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On the Number of Functions Realized by Cascades and Disjunctive Networks

JON T. BUTLER, MEMBER, IEEE

Abstract—**In this paper, the number of functions realized by certain networks of two-input one-output gates are presented. Two networks are considered; one is the disjunctive network, which is characterized by the restriction that each gate output and each network input connect to exactly one gate input. The other network, the cascade, is the special case of the disjunctive networks in which each gate has at least one input which connects to a network input. For both networks, a recursion relation is derived for the number of realized switching functions dependent on exactly k variables. Both expressions have been solved by computer for k up to 15. Also, expressions are derived for the number of functions realized by cascades and disjunctive networks of two-input one-output cells, where each cell realizes any of the 16 functions on two variables.**

Index Terms—**Cascades, disjunctive networks, polyfunctional nets, switching function decomposition, switching function enumeration, universal cells.**

I. INTRODUCTION

In this paper several previously unsolved enumeration problems are considered. The first concerns the *cascade*, a network of two-input one-output gates, in which the following restrictions on interconnection apply.

1) The fan-out of each gate is one.

2) Each network input connects to the input of exactly one gate.

3) Each gate connects to at least one network input.

Fig. 1(b) shows an example of a five-input cascade. Cascades have a number of interesting properties and have been the subject of many papers; Maitra [1], Sklansky [2], Minnick [3], Stone [4], Papkonstantinou [5], Weiss [6], and Sklansky *et al.* [18]. In Section II, it is shown that cascade realizable functions are precisely the set of functions which can be decomposed into a specific form. Counting the number of functions with this property leads to a recursion relation for $N_{\text{cas}}(k)$, the number of cascade realizable functions dependent on k variables. This relation has been solved by computer for values of k up to 15, and the results indicate that $N_{\text{cas}}(k)$ varies approximately as $k!c_c{}^k b_c$, where c_c and b_c are constants.

Also considered here is the *disjunctive network*, a more general interconnection in which only restrictions 1) and 2) above apply. The two networks of Fig. 1 are examples of disjunctive networks. Disjunctive networks share some of the properties of cascades and have been studied by Levy *et al.* [7], Maruoka and Honda [8], and Butler and Breeding [9]. Because of the simplicity of gate interconnections, disjunctive networks lend themselves well to fault detection experiments. For example, the somewhat more general *fan-out free* network has been studied by Kohavi and Kohavi [10] and Berger and Kohavi [11]. Since disjunctive networks have nice fault detection properties, a prospective designer of networks would like to know how probable it is that a given function is realizable by such a network. Assuming all k variable functions are equally likely to require implementation, the probability that the function is disjunctively realizable is just the fraction of the total number of functions which are disjunctively realizable. It is shown in Section III that this fraction is quite small even for moderately sized networks. Also in Section III, it is shown that disjunctively realizable functions are precisely the set of functions with certain decompositions. As with cascade realizable functions, this leads to an expression for the number of functions realized. In particular, a recursion relation is derived for $N_{\text{dis}}(k)$, the number of disjunctively realizable functions dependent on exactly k variables.

In Section IV, the case of cascades and disjunctive networks of two-input one-output cells, where each cell realizes any of the 16 functions on two inputs, is considered. For both cascades and disjunctive networks of cells, a relation is developed for the number of functions realized when permutation of inputs is allowed.

II. THE NUMBER OF CASCADE REALIZABLE FUNCTIONS

It has been shown in Sklansky *et al.* [18] that B_k, the number of k variable symmetry types realized by cascades, is related by $B_k - 3B_{k-1} + B_{k-2} = 0$, $B_2 = 3$, and $B_1 = 1$. Solving this equation yields

$$B_k \simeq \frac{(2.62)^k}{2.24} \qquad \text{for large } k. \tag{1}$$

This result is now extended and a relation is developed for the exact number of cascade realizable functions.

Consider a cascade of two-input and one-output gates where each gate realizes one of the ten functions on two variables. Alternatively, in place of these gates one can

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The author was with the Air Force Avionics Laboratory, Wright-Patterson Air Force Base, Dayton, Ohio 45433. He is now with the Department of Electrical Engineering, Northwestern University, Evanston, Ill. 60201.

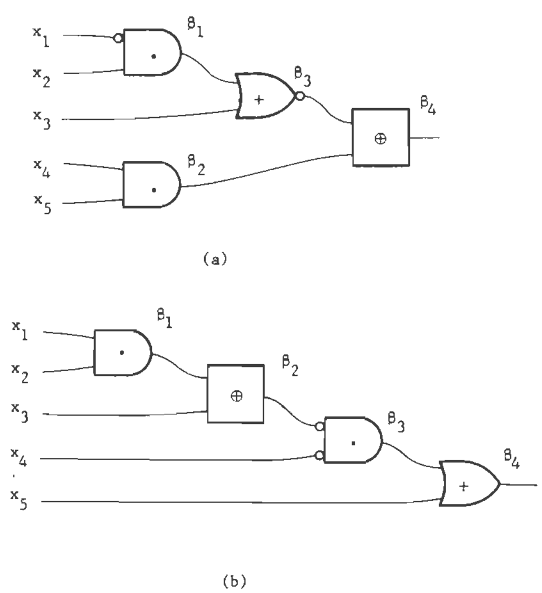

Fig. 1. Two examples of disjunctive network. (a) Noncascade. (b) Cascade.

have circuits consisting of a single AND, OR, or EXCLUSIVE OR gate plus possibly one or two inverters. Let $f(X)$ denote the function realized by a cascade, where $X = \{x_1, x_2, \cdots, x_k\}$ is the set of inputs. The procedure for calculating $N_{cas}(k)$ is based on the observation that each cascade realizable function, $f(X)$, dependent on k inputs can be expressed as[1]

$$f(X) = g(h(X - \{x_j\}), x_j) \qquad \text{for } 1 \leq j \leq k \quad (2)$$

where $g(h, x_j)$ is the function realized by the gate whose output is the network output and $h(X - \{x_j\})$ is the (cascade realizable) function realized by the other gates in the network. Thus, each cascade realizable function is a member of at least one set, S_j, where S_j includes all functions with the decomposition[2] of (2). Since the union, $S_1 \cup S_2 \cup \cdots \cup S_k$, is exactly the set of cascade realizable functions, the inclusion/exclusion principle[3] can be used to derive $N_{cas}(k)$ as follows

$$\begin{aligned} N_{cas}(k) &= | S_1 \cup S_2 \cup \cdots \cup S_k | \\ &= \sum_{i_1} | S_{i_1} | - \sum_{\substack{i_1, i_2 \\ i_1 \neq i_2}} | S_{i_1} \cap S_{i_2} | \\ &\quad + \sum_{\substack{i_1, i_2, i_3; i_1 \neq i_2 \\ i_2 \neq i_3; i_3 \neq i_1}} | S_{i_1} \cap S_{i_2} \cap S_{i_3} | + \cdots \\ &\quad + (-1)^k | S_1 \cap S_2 \cap \cdots \cap S_k | \end{aligned} \quad (3)$$

where

$$\sum_{i_1} | S_{i_1} |$$

is summed over all $i_1 \in \{1, 2, \cdots, k\}$,

$$\sum_{\substack{i_1, i_2 \\ i_1 \neq i_2}} | S_{i_1} \cap S_{i_2} |$$

is summed over all pairs $i_1, i_2 \in \{1, 2, \cdots, k\}$ such that $i_1 \neq i_2$, etc.

