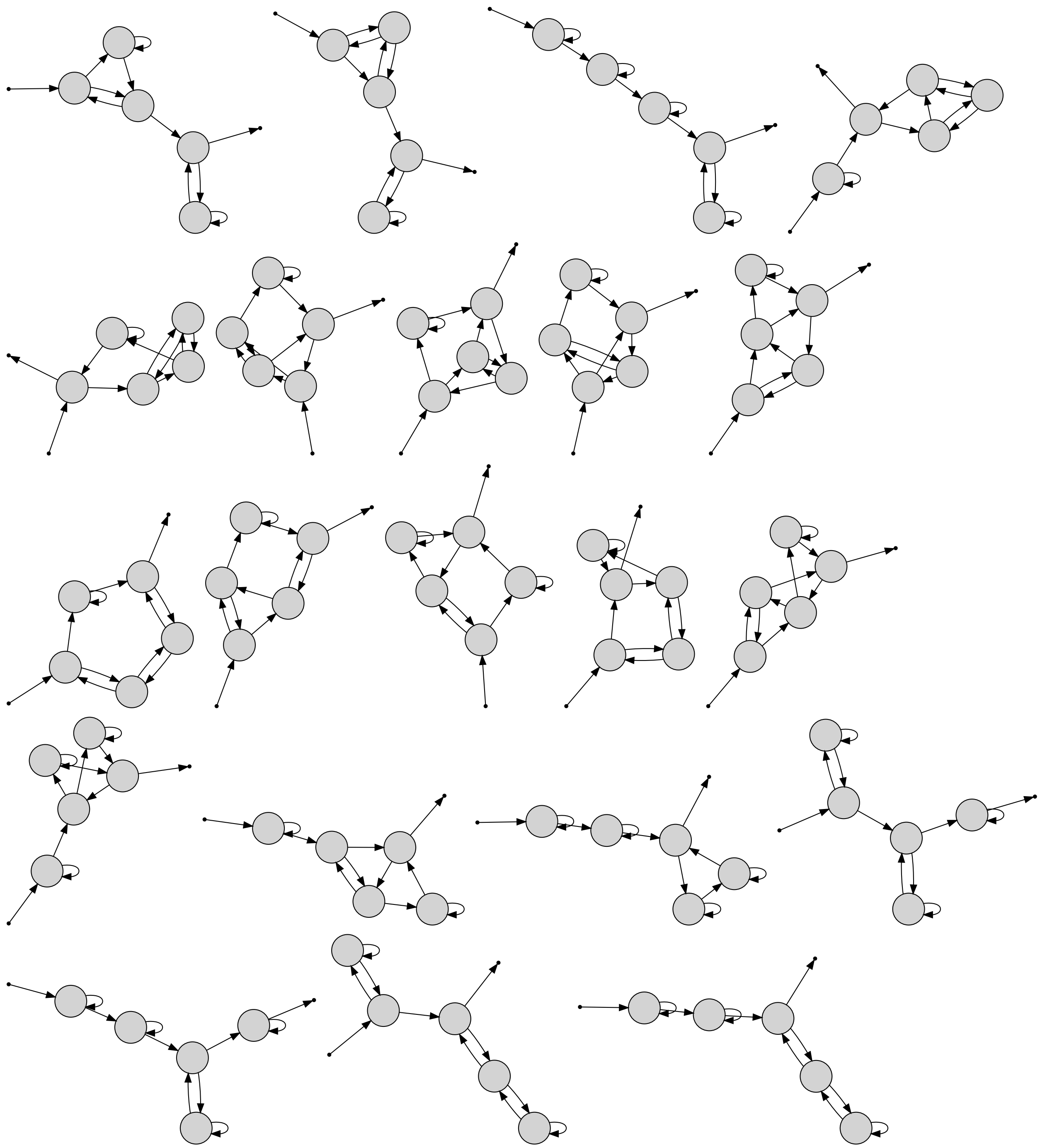
2.1.5. *6 nodes.* There are 33 graphs on 6 nodes (5 without loops):