$$| S_{i_1} \cap S_{i_2} \cap \cdots \cap S_{i_p} |,$$

of course, is the number of functions with p decompositions of the form shown in (2), where $j = i_1, i_2, \cdots, i_p$. From Theorem 4.9 of Curtis [13], a function with these decompositions also has the following form:

$$f(X) = H(X - \{x_{i_1}, x_{i_2}, \cdots, x_{i_p}\}) \bigcirc x_{i_1}^* \bigcirc x_{i_2}^* \bigcirc \cdots \bigcirc x_{i_p}^* \quad (4)$$

where $\bigcirc \in \{\cdot, +, \oplus\}$, $x_{i_j}^* \in \{x_{i_j}, \bar{x}_{i_j}\}$, and $i_a \neq i_b$ for $a \neq b$.

For some assignment of 0's and 1's to the variables x_{i_1}, x_{i_2}, $\cdots$, and x_{i_p}, $f(X)$ is exactly $H(X - \{x_{i_1}, x_{i_2}, \cdots, x_{i_p}\})$. Consider a cascade which realizes $f(X)$, in which the inputs x_{i_1}, x_{i_2}, $\cdots$, and x_{i_p} are replaced by 0's and 1's according to this assignment. Such a network realizes H and contains redundant gates, those gates which connect to 0's and 1's. If $p \leq k - 2$, there is at least one nonredundant gate, and so the functions realized by all redundant gates can be incorporated into adjacent nonredundant gates without changing the function realized, thus eliminating nonredundant gates. Since the resulting network is a cascade which realizes $H(X - \{x_1, x_2, \cdots, x_p\})$, we have the following result.

Lemma 1: A cascade realizable function, $f(X)$, which can be decomposed in all of the following p ways

$$\begin{aligned} f(X) &= g_1(h_1(X - \{x_{i_1}\}), x_{i_1}) \\ f(X) &= g_2(h_2(X - \{x_{i_2}\}), x_{i_2}) \\ &\ \ \vdots \\ f(X) &= g_p(h_p(X - \{x_{i_p}\}), x_{i_p}), \end{aligned} \quad (5)$$

also has the decomposition

$$f(X) = H(X - \{x_{i_1}, x_{i_2}, \cdots, x_{i_p}\}) \bigcirc x_{i_1}^* \bigcirc x_{i_2}^* \bigcirc \cdots \bigcirc x_{i_p}^* \quad (6)$$

where H is cascade realizable for $1 \leq p \leq k - 2$.

Trivially, the converse of Lemma 1 is true, and so

$$| S_{i_1} \cap S_{i_2} \cap \cdots \cap S_{i_p} |$$

is exactly the number of functions of the form given in (6). For each of the $N_{cas}(k - p)$ choices for H in (6), if $\bigcirc = \cdot$ or $+$, the 2^p ways to complement or leave uncomplemented the variables x_{i_1}, x_{i_2}, $\cdots$, and x_{i_p} result in a distinct $f(X)$. Therefore, for $\bigcirc = \cdot$ and $+$, there are

[1] Except where noted, the switching functions in this paper are nonvacuous in the variables of the argument (depend on all variables). Set notation is used to express the variables of the argument when more than one variable is involved. Thus, $h(X - \{x_i, x_j\})$, for example, denotes a function h dependent on all variables in X except x_i and x_j.

[2] A discussion of switching function decomposition appears in Kohavi [12, pp. 103–114] and Curtis [13].

[3] A discussion of the inclusion/exclusion principle appears in Liu [14, pp. 96–106].

$2^{p+1}N_{cas}(k-p)$ distinct functions. Lemma 1 of Butler and Breeding [9] shows that if $H(X - \{x_{i_1}, x_{i_2}, \cdots, x_{i_p}\})$ is a cascade realizable function on k variables, then so also is the complement function $\bar{H}(X - \{x_{i_1}, x_{i_2}, \cdots, x_{i_p}\})$. From this and the fact that $\bar{a} \oplus b = \overline{a \oplus b}$, it follows that there are $N_{cas}(k-p)$ cascade realizable functions when $\bigcirc = \oplus$. Thus,

$$|S_{i_1} \cap S_{i_2} \cap \cdots \cap S_{i_p}| = (2^{p+1}+1)N_{cas}(k-p) \quad \text{for } 1 \le p \le k-2. \quad (7)$$

For a function $f(X)$ with the decomposition of (5), where $p = k-1$ and $p = k$, it can be seen, again from Lemma 4.9 of Curtis [13], that $f(X)$ also has the decomposition

$$f(X) = x_1^* \bigcirc x_2^* \bigcirc \cdots \bigcirc x_k^*. \quad (8)$$

Since the converse is true trivially, the number of cascade realizable functions for these cases is given as

$$|S_{i_1} \cap S_{i_2} \cap \cdots \cap S_{i_{k-1}}| = |S_1 \cap S_2 \cap \cdots \cap S_k| = (2^k+1)2. \quad (9)$$

Comparing (9) and (7), it is appropriate to let

$$N_{cas}(1) = 2. \quad (10)$$

Since there are $\binom{k}{p}$ ways to choose the p subscripts in

$$|S_{i_1} \cap S_{i_2} \cap \cdots \cap S_{i_p}|$$

and for each choice (7) or (9) apply

$$\sum_{i_1, i_2, \cdots, i_p} |S_{i_1} \cap S_{i_2} \cap \cdots \cap S_{i_p}| = \binom{k}{p}(2^{p+1}+1)N_{cas}(k-p) \quad \text{for } 1 \le p \le k-1. \quad (11)$$

The substitution of (11) and (9) into (3) yields the following.

Theorem 1: The number of cascade realizable functions, $N_{cas}(k)$ dependent on k variables is

$$N_{cas}(k) = \sum_{p=1}^{k-1} (-1)^{p+1} \binom{k}{p} (2^{p+1}+1)N_{cas}(k-p) - (-1)^k(2^k+1)N_{cas}(1) \quad (12)$$

where $N_{cas}(1) = 2$.

A computer program has been written which computes the value of $N_{cas}(k)$ as given by (12) for values of k up to 15. The result of this computation is shown in Table I. Although an exact explicit expression for $N_{cas}(k)$ has not been discovered, the data of Table I indicate the following.

Conjecture:

$$N_{cas}(k) \simeq k! c_c^k b_c \quad \text{for large } k \quad (13)$$

where

TABLE I
NUMBER OF k-VARIABLE CASCADE REALIZABLE FUNCTIONS

k	$N_{cas}(k)$
2	10
3	114
4	1,842
5	37,226
6	902,570
7	25,530,658
8	825,345,250
9	30,016,622,298
10	1,212,957,186,330
11	53,916,514,446,482
12	2,614,488,320,210,258
13	137,345,270,749,953,610
14	7,770,078,330,925,987,210
15	470,977,659,902,530,345,986

$$c_c = 4.04095\cdots$$

$$b_c = 0.28790\cdots.$$

Indeed, if (13) is substituted into (12) we have

$$k! c_c^k b_c = \sum_{p=1}^{k-1} (-1)^{p+1} \frac{k!}{(k-p)!p!} \cdot (2^{p+1}+1)(k-p)! c_c^{k-p} b_c - (-1)^k(2^k+1)c_c b_c.$$

Rearranging,

$$1 = -2\sum_{p=1}^{k-1} \frac{(2/c_c)^p}{p!} - \sum_{p=1}^{k-1} \frac{(1/c_c)^p}{p!} - \frac{(-1)^k(2^k+1)}{c_c^{k-1}k!}.$$

When k approaches ∞, the right term approaches 0 and the other two terms become the series equivalents of exponential functions. Thus, in the limit,

$$2 = 2\exp(2/c_c) + \exp(1/c_c). \quad (14)$$

Solving for c_c yields

$$c_c = \left(\ln \frac{4}{\pm(17)^{1/2} - 1}\right)^{-1}.$$

Choosing the negative sign, yields infinitely many complex values of c_c. Choosing the + sign gives $c_c = 4.04095\cdots$ the value consistent with the data of Table I. By way of comparison, the values of $N_{cas}(k)$ as obtained from (13) approximate the exact values as obtained by (12) to within 6.0 percent, 0.01 percent, 0.03 percent, and 0.0002 percent for $k = 2$, 3, 4, and 5, respectively.

III. THE NUMBER OF DISJUNCTIVE REALIZABLE FUNCTIONS

In this section an expression is developed for $N_{dis}(k)$, the number of functions realized by k-input disjunctive networks. For the case of the general combinatorial network of two-input one-output gates, there are three gate types.

Type 1: Both gate inputs connect to a network input.

Type 2: One gate input only connects to a network input.

Type 3: Neither gate input connects to a network input.

It is assumed that the networks are nondegenerate in that gate inputs not connected to network inputs connect to the outputs of other gates. In Fig. 1(a), for example, gates β_1 and β_2 are Type 1, β_3 is Type 2, and β_4 is Type 3.

Lemma 2: A combinatorial network η of two-input one-output gates has at least one Type 1 gate.

Proof: Let a *path* $P_n = (\beta_1,\beta_2,\cdots,\beta_n)$ in η denote a sequence of n gates, such that the output of β_i connects to an input of β_{i+1}, for $1 \leq i \leq n-1$. Since there are no closed loops in a combinatorial network, no cell appears more than once in any path. Thus, it is appropriate to consider a longest path $P_m' = (\beta_1',\beta_2',\cdots,\beta_m')$, where $m \geq n$ for all other paths P_n in η. But this implies β_1' is neither Type 2 nor Type 3, for otherwise there would be a longer path in η. Thus, β_1' is Type 1. Q.E.D.

From Lemma 2, it follows that a disjunctive network η can be decomposed as shown in Fig. 2, where η' is a disjunctive network containing one fewer gate than η. Thus, a disjunctively realizable function $f(X)$ has the following decomposition.

$$f(X) = h(g(x_{i_1},x_{i_2}),x_{i_3},\cdots,x_{i_k}) \qquad \text{for } i_j \in \{1,2,\cdots,k\} \tag{15}$$

where g is realized by gate β of Fig. 2 and h is realized by η'. Thus, each disjunctively realizable function is a member of at least one set $S_{i_1i_2}$, where $S_{i_1i_2}$ includes all functions with the decomposition of (15). The application of the inclusion/exclusion principle yields for the number of disjunctively realizable functions

$$\begin{aligned} N_{\text{dis}}(k) &= | S_{12} \cup S_{13} \cup \cdots \cup S_{k-1k} | \\ &= \sum_{\substack{a,b \\ a\neq b}} | S_{ab} | - \sum_{\substack{a,b,c,d \\ a\neq b, c\neq d}} | S_{ab} \cap S_{cd} | \\ &\quad + \cdots (-1)^{\binom{k}{2}+1} | S_{12} \cap S_{13} \cap \cdots \cap S_{k-1k} |. \end{aligned} \tag{16}$$

Each term $| S_{ab} \cap S_{cd} \cap \cdots \cap S_{ef} |$ in (16) represents the number of functions with decompositions of the form (15), where $(i_1,i_2) = (a,b),(c,d),\cdots,(e,f)$. Let D, a *decomposition set* of $f(X)$, denote this set of decompositions, and let $G_D(V,E)$, a *decomposition graph* of $f(X)$, be an undirected [4]graph with vertex set $V = \{x_1,x_2,\cdots,x_k\}$ and edge set E defined as follows; edge $e_{ab} \in E$ if and only if $h(g(x_a,x_b), X - \{x_a,x_b\}) \in D$. $G_D(V,E)$ consists of m disjoint connected subgraphs $G_1(V_1,E_1)$, $G_2(V_2,E_2)$, $\cdots$, and $G_m(V_m, E_m)$, such that $V_1 \cup V_2 \cup \cdots \cup V_m = V$ and $E_1 \cup E_2 \cup \cdots \cup E_m = E$. Let $G_D(V,E)$ be undefined if D is null.

Theorem 2: Let $f(X)$ be a function decomposable according to the decompositions in a decomposition set D. For each subgraph $Gi(Vi,Ei)$ in $G_D(V,E)$ with the property $| Vi | \geq 2$, $f(X)$ has the decomposition

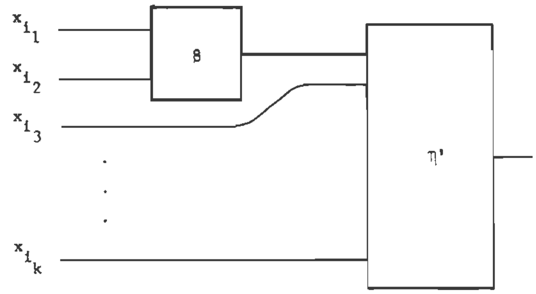

Fig. 2. Decomposition of a disjunctive network into a gate and another disjunctive network.

$$f(X) = H(x_{i_1}^* \bigcirc x_{i_2}^* \bigcirc \cdots \bigcirc x_{i_p}^*,x_{i_{p+1}},\cdots,x_{i_k}) \tag{17}$$

where $x_{i_1},x_{i_2},\cdots,x_{i_p} \in Vi$.