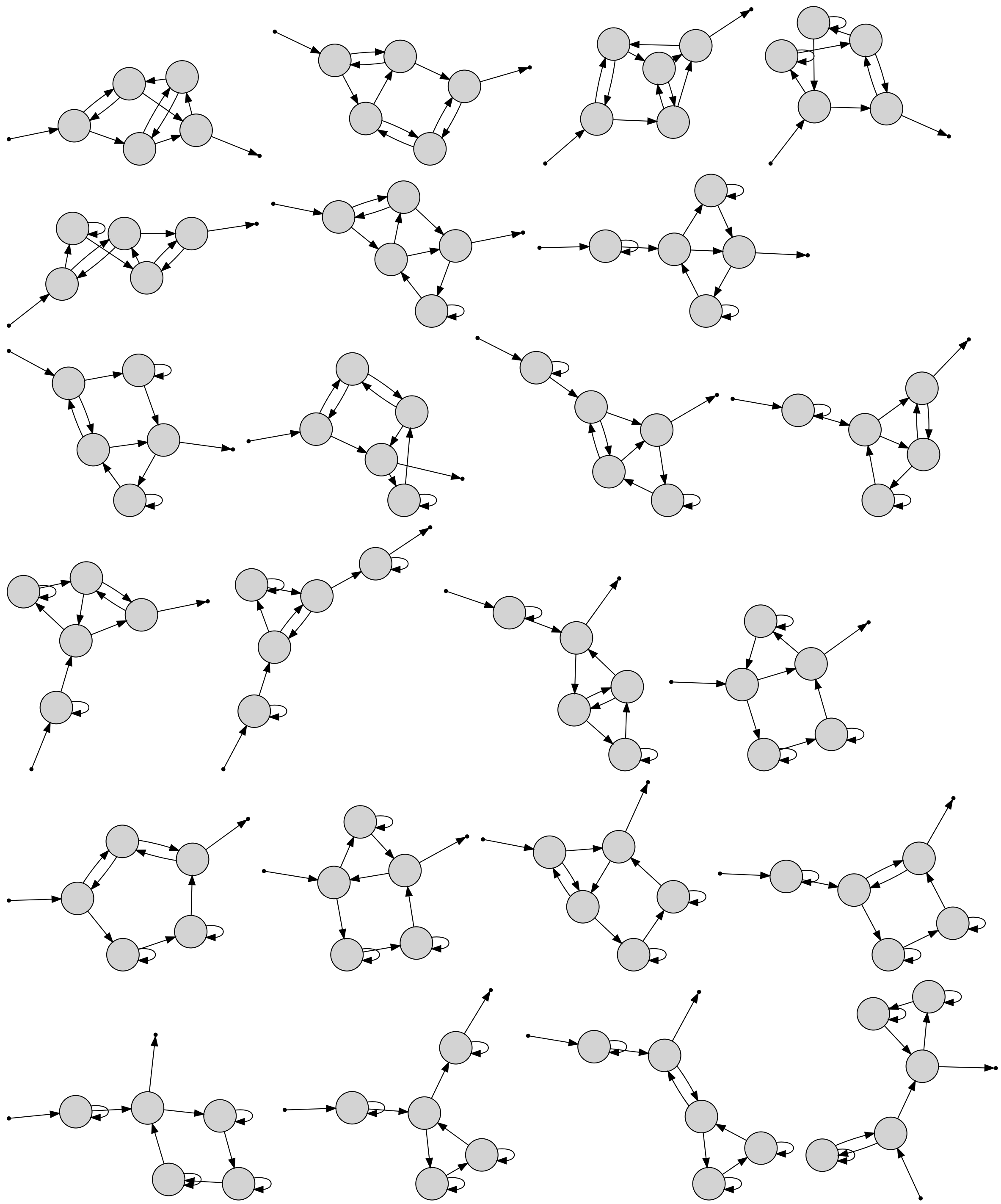


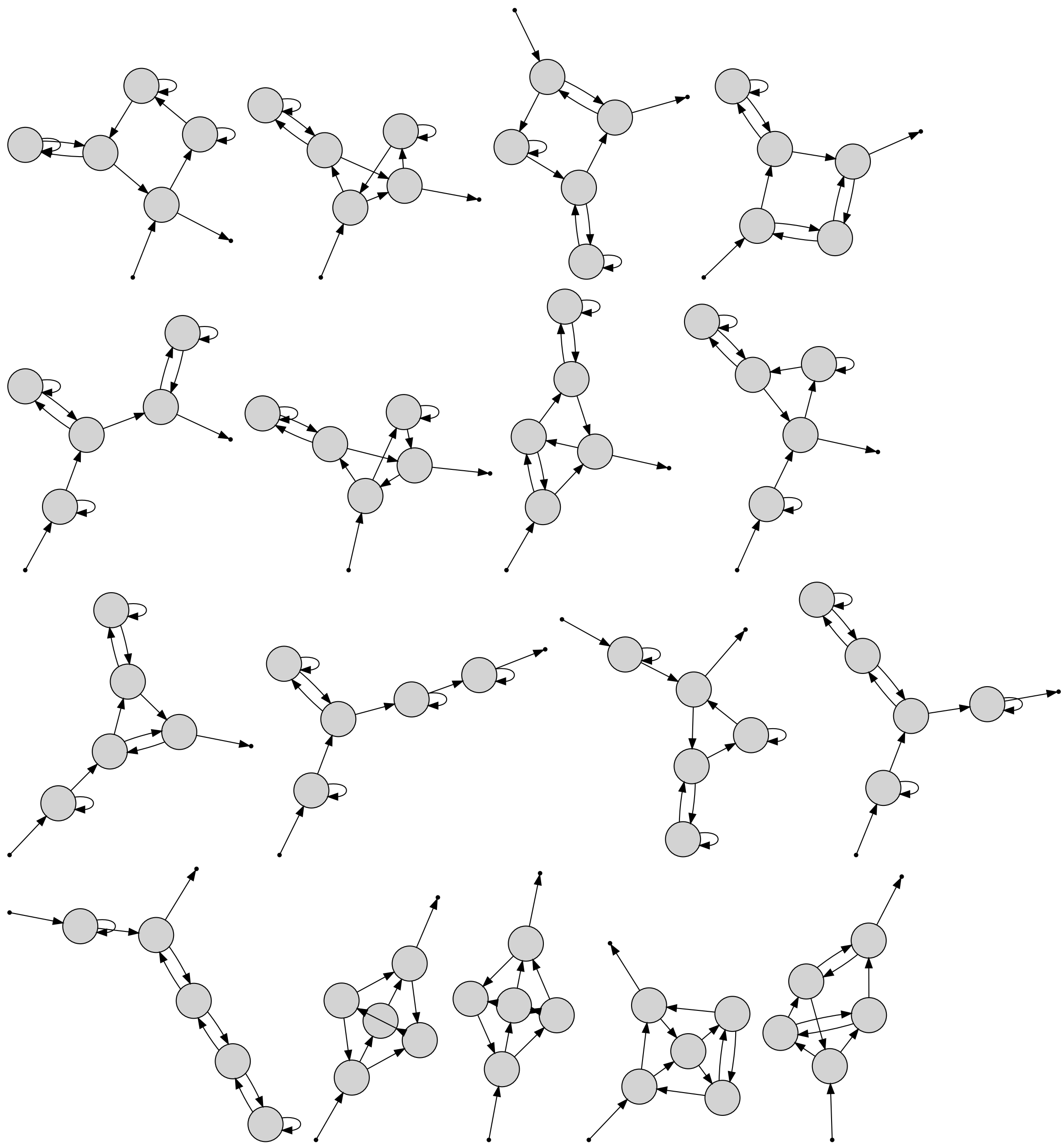


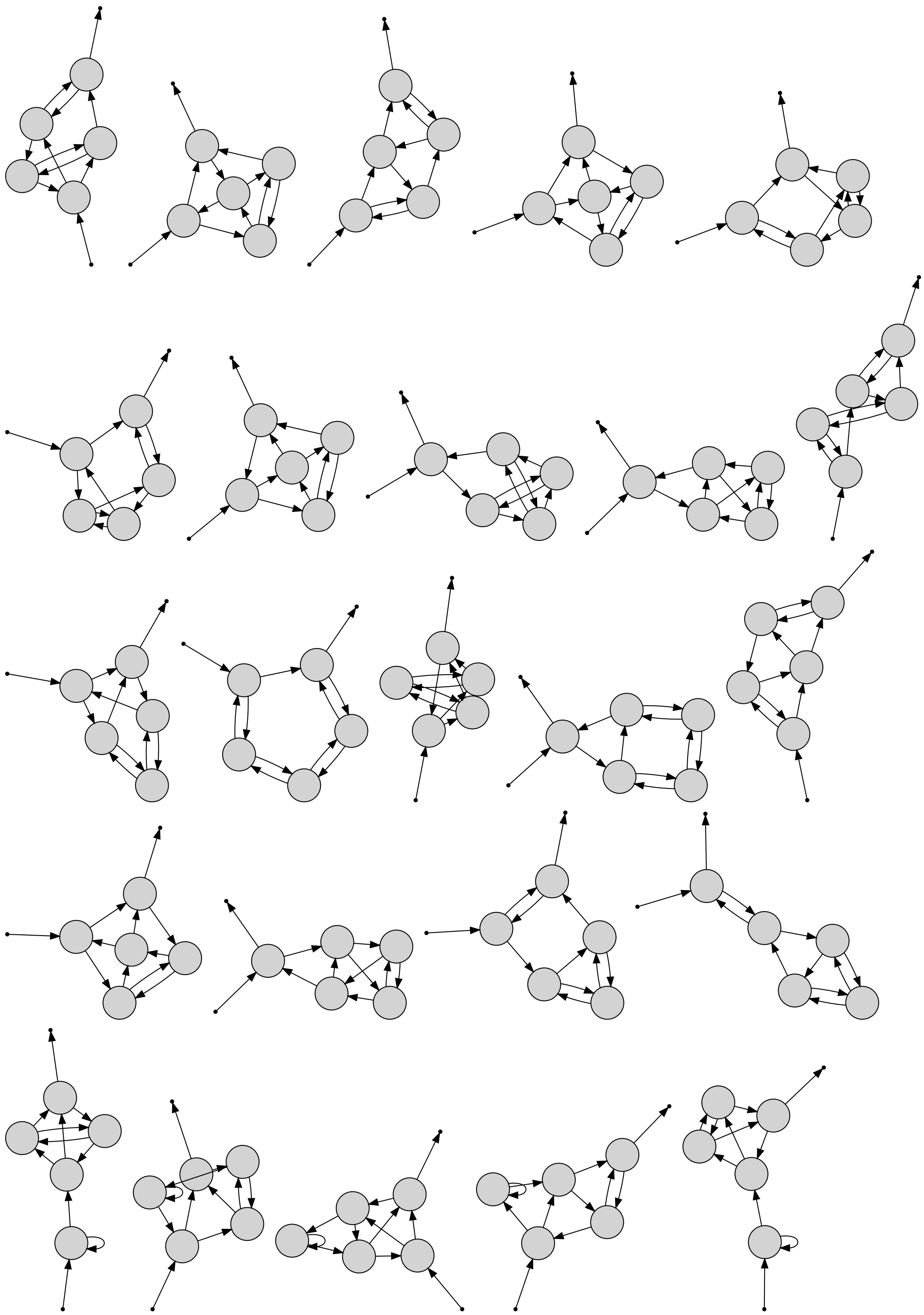
2.1.6. *7 nodes.* There are 184 graphs on 6 nodes (26 without loops):