Proof: If $| Vi | = 2$ there is only one decomposition of the form shown in (15). Since $g(x_{i_1},x_{i_2})$ is realized by a gate, it depends on both x_{i_1} and x_{i_2}. Thus,

$$g(x_{i_1},x_{i_2}) = x_{i_1}^* \bigcirc x_{i_2}^*. \tag{18}$$

Substituting (18) into (15) yields (17), proving the assertion.

For $| Vi | \geq 3$ there must be at least vertex $x_{i_1} \in Vi$ which is incident to at least two edges $e_{i_1i_2}$ and $e_{i_1i_3}$; otherwise $Gi(Vi,Ei)$ would be disconnected. Thus, $f(X)$ has at least two decompositions,

$$\begin{aligned} f(X) &= h_1(g_1(x_{i_1},x_{i_2}),X - \{x_{i_1},x_{i_2}\}) \\ f(X) &= h_2(g_2(x_{i_1},x_{i_3}),X - \{x_{i_1},x_{i_3}\}). \end{aligned} \tag{19}$$

Applying Theorem 4.5 of Curtis [13], shows that $f(X)$ also has the decomposition

$$f(X) = H(x_{i_1}^* \bigcirc x_{i_2}^* \bigcirc x_{i_3}^*,X - \{x_{i_1},x_{i_2},x_{i_3}\}). \tag{20}$$

If $| Vi | = 3$ the assertion is proved. If $| Vi | > 3$ there is at least one vertex, call it x_{i_4}, which is adjacent[5] to x_{i_1}, x_{i_2}, or x_{i_3}; otherwise $Gi(Vi,Ei)$ would be disconnected. Assume without loss of generality, that x_{i_4} is adjacent to x_{i_1}. Thus, $f(X)$ also has the decomposition

$$f(X) = h_3(g_3(x_{i_1},x_{i_4}),X - \{x_{i_1},x_{i_4}\}). \tag{21}$$

From the application of Theorem 4.5 of Curtis [13] to (20) and (21) it follows that

$$f(X) = H'((x_{i_2}^* \square x_{i_3}^*) \triangle x_{i_1}^* \triangle x_{i_4}^*, X - \{x_{i_1},x_{i_2},x_{i_3},x_{i_4}\}) \tag{22}$$

where $\square,\triangle \in \{\cdot,+,\oplus\}$. From (22), it follows that

$$f(X) = H''((x_{i_2}^* \square x_{i_3}^*) \triangle x_{i_1}^*,X - \{x_{i_1},x_{i_2},x_{i_3}\}). \tag{23}$$

From Corollary 3.1A of Curtis [13], a function with a simple disjunctive decomposition can have only two decompositions involving the same variables. Both decompositions have the form expressed in (20). Thus, the argument of H in (20) involving x_{i_1}, x_{i_2}, and x_{i_3} is equal to the corresponding argument of H'' in (23). Thus,

[4] Graph theoretical concepts are discussed Liu [14, pp. 167–182] and in Harary [15].

[5] Two vertices x_a and x_b are adjacent if there exists an edge e_{ab} incident to both.

$$\square = \triangle = \bigcirc. \quad (24)$$

Therefore, (22) can be written as follows:

$$f(X) = H'(x_{i_1}{}^* \bigcirc x_{i_2}{}^* \bigcirc x_{i_3}{}^* \bigcirc x_{i_4}{}^*, X - \{x_{i_1}, x_{i_2}, x_{i_3}, x_{i_4}\}). \quad (25)$$

If $|V_i| = 4$, the assertion is proved. Otherwise, the above procedure can be applied starting with (25) to yield the assertion for any p. Q.E.D.

If $G_D(V,E)$ has more than one disjoint connected subgraph with the property $|V_i| > 2$, then D can be broken down into disjoint nonempty subsets, $D_1, D_2, \cdots$, and D_s, where the decompositions in subset D_i correspond to edges in these subgraphs. For a function $f(X)$ with such a decomposition set, it can be concluded from Theorem 2 that $f(X)$ also has all of the following decompositions:

$$f(X) = H_1(x_{i_1}{}^* \bigcirc x_{i_2}{}^* \bigcirc \cdots \bigcirc x_{i_a}{}^*, X - \{x_{i_1}, x_{i_2}, \cdots, x_{i_a}\})$$

$$f(X) = H_2(x_{i_{a+1}}{}^* \triangle x_{i_{a+1}}{}^* \triangle \cdots \triangle x_{i_{a+b}}{}^*, X - \{x_{i_{a+1}}, x_{i_{a+2}}, \cdots, x_{i_{a+b}}\})$$

$$\vdots$$

$$f(X) = H_s(x_{i_{a+b+\cdots+y+1}}{}^* \square x_{i_{a+b+\cdots+y+2}}{}^* \square \cdots \square x_{i_{a+b+\cdots+y+z}}{}^*, X - \{x_{i_{a+b+\cdots+y+1}}, x_{i_{a+b+\cdots+y+2}}, \cdots, x_{i_{a+b+\cdots+y+z}}\}). \quad (26)$$

Applying Theorem 4.10 of Curtis [13], yields the following result.

Theorem 3: Let $f(X)$ be a function decomposable according the decompositions in a decomposition set D. Then, $f(X)$ is also decomposable as follows:

$$f(X) = H(x_{i_1}{}^* \bigcirc x_{i_2}{}^* \bigcirc \cdots \bigcirc x_{i_a}{}^*, x_{i_{a+1}}{}^* \triangle x_{i_{a+2}}{}^* \triangle \cdots \triangle x_{i_{a+b}}{}^*, \cdots, x_{i_{a+b+\cdots+y+1}} \square x_{i_{a+b+\cdots+y+2}} \square \cdots \square x_{i_{a+b+\cdots+y+z}}, \cdots, x_{i_k}) \quad (27)$$

where

$$x_{i_1}, x_{i_2}, \cdots, x_{i_a} \in V_1; x_{i_{a+1}}, x_{i_{a+2}}, \cdots, x_{i_{a+b}} \in V_2; \cdots;$$

and

$$x_{i_{a+b+\cdots+y+1}}, x_{i_{a+b+\cdots+y+2}}, \cdots, x_{i_{a+b+\cdots+y+z}} \in V_s$$

for V_j, the jth disjoint connected subgraph vertex set with the property $|V_j| \geq 2$, $1 \leq j \leq s$.

Now consider the case where the function $f(X)$ of Theorem 3 is realized by a disjunctive network η. For at least one assignment of 0's and 1's to

$$x_{i_2}, \cdots, x_{i_a}; x_{i_{a+2}}, \cdots, x_{i_{a+b}}; \cdots; x_{i_{a+b+\cdots+y+2}}, \cdots x_{i_{a+b+\cdots+y+z}}$$

the functions

$$x_{i_1}{}^* \bigcirc x_{i_2}{}^* \bigcirc \cdots \bigcirc x_{i_a}{}^*, x_{i_{a+1}}{}^* \triangle x_{i_{a+2}}{}^* \triangle \cdots \triangle x_{i_{a+b}}{}^*, \cdots, x_{i_{a+b+\cdots+y+1}}{}^* \square x_{i_{a+b+\cdots+y+2}}{}^* \square \cdots \square x_{i_{a+b+\cdots+y+z}}{}^*$$

reduce, respectively, to

$$x_{i_1}{}^*, x_{i_{a+1}}{}^*, \cdots, x_{i_{a+b+\cdots+y+1}}{}^*.$$

For this same assignment of 0's and 1's to the input variables of η, η realizes

$$H(x_{i_1}{}^*, x_{i_{a+1}}{}^*, \cdots, x_{i_{a+b+\cdots+y+1}}{}^*, \cdots, x_{i_k}). \quad (28)$$

Those gates in η which connect to inputs fixed by the assignment to 0 or 1 are redundant, realizing $c = y^*$, where c is the gate output and y is the other input. If H has at least two arguments, then η has at least one nonredundant gate. Therefore, the functions realized by redundant gates can be incorporated into the functions realized by nonredundant gates, eliminating the former. Call the new network η'. η' realizes, of course, (28) and is disjunctive, since the fan-out of each gate is still one. If x_{i_1}, for example, appears complemented in (28), then a new network can be formed which realizes[6]

$$H(x_{i_1}, x_{i_{a+1}}{}^*, \cdots, x_{i_{a+b+\cdots+y+1}}{}^*, \cdots, x_{i_k})$$

by changing the function of the gate to which x_{i_1} connects. Similarly, all other gates connecting to inputs which are complemented in (28) can be so modified. This forms η'', a disjunctive network realizing

$$H(x_{i_1}, x_{i_{a+1}}, \cdots, x_{i_{a+b+\cdots+y+1}}, \cdots, x_{i_k}).$$

This proves the following.

Corollary 3.1: Let $f(X)$ be a disjunctively realizable function decomposable according to the decompositions in a decomposition set D. Then, $f(X)$ is also decomposable as:

$$f(X) = H(x_{i_1}{}^* \bigcirc x_{i_2}{}^* \bigcirc \cdots \bigcirc x_{i_a}{}^*, x_{i_{a+1}}{}^* \triangle x_{i_{a+2}}{}^* \triangle \cdots \triangle x_{i_{a+b}}{}^*, \cdots, x_{i_{a+b+\cdots+y+1}}{}^* \square x_{i_{a+b+\cdots+y+2}}{}^* \square \cdots \square x_{i_{a+b+\cdots+y+z}}{}^*, \cdots, x_{i_k}) \quad (29)$$

where H is a disjunctively realizable function and where

$$x_{i_1}, x_{i_2}, \cdots, x_{i_a} \in V_1; x_{i_{a+1}}, x_{i_{a+2}}, \cdots, x_{i_{a+b}} \in V_2; \cdots; x_{i_{a+b+\cdots+y+1}}, x_{i_{a+b+\cdots+y+2}}, \cdots, x_{i_{a+b+\cdots+y+z}} \in V_s,$$

where V_j is the jth disjoint connected subgraph vertex set such that $|V_j| \geq 2$, $1 \leq j \leq s$.

From Lemma 4 of Butler and Breeding [9], there exist disjunctive networks which realize the functions

$$x_{i_1}{}^* \bigcirc x_{i_2}{}^* \bigcirc \cdots \bigcirc x_{i_a}{}^*, x_{i_{a+1}}{}^* \triangle x_{i_{a+2}}{}^* \triangle \cdots \triangle x_{i_{a+b}}{}^*, \cdots,$$

and

$$x_{i_{a+b+\cdots+y+1}}{}^* \square x_{i_{a+b+\cdots+y+2}}{}^* \square \cdots \square x_{i_{a+b+\cdots+y+z}}{}^*.$$

Further, since there exists a disjunctive network realizing H, the interconnection of networks suggested by (29) results in a disjunctive network realizing $f(X)$. Thus, it follows that $|S_{ab} \cap S_{cd} \cap \cdots \cap S_{ef}|$, the number of disjunctively realizable functions with decomposition set D,

[6] This is a direct consequence of Lemma 2 of Butler and Breeding [9].

is precisely the number of functions with the decomposition of (29). To determine $|S_{ab} \cap S_{cd} \cap \cdots \cap S_{ef}|$, observe that there are $N_{dis}(k-s)$ ways to choose H. For each choice of H, when $\bigcirc = \cdot$ or $+$ there are 2^{a+1} ways to complement or leave uncomplemented the variables x_{i_1}, x_{i_2}, $\cdots$, and x_{i_a}, and when $\bigcirc = \oplus$ there are only two functions $x_{i_1} \oplus x_{i_2} \oplus \cdots \oplus x_{i_a}$ and

$$\overline{x_{i_1} \oplus x_{i_2} \oplus \cdots \oplus x_{i_a}}.$$

Thus, a total of $2^{a+1} + 2$ choices exist. However, from Lemma 2 of Butler and Breeding [9], it follows that if $H(a,\cdots,z)$ is realized by a disjunctive network, then so also is $H(\bar{a},\cdots z)$. Since for every function represented by

$$x_{i_1}{}^* \bigcirc x_{i_2}{}^* \bigcirc \cdots \bigcirc x_{i_a}{}^*,$$

the complement function is also represented by

$$x_{i_1}{}^* \bigcirc x_{i_2}{}^* \bigcirc \cdots \bigcirc x_{i_a}{}^*,$$

only one-half of the $2^{a+1} + 2$ or $2^a + 1$ choices for

$$x_{i_1}{}^* \bigcirc x_{i_2}{}^* \bigcirc \cdots \bigcirc x_{i_a}{}^*$$

result in a unique $f(X)$. In the same way, independent choices can be made for

$$x_{i_{a+1}}{}^* \triangle x_{i_{a+2}}{}^* \triangle \cdots \triangle x_{i_{a+b}}{}^*, \cdots,$$

and

$$x_{i_{a+b+\cdots+y+1}}{}^* \square x_{i_{a+b+\cdots+y+2}}{}^* \square \cdots \square x_{i_{a+b+\cdots+y+z}}{}^*.$$