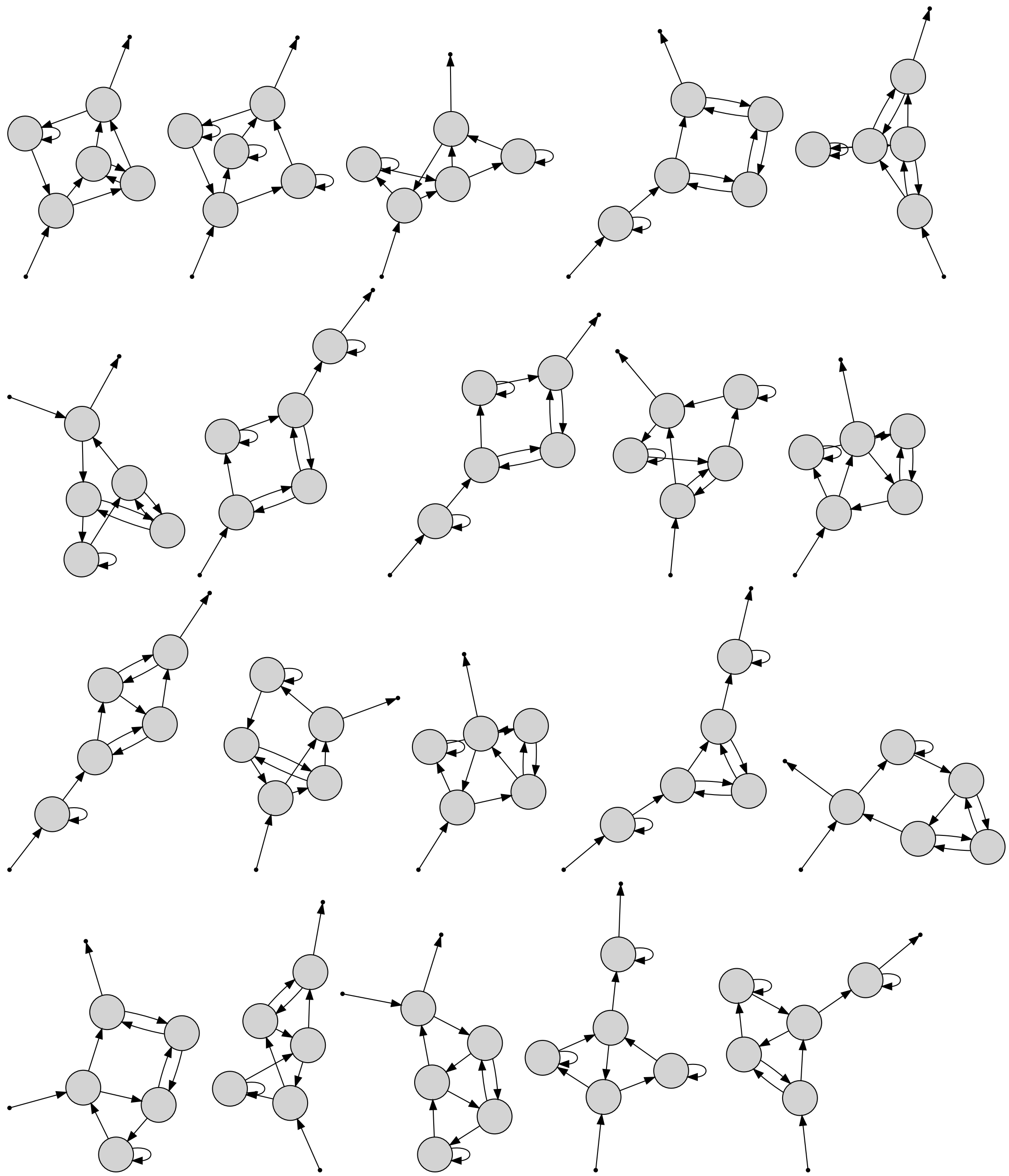


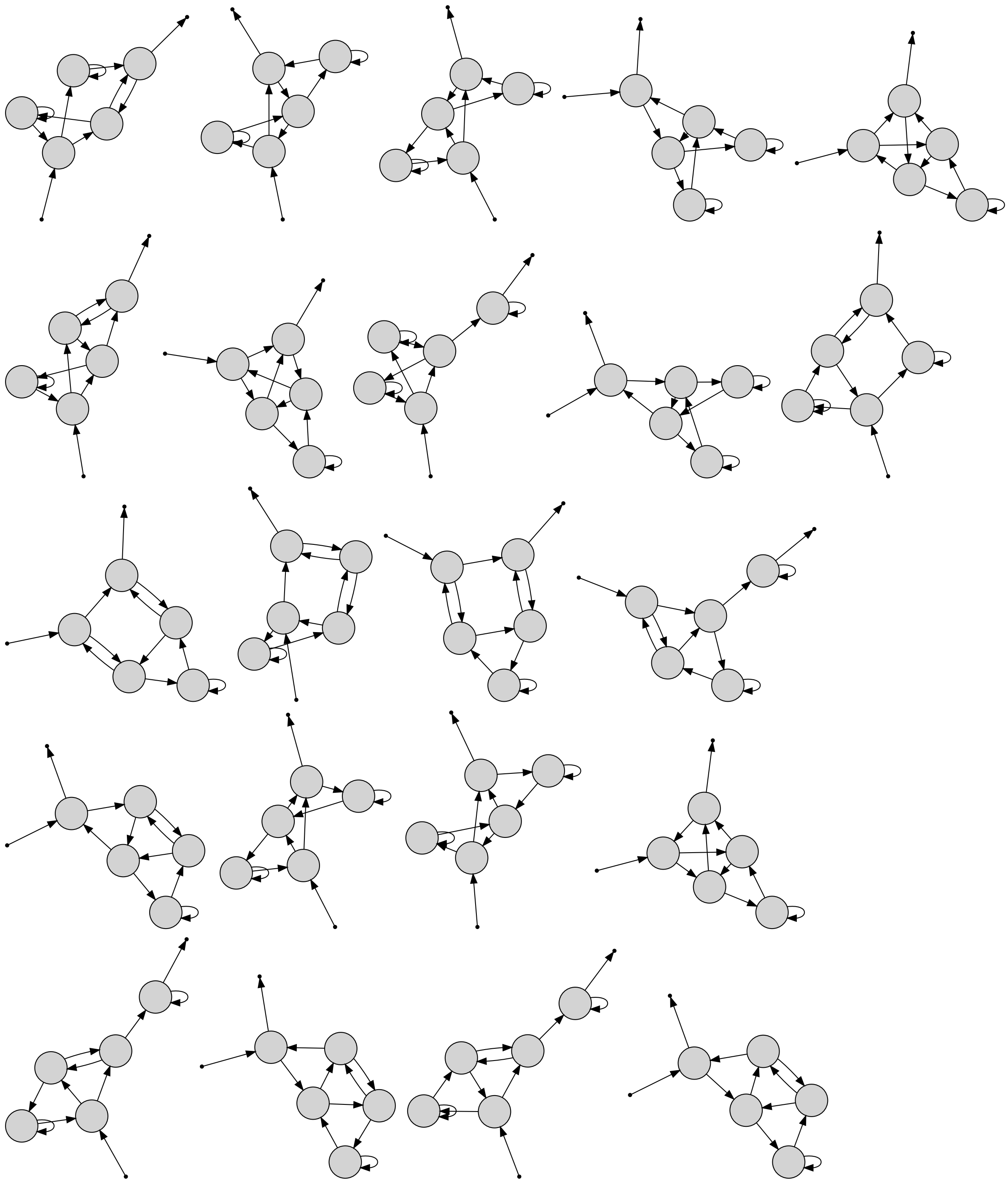


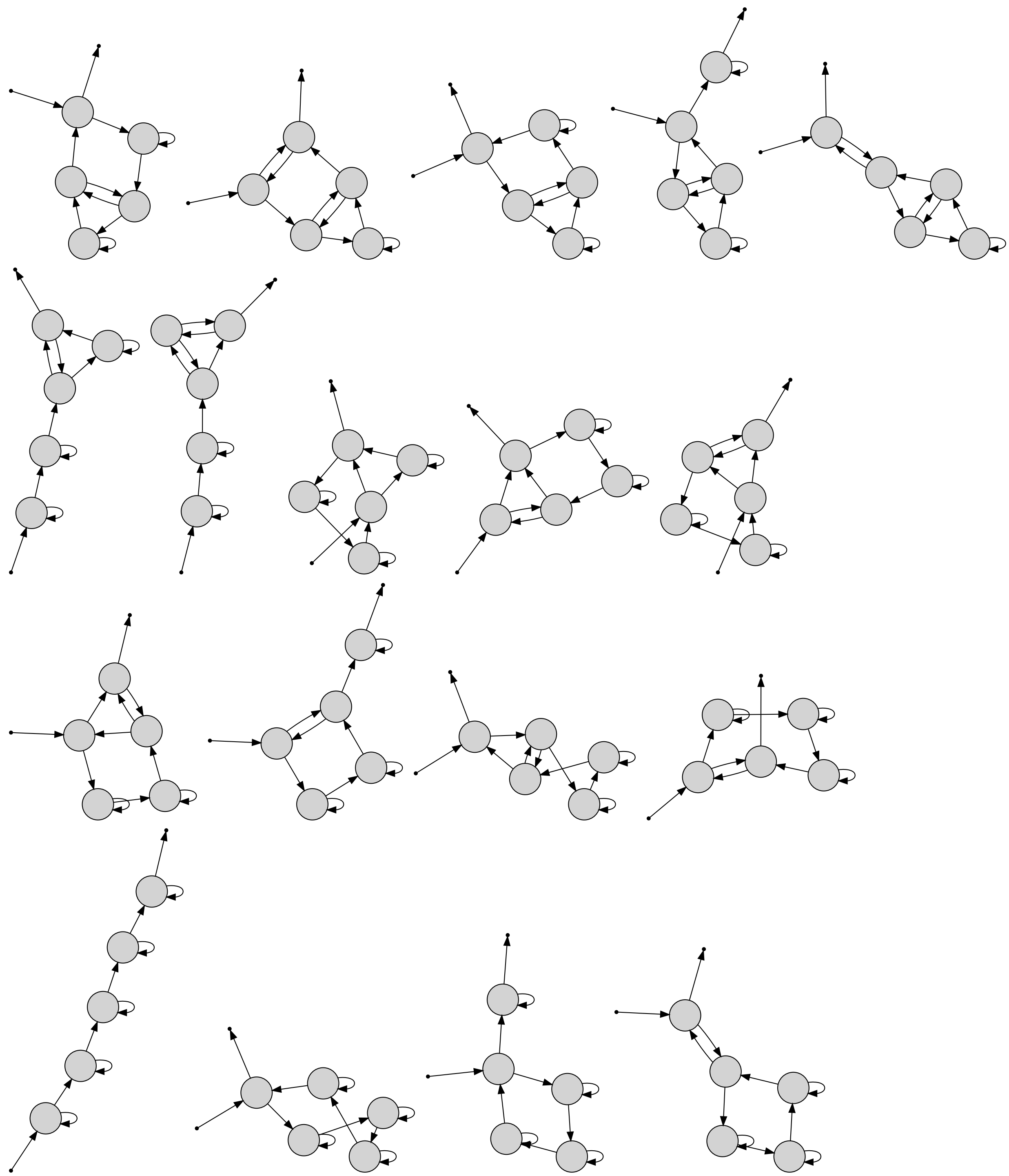


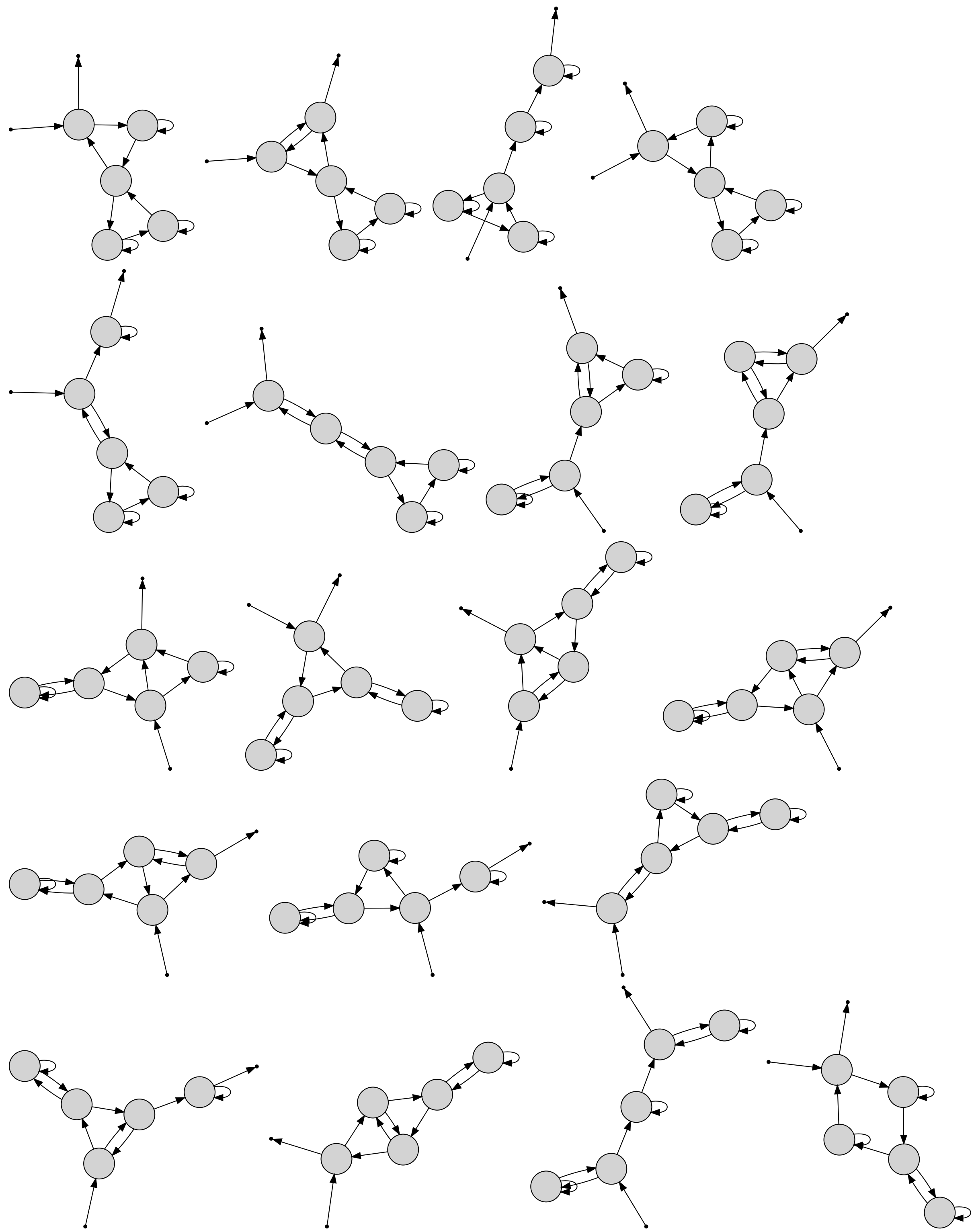


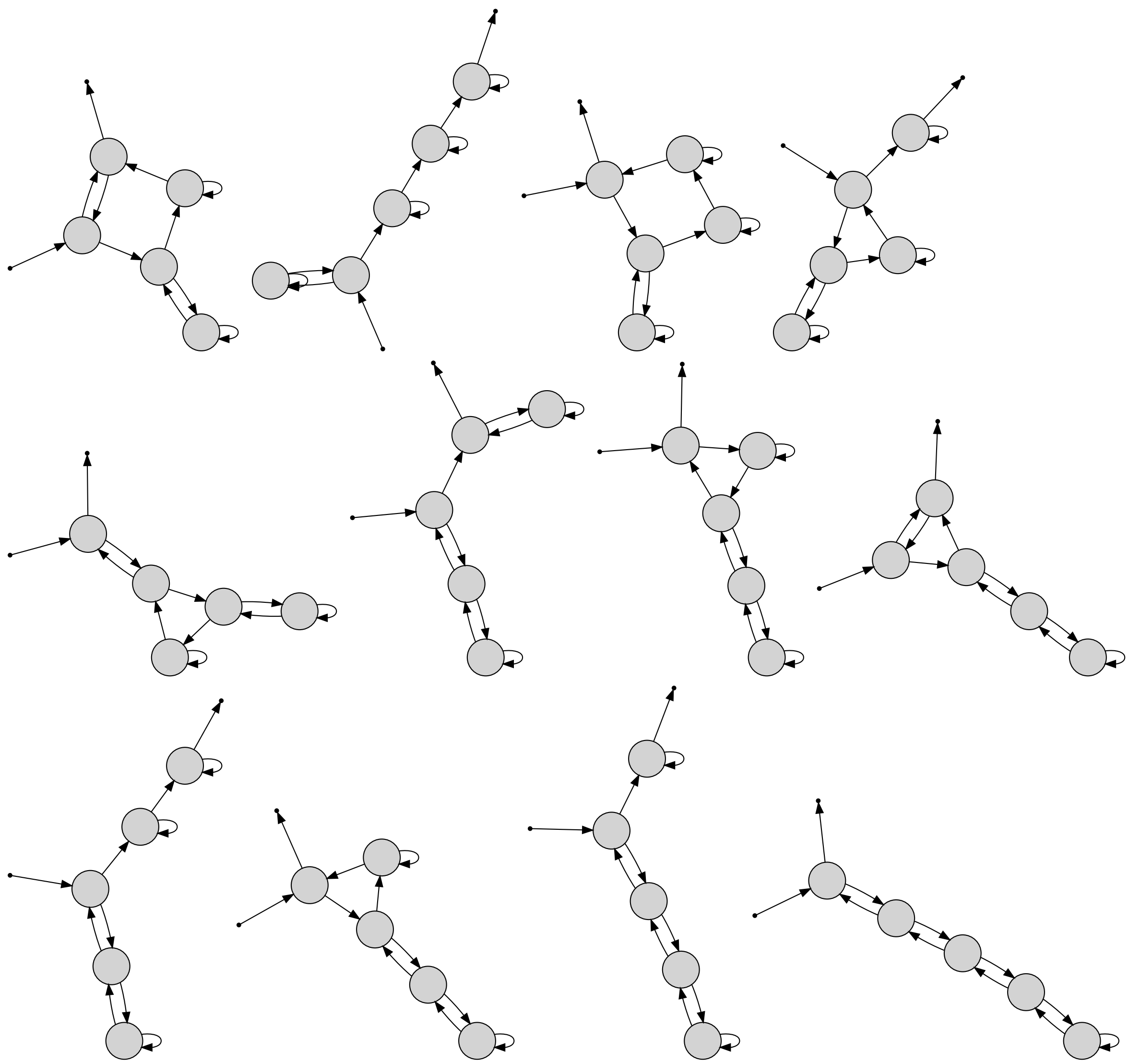












The bipole graphs with more than one component are created by taking any of the $U_c^{(2)}$ graphs and placing a sufficient number of $U^{(0)}$ graphs at their side until the total number of nodes is reached. If the generating functions are

$$U^{(0)}(x) \equiv \sum_{n\ge 0} U^{(0)}(n)x^n; \tag{1}$$

$$U_c^{(2)}(x) \equiv \sum_{n\ge 0} U_c^{(2)}(n)x^n; \tag{2}$$

then

$$U^{(2)}(x) \equiv \sum_{n\ge 0} U^{(2)}(n)x^n = U_c^{(2)}(x)U^{(0)}(x). \tag{3}$$

In the same fashion for the subsets of graphs without loops:

$$U^{(0)}_{\not L}(x) \equiv \sum_{n\ge 0} U^{(0)}_{\not L}(n)x^n; \tag{4}$$

$$U^{(2)}_{c\not L}(x) \equiv \sum_{n\ge 0} U^{(2)}_{c\not L}(n)x^n; \tag{5}$$

then

$$U^{(2)}_{\not L}(x) \equiv \sum_{n\ge 0} U^{(2)}_{\not L}(n)x^n = U^{(2)}_{c\not L}(x)U^{(0)}_{\not L}(x). \tag{6}$$

As expected the corresponding relation for the labeled graphs is exploiting the exponential generating functions defined as

$$L^{(0)}(x) \equiv \sum_{n\ge 0} L^{(0)}(n)\frac{x^n}{n!}; \tag{7}$$

$$L^{(2)}_c(x) \equiv \sum_{n\ge 0} L^{(2)}_c(n)\frac{x^n}{n!}; \tag{8}$$

$$L^{(2)}(x) \equiv \sum_{n\ge 0} L^{(2)}(n)\frac{x^n}{n!}; \tag{9}$$

to enumerate the graphs with any set of components by their product:

$$L^{(2)}(x) = L^{(2)}_c(x)L^{(0)}(x). \tag{10}$$

For the subset of loopless graphs in the same fashion

$$L^{(0)}_{\not L}(x) \equiv \sum_{n\ge 0} L^{(0)}_{\not L}(n)\frac{x^n}{n!}; \tag{11}$$

$$L^{(2)}_{c\not L}(x) \equiv \sum_{n\ge 0} L^{(2)}_{c\not L}(n)\frac{x^n}{n!}; \tag{12}$$

$$L^{(2)}_{\not L}(x) \equiv \sum_{n\ge 0} L^{(2)}_{\not L}(n)\frac{x^n}{n!}; \tag{13}$$

enumerate the graphs with any set of components by their product:

$$L^{(2)}_{\not L}(x) = L^{(2)}_{c\not L}(x)L^{(0)}_{\not L}(x). \tag{14}$$

3. Digraphs with Two Cuts: Quadripole Digraphs

Cutting *two* arcs in the regular digraphs generates another kind of fairly regular digraphs: they have two nodes with indegree 1 and outdegree 0, two nodes with indegree 0 and outdegree 1; the remaining $n-4$ nodes have indegree and outdegree 2. There is potentially a relation between the 2-marked line graphs of these regular digraphs and the "fairly" digraphs considered here, but this relation and the implications of symmetries are not explored here.

The number of unlabeled quadripole digraphs of this kind is $U^{(4)}(n)$ in Table 5, the connected unlabeled digraphs is $U^{(4)}_c(n)$, and the labeled and labeled connected are $L^{(4)}(n)$ and $L^{(4)}_c(n)$.

The subsets of loopless graphs of the same types are listed in Table 6.

n	4	5	6	7	8	9	10
$U^{(4)}$	1	2	7	29	144	854	
$U_c^{(4)}$	0	1	3	15	81	538	
$L^{(4)}$	12	150	3060	95130	2288160		
$L_c^{(4)}$	0	30	1080	44100	4147920		

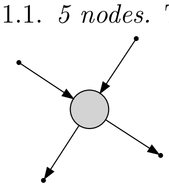
TABLE 5. Weakly connected or not necessarily connected quadripole digraphs with n nodes.

n	4	5	6	7	8	9	10
$U_{\not L}^{(4)}$	1	1	1	6	26	138	
$U_{c\not L}^{(4)}$	0	1	1	4	18	104	
$L_{\not L}^{(4)}$	12	30	360	15540	659400		
$L_{c\not L}^{(4)}$	0	30	360	20080	468720		

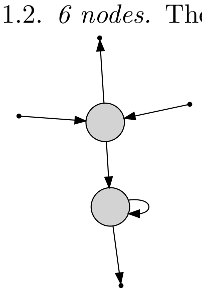
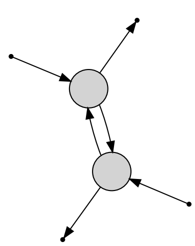
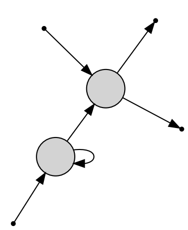
TABLE 6. Weakly connected or not necessarily connected loopless quadripole digraphs with n nodes.

3.1. **Illustrations Connected quadripolar Digraphs.** The following are illustrations of the $U_c^{(4)}$ graphs, including the $U_{c\not L}^{(4)}$ without loops. The 2 "external" input and 2 "external" output nodes are shown as small dots, and the $n-4$ nodes as grey circles.