Thus, for $k - s \geq 2$,

$$|S_{ab} \cap S_{cd} \cap \cdots \cap S_{ef}| = (2^a + 1)(2^b + 1)\cdots(2^z + 1)N_{dis}(k-s). \quad (30)$$

For $k - s = 1$, Theorem 3 shows that $f(X)$ has the decomposition

$$H(x_1{}^* \bigcirc x_2{}^* \bigcirc \cdots \bigcirc x_k{}^*). \quad (31)$$

H in (31) must depend on the argument, and, thus it can have only two values, $H(a) = a$ or $H(a) = \bar{a}$. For $k - s = 1$ the number of disjunctive realizable functions is

$$(2^k + 1)2. \quad (32)$$

Comparing (30) and (32), it is appropriate to choose $N_{dis}(1) = 2$. Thus,

$$|S_{12} \cap S_{13} \cap \cdots \cap S_{1k} \cap \cdots \cap S_{k-1k}| = (2^k + 1)N_{dis}(1). \quad (33)$$

Each term in (16), the summation for $N_{dis}(k)$, corresponds to a distinct decomposition graph on k vertices whose contribution to the sum is given by (30) and (33). Since all graphs except the one containing no edges are represented by a term in (16), the value of $N_{dis}(k)$ is obtained by enumerating graphs on k vertices, calculating the contribution of each, and summing the result. To facilitate this computation, we make use of a result from graph theory.

Consider the sum

$$d(k) = \sum_{\{G(V,E)\}} W(G(V,E)) \quad (34)$$

where $W(G(V,E))$ is a weight associated with graph $G(V,E)$ and the sum goes over *all* graphs on $|V| = k$ labeled vertices.

Corresponding to (34) is the counting series

$$D(x) = \sum_{k=1}^{\infty} d(k) \frac{x^k}{k!}. \quad (35)$$

It is shown in Ford and Uhlenbeck [16] that if

1) $W(G(V,E))$ is independent of the labeling of $G(V,E)$ and

2) $W(G(V,E))$ is the product of weights assigned to disjoint the connected subgraphs of $G(V,E)$, then

$$D(x) = \exp[C(X)] - 1, \quad (36)$$

where

$$C(x) = \sum_{k=1}^{\infty} c(j) \frac{x^j}{j!} \quad (37)$$

for

$$c(j) = \sum_{\{G(V,E)\}} W(G(V,E)), \quad (38)$$

where the sum of (38) goes over all *connected graphs* with j labeled vertices.

For the problem at hand let $G_1(V_1,E_1)$, $G_2(V_2,E_2)$, $\cdots$, and $G_t(V_t,E_t)$ denote the disjoint connected parts of graph $G(V,E)$, and let

$$W(G(V,E)) = W(G_1(V_1,E_1)) \cdot W(G_2(V_2,E_2)) \cdot \cdots \cdot W(G_t(V_t,E_t)) \quad (39)$$

where

$$W(G_i(V_i,E_i)) = N \qquad v_i = 1$$
$$= (-1)^{e_i}(2^{v_i} + 1)N \qquad v_i > 1 \quad (40)$$

for N, a dummy variable, and $v_i = |V_i|$, $e_i = |E_i|$.

Comparing (39) and (40) with (30), (31), and (16) yields

$$N_{dis}(k) = [-d(k) + N^k : N_{dis}(j) \to N^j, \quad j = 1,2,\cdots k-1] \quad (41)$$

where $N_{dis}(j) \to N^j$ means substitute each occurrence of N^j in the expression by $N_{dis}(j)$. The minus sign preceding $d(k)$ appears because the contribution of each term in (41) is the negative of the corresponding contribution to $N_{dis}(k)$ as given by (16). The N^k term in (41) appears because in the expression for $N_{dis}(k)$, the contribution of the graph containing no edges, is 0. Substituting (40) into (38) yields

$$c(j) = N \qquad \text{for } j = 1 \quad (42)$$

$$c(j) = \sum_{\{G(V,E)\}} (-1)^e(2^j + 1)N \qquad \text{for } j > 1. \quad (43)$$

Since the sum in (43) goes over all connected graphs with vertices, the expression can be written as

$$c(j) = (2^j + 1)N \sum_{e=0}^{\binom{j}{2}} (-1)^e \gamma(j,e) \quad (44)$$

where $\gamma(j,e)$ is the number of connected labeled graphs with e edges and j vertices. This number is[7]

$$\gamma(j,e) = \sum_{r=1}^{j} \frac{(-1)^{r+1}}{r} \int_{\{ji\}} \frac{j!}{j_1!j_2!\cdots j_r!} \begin{pmatrix} \sum_{i=1}^{r} \frac{1}{2} ji(ji-1) \\ e \end{pmatrix} \quad (45)$$

where $\int_{\{ji\}}$, the round sum, goes over all compositions $(j_1 + j_2 + \cdots + j_r = j)$ of j, for $ji \geq 1$.

Substituting (44) into (45) and rearranging yields

$$c(j) = (2^j + 1)N \left\{ \left[\sum_{r=1}^{j-1} \frac{(-1)^{r+1}}{r} \int_{\{ji\}} \frac{j!}{j_1!j_2!\cdots j_r!} \cdot \sum_{e=0}^{\binom{j}{2}} (-1)^e \begin{pmatrix} \sum_{i=1}^{r} \frac{1}{2} ji(ji-1) \\ e \end{pmatrix} \right] + (-1)^{j+1}(j-1)! \sum_{e=0}^{\binom{j}{2}} (-1)^e \begin{pmatrix} 0 \\ e \end{pmatrix} \right\}. \quad (46)$$

The sum over e in the summation over r is 0 for all ji while the sum over e in the rightmost term is 1. Thus, (46) reduces to

$$c(j) = (-1)^{j+1}(j-1)!(2^j + 1)N \quad (47)$$

for $j > 1$. Substituting (42) and (47) into (37) yields for the counting series on connected graphs

$$C(x) = N\{x - \tfrac{5}{2}x^2 + \cdots + (-1)^{j+1}[(2^j + 1)/j]x^j + \cdots\}. \quad (48)$$

Expressing exp $[C(x)]$ as an infinite series and substituting into (36) yields

$$D(x) = C(x) + 1/2C^2(x) + \cdots + (1/m!)C^m(x) + \cdots. \quad (49)$$

The coefficient of x^k in (49) is $d(k)/k!$ and so $d(k)$ can be expressed as follows

[7] For a derivation of $\gamma(j,e)$, see Ford and Uhlenbeck [16].

TABLE II
NUMBER OF k-VARIABLE DISJUNCTIVELY REALIZABLE FUNCTIONS

k	$N_{dis}(k)$
2	10
3	114
4	2,154
5	56,946
6	1,935,210
7	80,371,122
8	3,944,568,042
9	223,374,129,138
10	14,335,569,726,570
11	1,028,242,536,825,906
12	81,514,988,432,370,666
13	7,077,578,056,972,377,714
14	667,946,328,512,863,533,930
15	68,080,118,128,074,301,929,138

$$d(k) = k! \sum_{j=1}^{k} \frac{N^j}{j!} \int_{\{pi\}} D(p_1)D(p_2)\cdots D(p_j) \quad (50)$$

where

$$D(pi) = 1 \qquad \text{for } pi = 1$$
$$= (-1)^{pi+1} \frac{2^{pi} + 1}{pi} \qquad pi > 1.$$

Substituting (50) into (41) yields the following.

Theorem 4: The number of disjunctively realizable functions $N_{dis}(k)$ dependent on k variables is

$$N_{dis}(k) = -k! \sum_{j=1}^{k-1} \frac{N_{dis}(j)}{j!} \int_{\{pi\}} D(p_1)D(p_2)\cdots D(p_j) \quad (51)$$

where $N_{dis}(1) = 2$.

In expanded form (51) is

$$N_{dis}(k) = k(k-1)\tfrac{5}{2}N_{dis}(k-1) - k(k-1)(k-2) \cdot [3 + (k-3)(25/8)]N_{dis}(k-2) + \cdots + (-1)^k(k-1)!(2^k + 1)N_{dis}(1). \quad (52)$$

Table II shows $N_{dis}(k)$ for $2 \leq k \leq 15$ as computed from (52) on a digital computer. These data appear to behave in the same way as $N_{cas}(k)$; that is, for large k an approximation to (51) may be $N_{dis}(k) = k!c_d{}^k b_d$. However, if this is the case, the quality of the approximation for $k \leq 15$ is significantly inferior to the approximation for $N_{cas}(k)$. Thus, with the data at hand, c_d and b_d can only be roughly estimated as 6.8 and 0.0171, respectively. For these values, the approximation is accurate to within 84.2 percent, 71.7 percent, 59.2 percent, and 47.5 percent of the exact values as given by (51) for $k = 2$, 3, 4, and 5, respectively.

It is interesting, also, to compare the number of disjunctively realizable functions with the total number of switching functions. From Harrison [19] the total number of functions, $\alpha(k)$ dependent on exactly k variables is

$$\alpha(k) = \sum_{i=0}^{k} \binom{k}{i} (-1)^{k-i} 2^{2^i}.$$

$\alpha(k)$ approximates 2^{2^k} for large k. The percentage of functions which are disjunctively realizable is 100 percent, 52 percent, 3 percent, 10^{-3} percent, 10^{-11} percent, 10^{-29} percent, and 10^{-56} percent for $k = 2, 3, 4, 5, 6, 7$, and 8, respectively. Thus, even for moderate k, the fraction of switching functions which are disjunctively realizable is extremely small.

IV. THE NUMBER OF FUNCTIONS REALIZED BY CASCADES AND DISJUNCTIVE NETWORKS OF CELLS

In this section, the number of functions realized by networks of two-input one-output cells is considered, where each cell realizes the input/output relation

$$c = k_0\bar{a}\bar{b} + k_1\bar{a}b + k_2a\bar{b} + k_3ab \qquad (54)$$

for c, the output and a and b, the cell inputs. Since there are two ways to assign 0's and 1's to each of the coefficients in (44), there are $2^4 = 16$ different assignments in all. Each assignment results in a unique switching function $c = f(a,b)$ on two or fewer variables. A cell which can realize any of the 16 switching functions is described as *universal*.

Maitra [1] has shown that the number of functions realized by a k-input cascade of universal cells with a fixed assignment of input variables is

$$t_{\text{cas}}(k) = \frac{2\cdot 6^k + 8}{5} \qquad (55)$$

where $t_{\text{cas}}(k)$ includes the number of functions which depend on k, $k-1$, $k-2$, etc., variables. In a cascade of cells where permutation of the inputs is allowed, all cascade realizable functions dependent on k inputs, $N_{\text{cas}}(k)$, are realized. This follows from the fact that for each cascade realizable function $f(X)$ on k variables there is an assignment of variables to the inputs of the cascade and an assignment of functions to the cells such that $f(X)$ is realized. A similar statement is true for cascade realizable functions dependent on $k-i$ inputs for $k-i \geq 2$. There are $\binom{k}{k-1} N_{\text{cas}}(k-1)$ such functions. Considering functions dependent on one or no variables, there are, in all $2k(x_1,\bar{x}_1,x_2,\bar{x}_2,\cdots,x_k,\bar{x}_k)$ and two (0,1) functions, respectively. From Lemma 4 of Butler and Breeding [9] these functions are realized by the network. Taking $N_{\text{cas}}(1) = 2$ and $N_{\text{cas}}(0) = 2$, these amount to $\binom{k}{1}N_{\text{cas}}(1)$ and $\binom{k}{0}N_{\text{cas}}(0)$ functions, respectively. Since no other functions are realized by a cascade of cells, the number of functions realized when input permutations are allowed is

TABLE III
NUMBER OF FUNCTIONS REALIZED BY k-INPUT CASCADES OF CELLS ALLOWING PERMUTATION OF INPUT LABLES

k	$T_{cas}(k)$
2	16
3	152
4	2,368
5	47,688
6	1,156,000
7	32,699,080
8	1,057,082,752
9	38,444,581,640
10	1,553,526,946,144
11	69,054,999,618,888
12	3,348,574,955,346,496
13	175,908,582,307,762,312
14	9,951,733,002,164,182,048
15	603,217,074,746,723,736,776

$$T_{\text{cas}}(k) = \sum_{p=0}^{k} \binom{k}{p} N_{\text{cas}}(k-p) \qquad (56)$$

where $N_{\text{cas}}(1) = N_{\text{cas}}(0) = 2$. Table III shows $T_{\text{cas}}(k)$ for $2 \leq k \leq 15$ as computed from (56). It is seen by comparing Tables I and III that most of the functions realized by cascades of cells in which the inputs can permuted are functions dependent on all k inputs. Thi in contrast with cascades with fixed input assignments. It has been shown by Butler and Breeding [9] that the number of functions realized by disjunctive networks and thus cascades of cells with fixed input assignments and fixed interconnection that depend on all k inputs is

$$n_{\text{cas}}(k) = n_{\text{dis}}(k) = \tfrac{2}{5}(5^k). \qquad (57)$$

Thus,

$$\lim_{k\to\infty} \frac{n_{\text{cas}}(k)}{t_{\text{cas}}(k)} = \lim_{k\to\infty} \frac{\frac{2}{5}(5^k)}{(2\cdot 6^k + 8)/5} = 0. \qquad (58)$$

Considering disjunctive networks of cells, it has been shown by Maruoka and Honda [8], that disjunctive networks with fixed input assignments and fixed interconnection realize a total of

$$t_{\text{dis}}(k) = \frac{2\cdot 6^k + 8}{5} \qquad (59)$$

functions. Thus, in such networks, most of the functions realized do not depend on all k inputs, also.