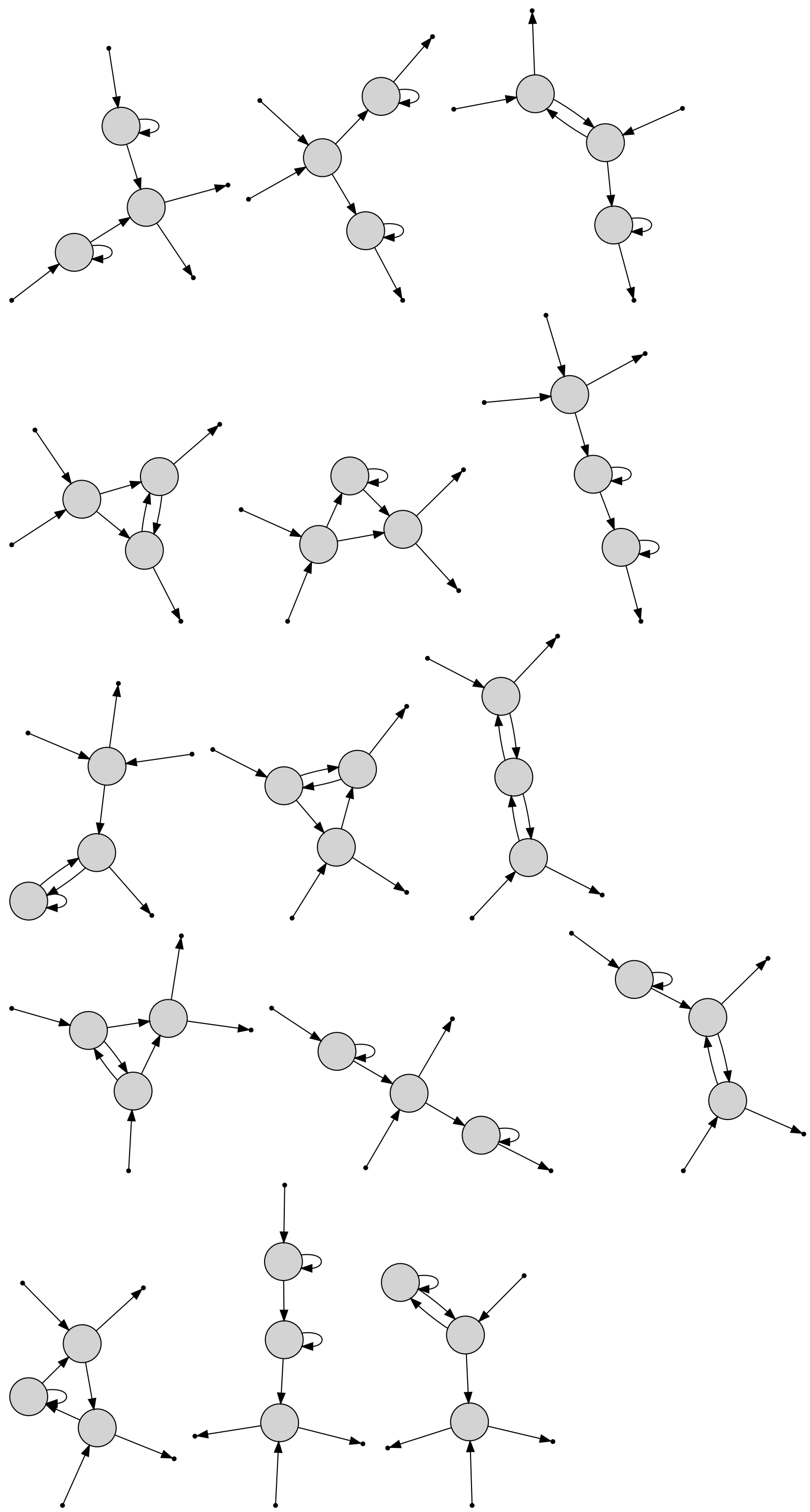
3.1.1. *5 nodes.* There is 1 graph with 5 nodes (1 without loops):



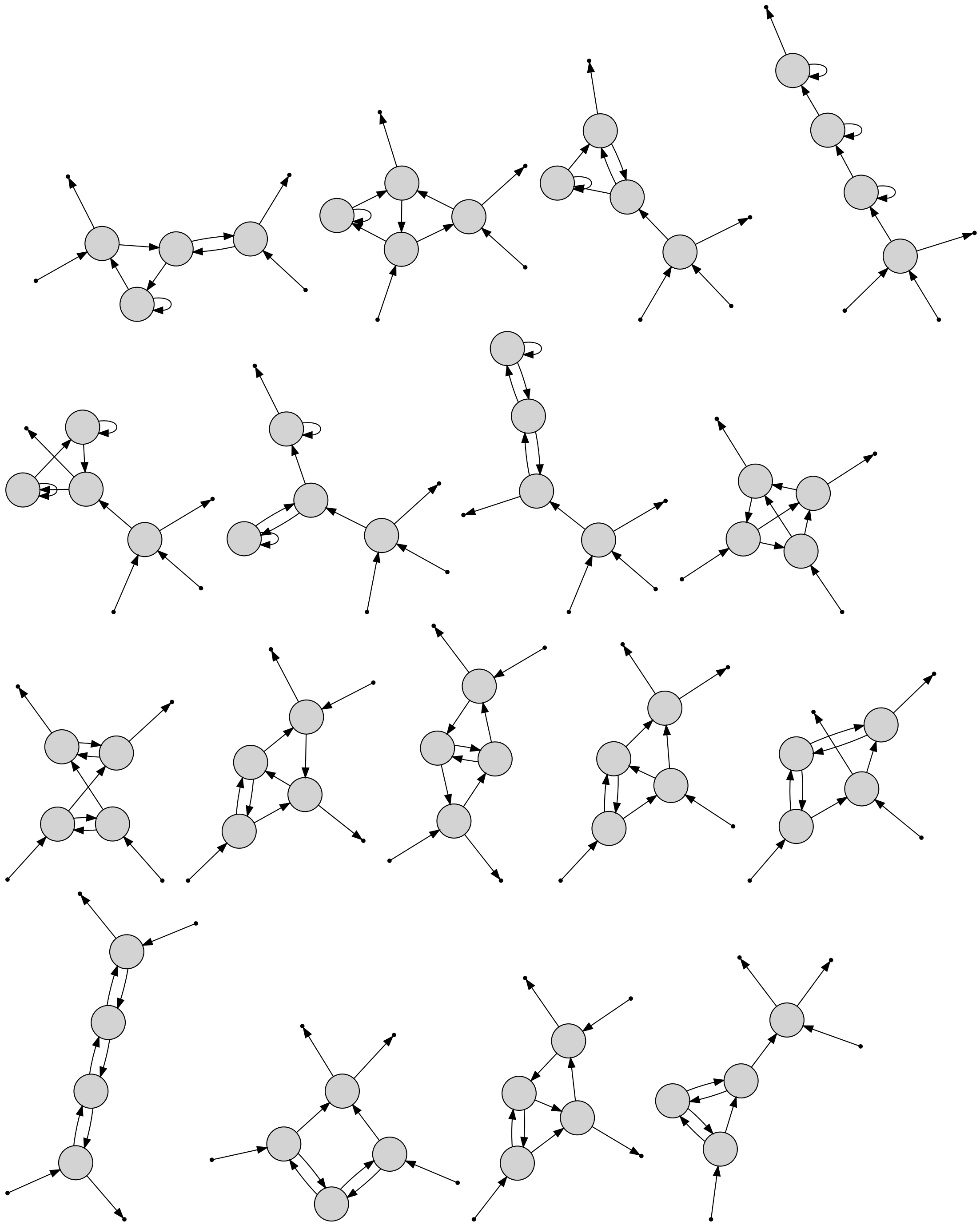
3.1.2. *6 nodes.* These are the 3 graphs with 6 nodes (1 without loops):