By a procedure similar to that applied to cascades, it is seen that the number of functions realized by a k-input disjunctive networks of cells in which both input assignments and interconnections are variable is

TABLE IV
NUMBER OF FUNCTIONS REALIZED BY k-INPUT DISJUNCTIVE NETWORKS OF CELLS ALLOWING PERMUTATION OF INPUT LABLES

k	$T_{dis}(k)$
2	16
3	152
4	2,680
5	68,968
6	2,311,640
7	95,193,064
8	4,645,069,336
9	261,938,616,104
10	16,756,882,325,464
11	1,198,897,678,224,232
12	94,851,206,834,082,200
13	8,221,740,727,881,348,520
14	774,839,374,768,829,174,104
15	78,880,995,816,162,599,086,568

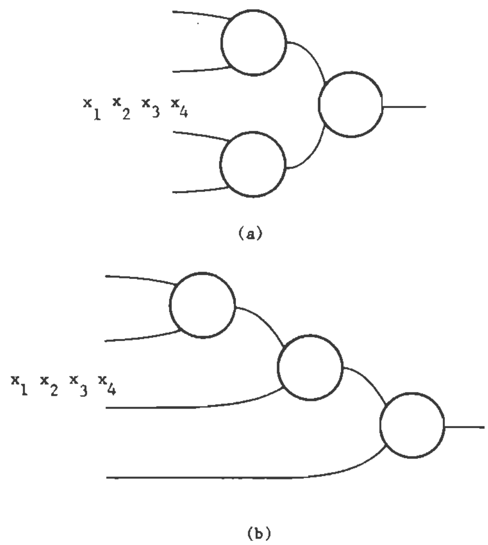

Fig. 3. Disjunctive network of cells.

$$t_{dis}(k) = \sum_{p=0}^{k} \binom{k}{p} N_{dis}(k-p). \quad (60)$$

Table IV shown $T_{dis}(k)$ for $2 \leq k \leq 15$ as computed from this expression. As with cascades, most of the realized functions for larger values of k are those functions which depend on all k inputs.

V. CONCLUSIONS AND COMMENTS

In this paper a counting technique has been demonstrated for the number of functions $N_{cas}(k)$ realized by k-input cascades of two-input one-output gates. This extends the result by Sklansky *et al.* [18] in which the number of cascade realizable symmetry types was calculated. A recursion relation is derived expressing N_{cas} as a function of $N_{cas}(k-1)$, $N_{cas}(k-2)$, $\cdots$, and $N_{cas}(1)$. This expression has been solved by computer for values of k up to 15. From the results, it has been surmised that for large k, $N_{cas}(k) \simeq k! c_c{}^k b_c$ for constant c_c and b_c. Additionally, a counting technique is demonstrated for $N_{dis}(k)$, the number of functions realized by k-input disjunctive networks. $N_{dis}(k)$ is expressed also as a recursion relation, which has been solved by computer for values of k up to 15. Using the expressions for $N_{cas}(k)$ and $N_{dis}(k)$, expressions are derived for $T_{cas}(k)$ and $T_{dis}(k)$, respectively, the number of functions realized by cascades and disjunctive networks of two-input one-output cells in which permutation of input assignments and interconnections are allowed.

It is interesting to compare the number of functions realized by fixed-interconnection disjunctive networks of cells with fixed-input assignments to networks with variable input assignments. As mentioned previously, Maruoka and Honda [8] have shown that all disjunctive networks of cells with fixed-input assignments realize a total of $(2 \cdot 6^k + 8)/5$ functions where k is the number of inputs. Thus, two k-input disjunctive networks of cells with fixed-input assignments realize the same number of functions. However, this is not true, in general, of disjunctive networks with variable input assignments. For example, consider the four-input cascade with variable input assignments as shown in Fig. 3(b). Table III shows that such network realizes 2368 functions. On the other hand, from Butler [17] the network of Fig. 3(a) realizes 1208 functions when permutation of input labels is permitted. Thus, when variable assignments are allowed, different fixed-interconnection disjunctive networks with the same number of variables realize, in general, a different number of functions.

It is interesting to note that the four-input cascade of Fig. 3(b) realizes $2368/2^{2^4}$ or 3.6 percent of the total number of four variable switching functions and 21/402 or 5.2 percent of the total number of symmetry types. Thus, four variable cascade symmetry types contain on the average fewer functions than general symmetry types do.

Although the design of cascades and disjunctive networks was not considered, this is indeed an important topic. Since the enumeration of functions depends on the observation of specific functional decompositions (Lemma 1 for cascades and Corollary 3.1 for disjunctive networks), it is reasonable to ask whether functional decomposition can be used to synthesize such networks. The answer is indeed yes. Curtis [13] shows the use of functional decomposition for general networks, and Butler [17] describes its use in the synthesis of disjunctive networks.

Another problem not considered in this paper is the "box of parts" problem. Given c universal two-input one-output cells, how many k-variable functions $N(c,k)$ can be realized by some interconnection of the c cells? Additionally, how many functions $T(c,k)$ can be realized which depend on k, $k-1$, etc., variables? In Butler [17], it is shown that there exists a k-input network realizing all 2^{2^k} functions on k or fewer variables which is composed of $c = \frac{3}{4}(2^k - 2)$ cells. Thus, $T(c,k) = 2^{2^k}$ for $c \geq \frac{3}{4}(2^k - 2)$. However, for general c and k more advanced counting procedures are required.

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Jon T. Butler (M'68) was born in Baltimore, Md., on December 26, 1943. He received the B.E.E. and M. Eng. (E.E.) degrees from Rensselaer Polytechnic Institute, Troy, N.Y., in 1966 and 1967, respectively. He received the Ph.D. (E.E.) degree from Ohio State University, Columbus, Ohio in 1973.

From 1967 to 1970 he served as an Officer in the Air Force working in the Air Force Avionics Laboratories, Dayton, Ohio, where he conducted research and managed contracts in the areas of bionics techniques and radar countermeasures. From 1973 to 1974 he was a National Research Council Postdoctoral Associate at the Avionics Laboratories. Presently, he is an Assistant Professor in the Department of Electrical Engineering, Northwestern University, Evanston, Ill. His research interests include switching theory, cellular automata, and pattern recognition.

Dr. Butler is a member of Tau Beta Pi, Eta Kappa Nu, and Sigma Xi.