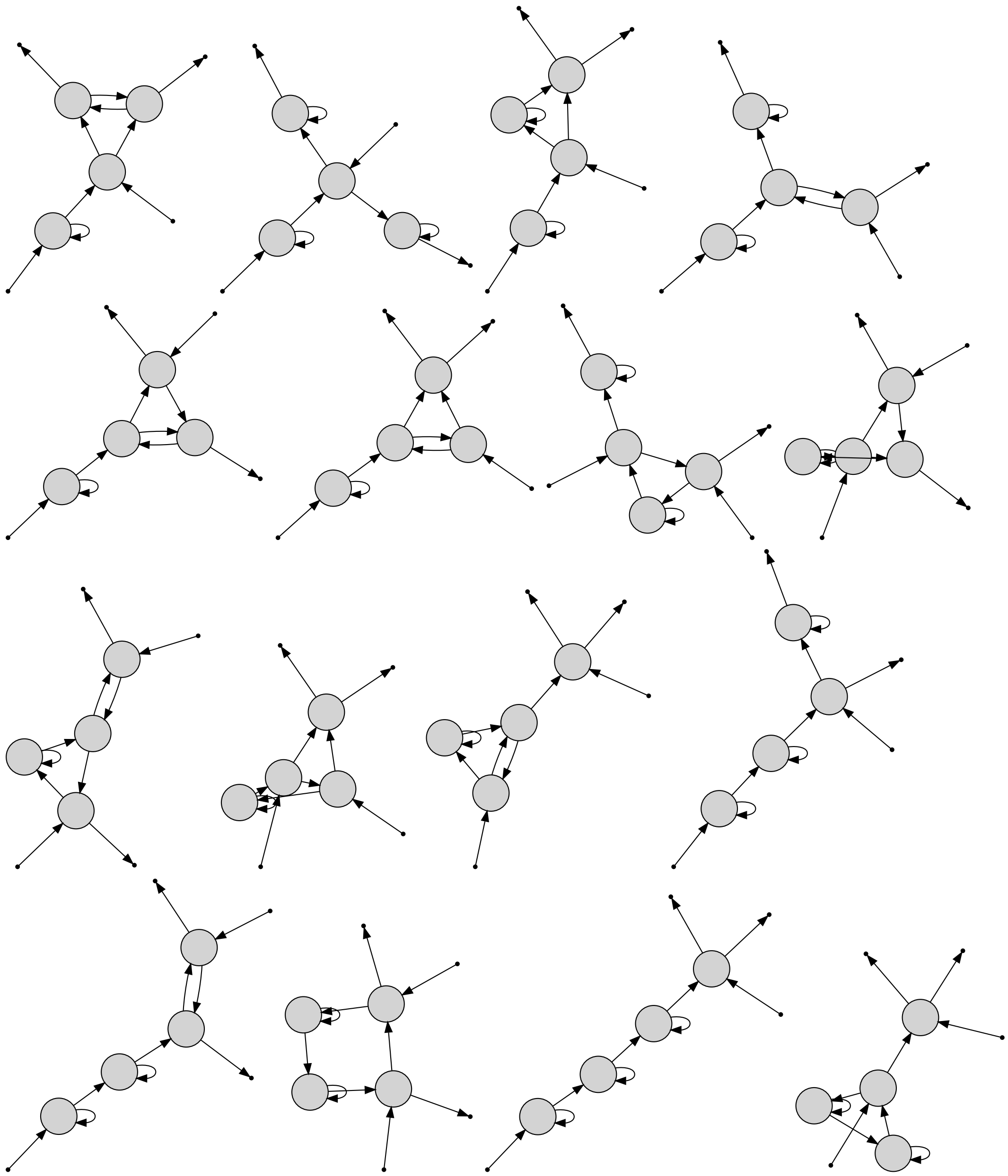




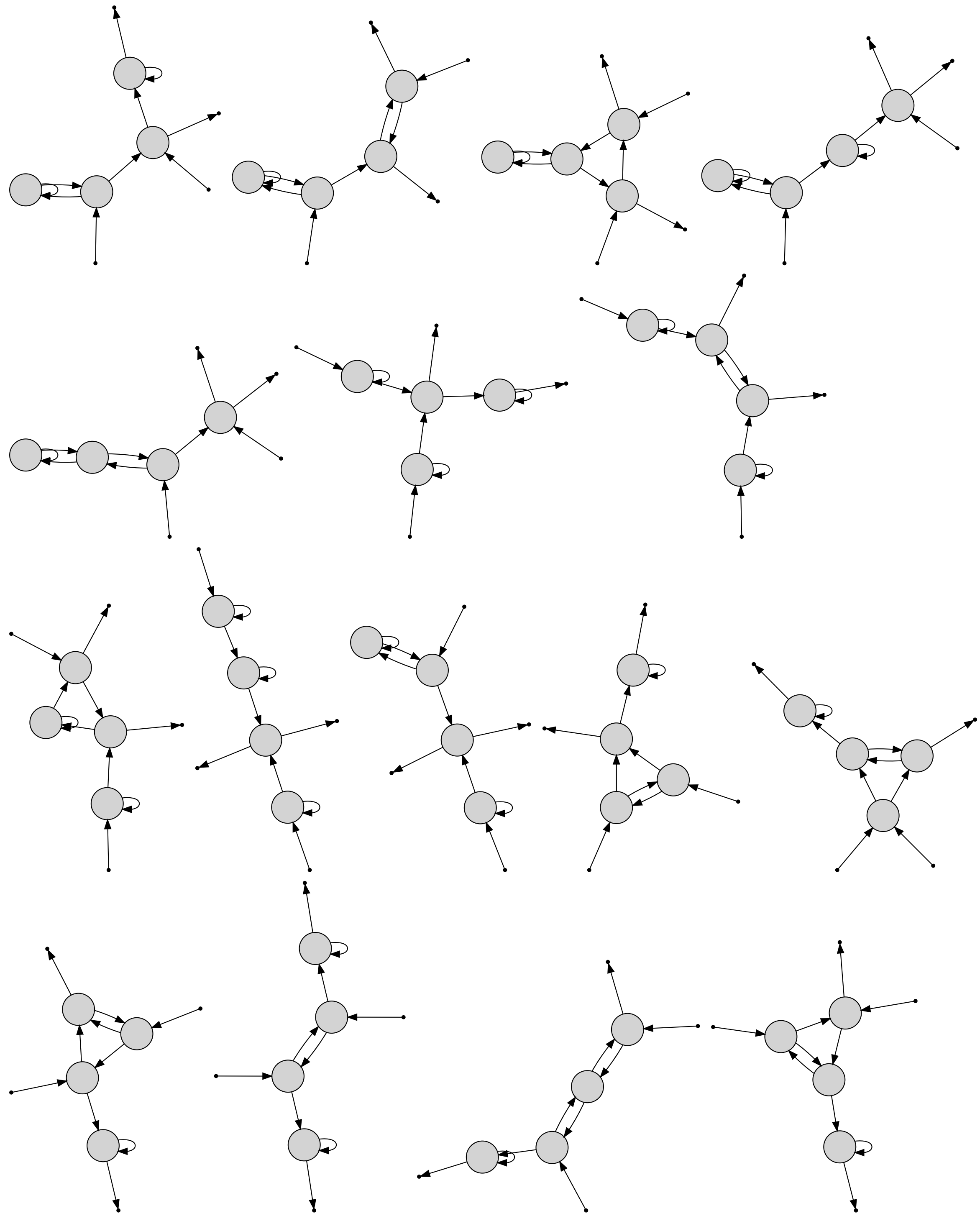
3.1.3. *7 nodes.* These are the 15 graphs with 7 nodes (4 without loops):

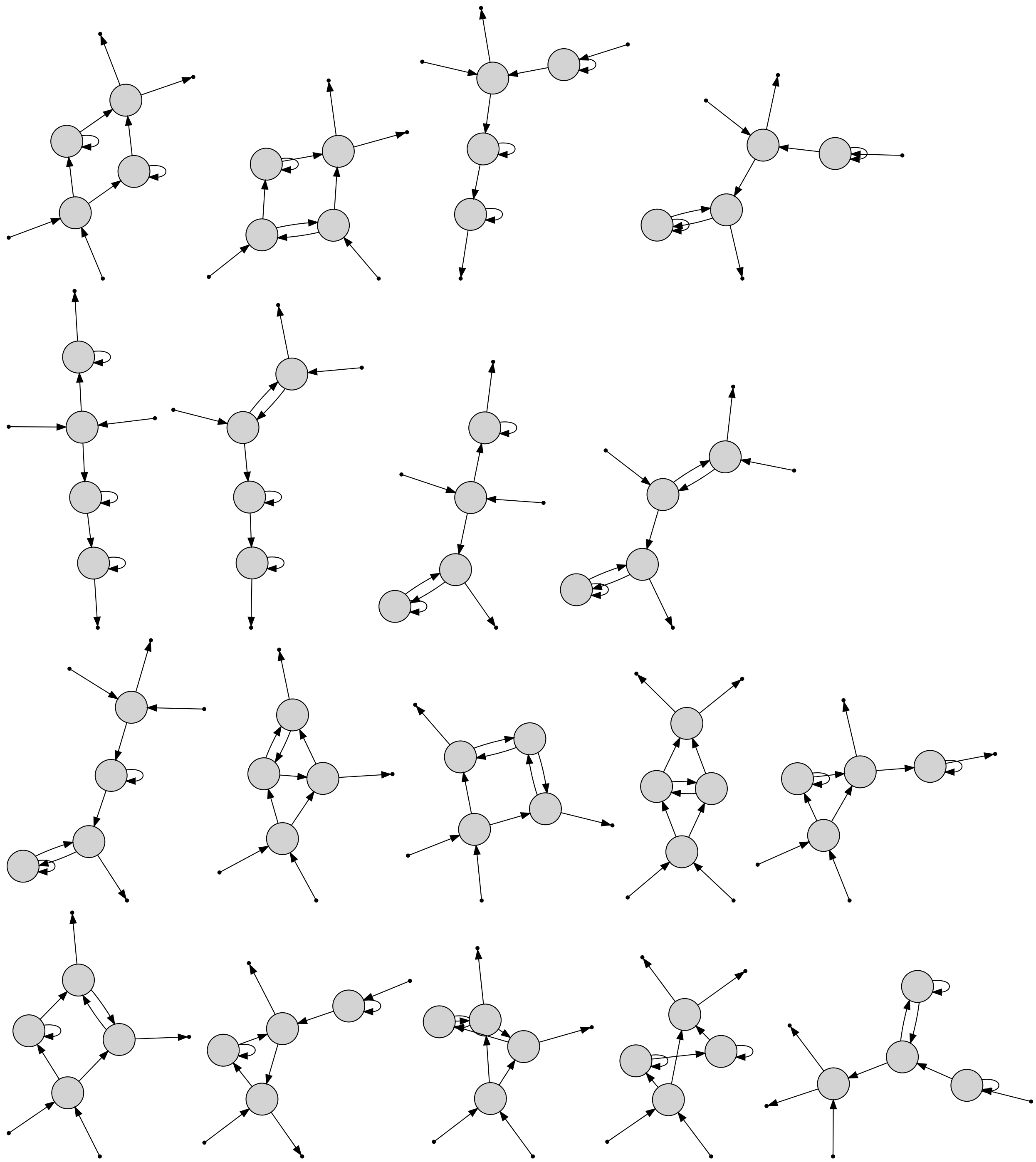


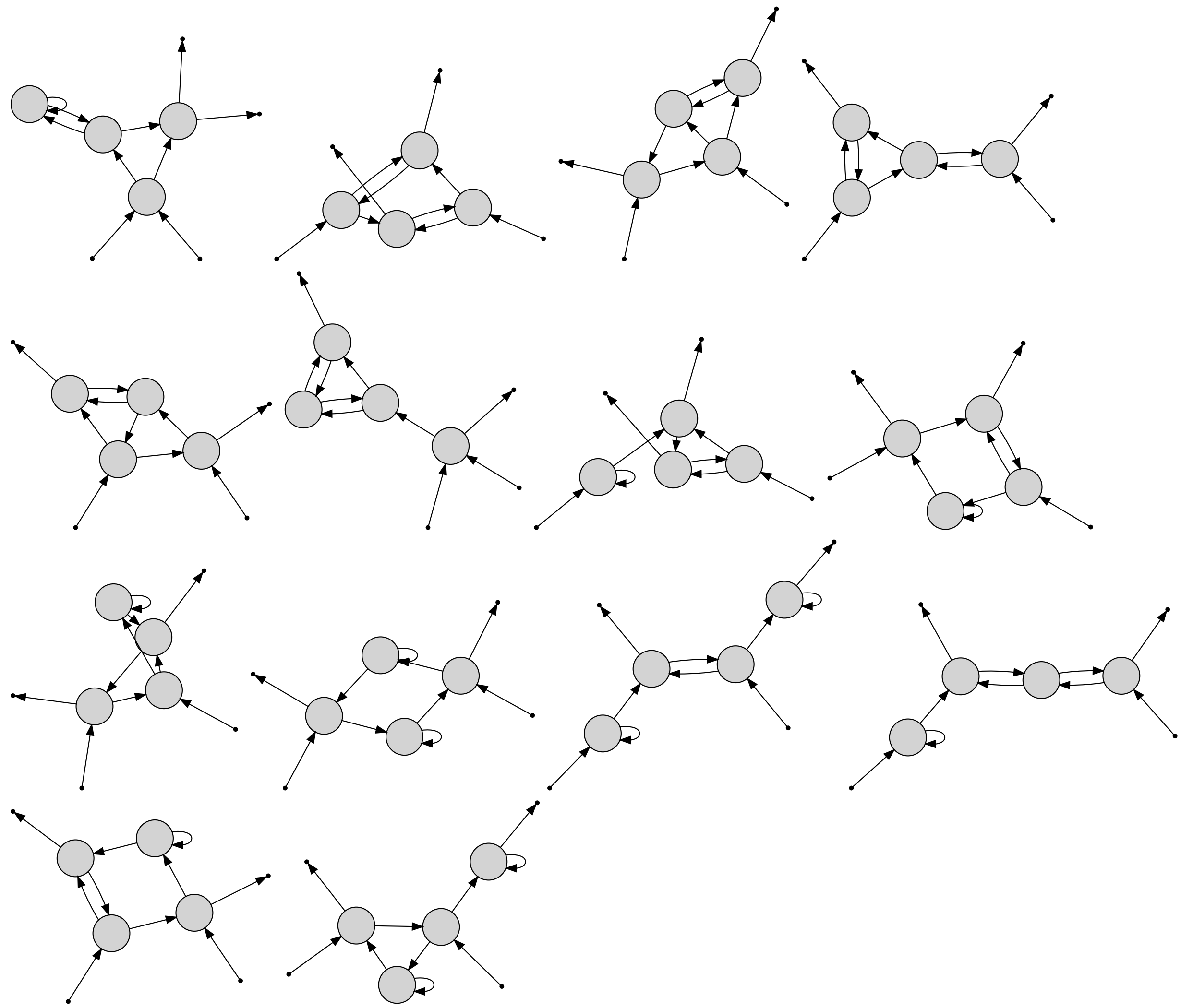
3.1.4. *8 nodes.* These are the 81 graphs with 8 nodes (18 without loops):











3.2. **Illustrations Disconnected Graphs.** The difference between the numbers of the first and second line in Table 5 alludes to the quadripole graphs with at least 2 components. They are of two types:

- One component is a connected quadripole graph of the set $U_c^{(4)}$ and zero or more components are 2-regular digraphs of the set $U^{(0)}$.
- Two components are connected bipoles of the set $U_c^{(2)}$ [with the counting function $[U_c^{(2)}(x^2)+U_c^{(2)}(x)^2]/2$ with the usual Polya-Index argument of the Symmetric Group], and zero or more components are 2-regular digraphs of the set $U^{(0)}$.

Defining the generating functions

$$U^{(4)}(x) \equiv \sum_{n\ge 0} U^{(4)}(n)x^n; \tag{15}$$

$$U_c^{(4)}(x) \equiv \sum_{n\ge 0} U_c^{(4)}(n)x^n; \tag{16}$$

this means

$$U^{(4)}(x) = U_c^{(4)}(x)U^{(0)}(x) + \frac{U_c^{(2)}(x^2) + {U_c^{(2)}(x)}^2}{2} U^{(0)}(x). \tag{17}$$

Registering only the loopless graphs yields in the equivalent fashion

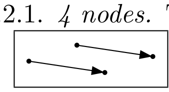
$$U_{\not L}^{(4)}(x) \equiv \sum_{n\ge 0} U_{\not L}^{(4)}(n)x^n; \tag{18}$$

$$U_{c\not L}^{(4)}(x) \equiv \sum_{n\ge 0} U_{c\not L}^{(4)}(n)x^n; \tag{19}$$

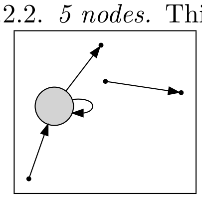
with the set of all graphs related to the connected ones via

$$U_{\not L}^{(4)}(x) = U_{c\not L}^{(4)}(x)U_{\not L}^{(0)}(x) + \frac{U_{c\not L}^{(2)}(x^2) + {U_{c\not L}^{(2)}(x)}^2}{2} U_{\not L}^{(0)}(x). \tag{20}$$

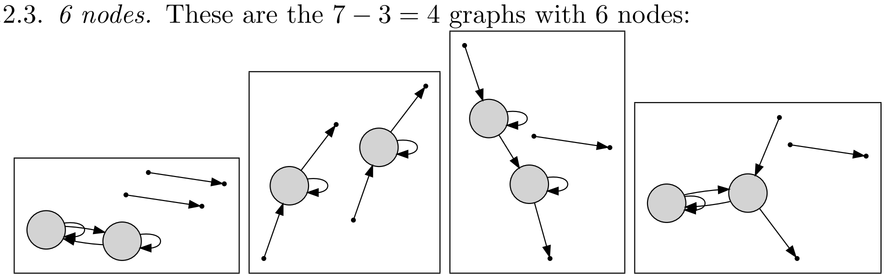
3.2.1. *4 nodes.* This is the $1-0=1$ graph with 4 nodes:



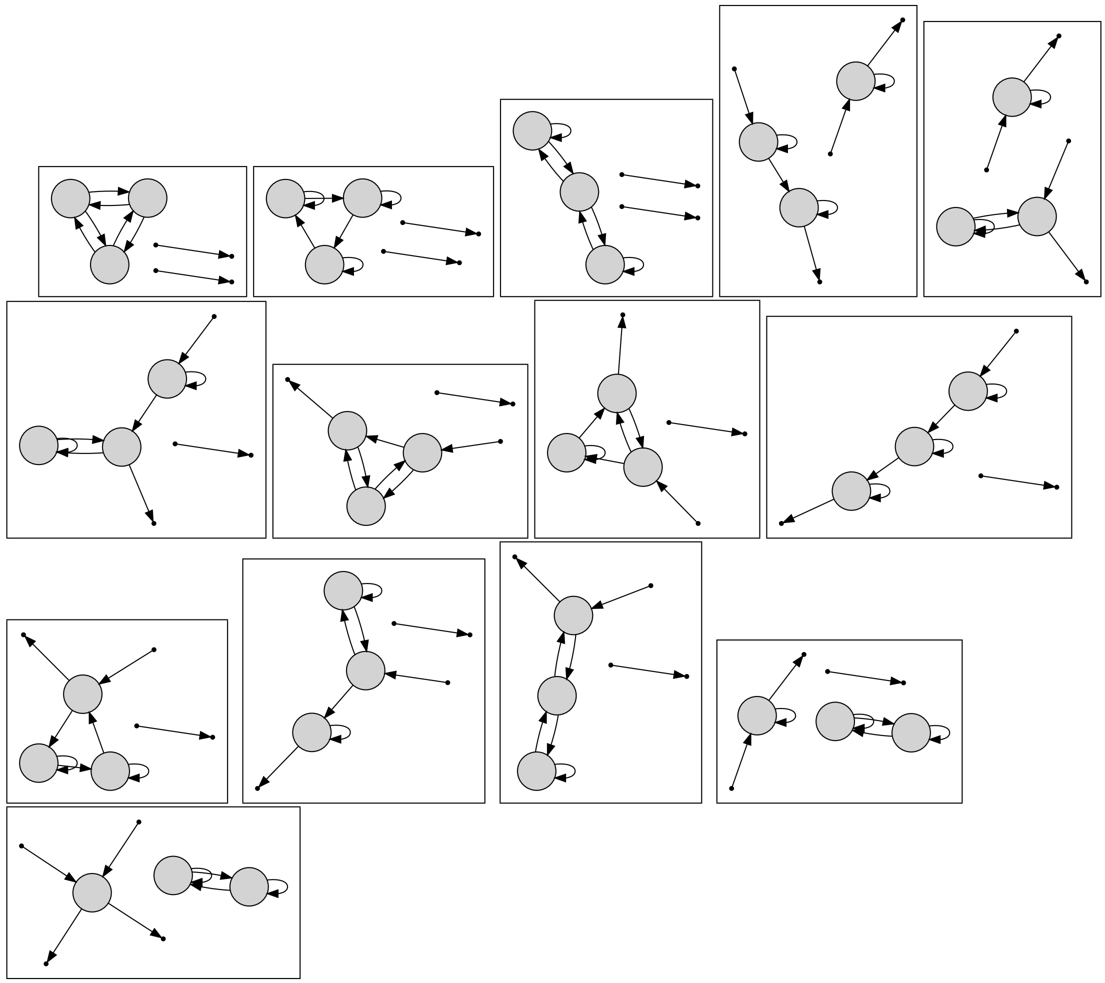
3.2.2. *5 nodes.* This is the $2-1=1$ graph with 5 nodes:



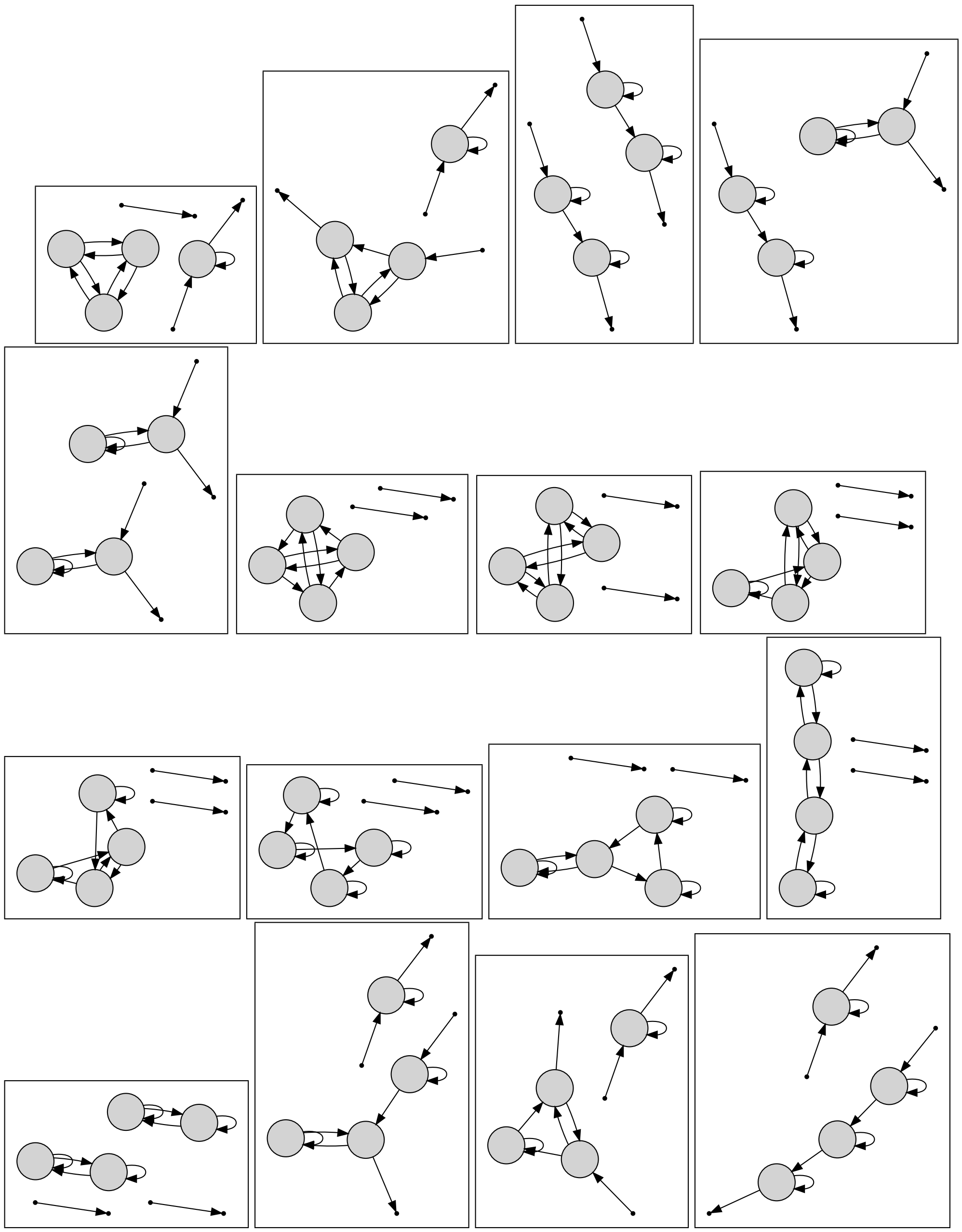
3.2.3. *6 nodes.* These are the $7-3=4$ graphs with 6 nodes:

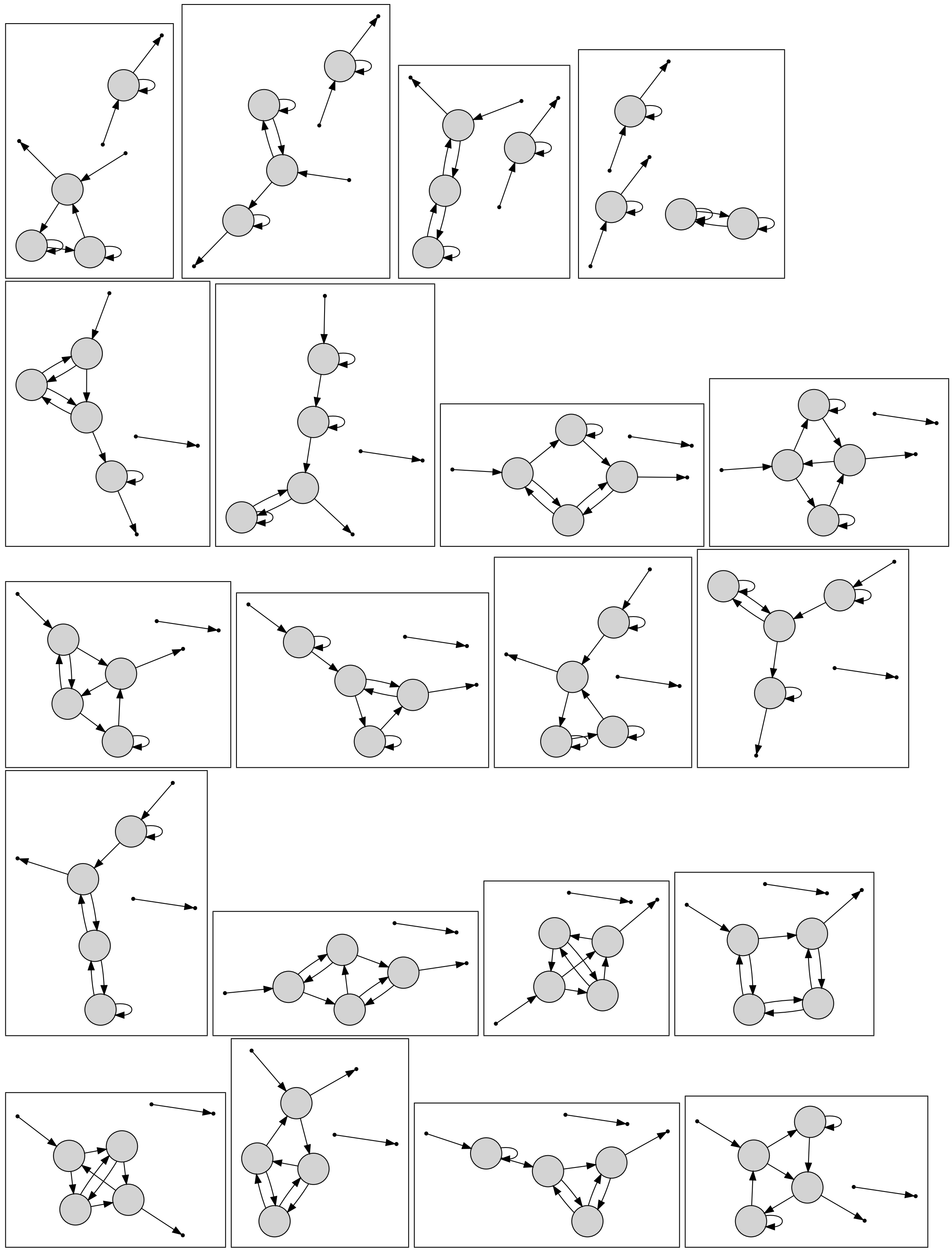


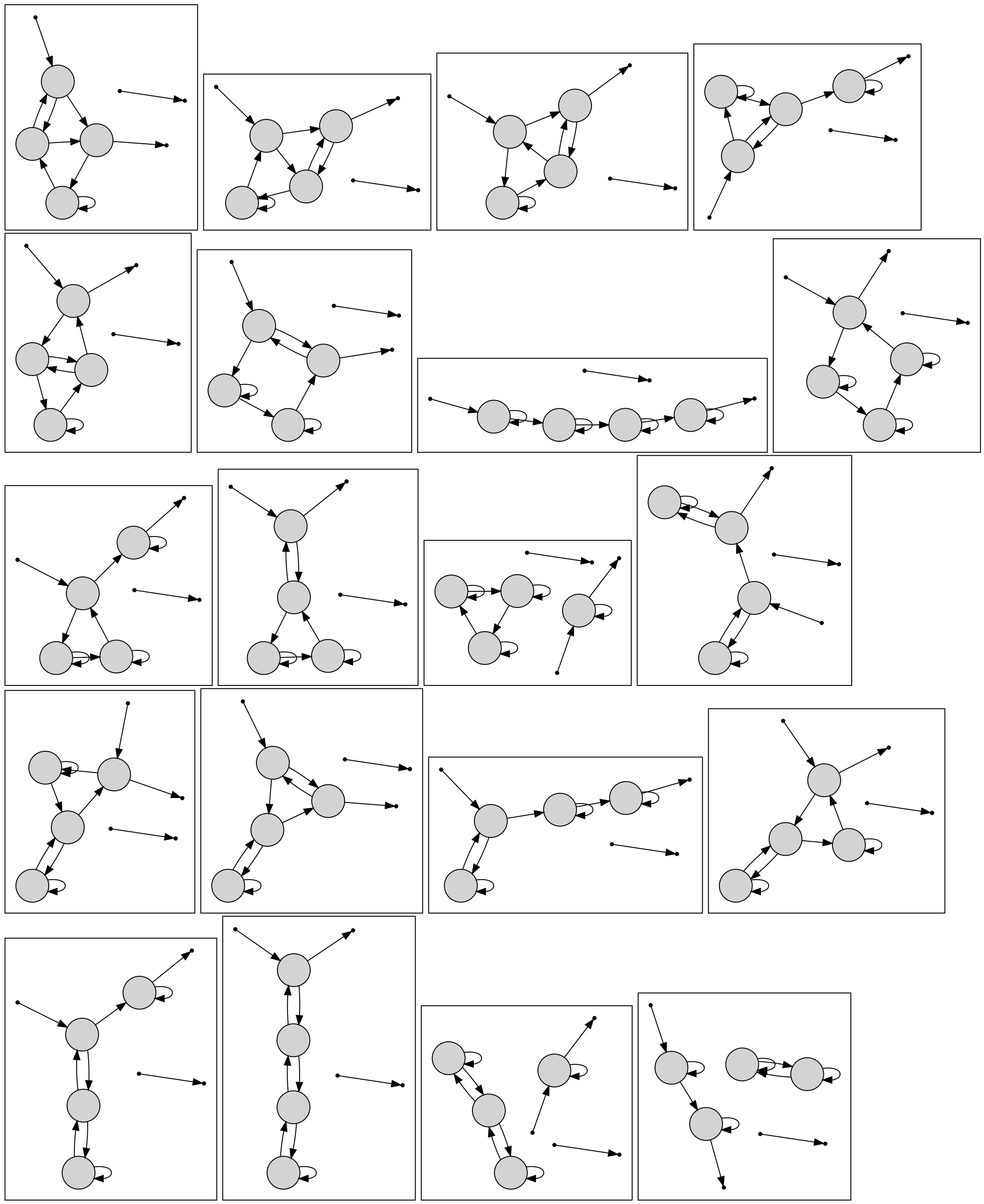
3.2.4. *7 nodes.* These are $29-15=14$ graphs with 7 nodes:

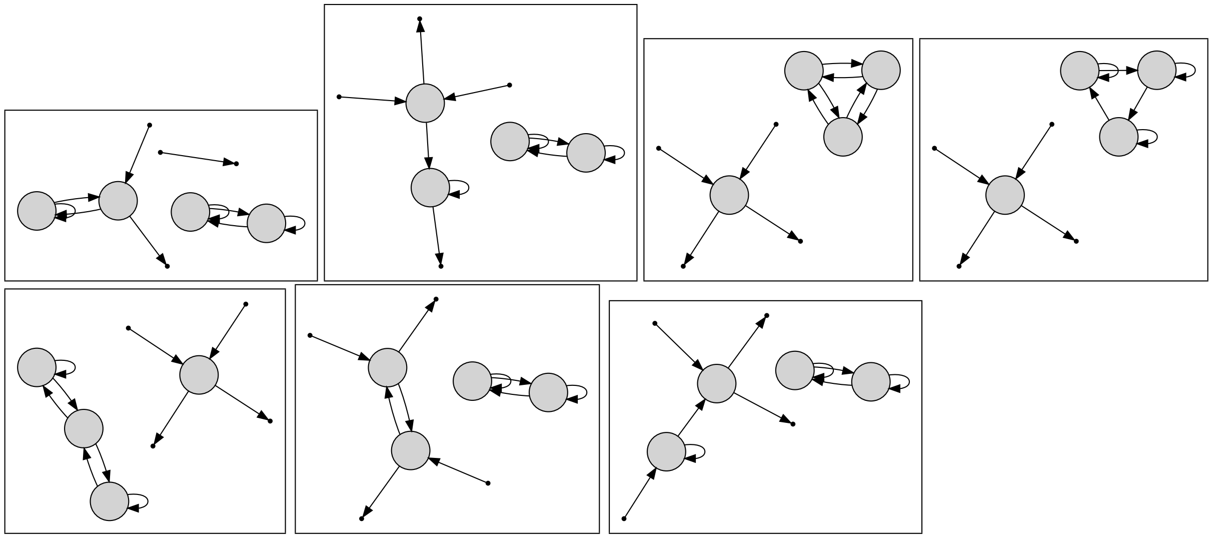


3.2.5. *8 nodes.* These are $144 - 81 = 63$ graphs with 8 nodes:









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