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With two cosine terms about a third and with three cosine terms about half as many shifts are needed as for an arbitrary $\delta$. The number of additions varies from about a fifth as many at $n = 1$ to about a fifteenth or sixteenth as many at $n = 5$.

It should be noted that if the processor used to implement this has 4 or 8 bit shifts, the number of shifts is greatly reduced. Also, it should be stated that this can be implemented in hardware with about 30 MSI TTL packs for the case $n = 7$, $r = 2$, $a = 1$, and $d = 16$.

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## Asymptotic Approximations for the Number of Fanout-Free Functions

EDWARD A. BENDER AND JON T. BUTLER

*Abstract*—**Expressions are derived for the approximate number of functions realized by various $n$-variable fanout-free networks. Six recently studied networks are considered. It is shown that the relative number of functions realized by two networks for small and large $n$ is quite different in certain cases.**

*Index Terms*—**Asymptotic approximations, cascades, combinatorial logic, fanout-free networks, function enumeration, switching functions.**

### I. Introduction

During the recent past, considerable interest has developed in the fanout-free network where each gate has a fanout of one. This interest has been motivated by the relative ease with which faults can be detected [1] and by technological restrictions, such as in magnetic bubble logic where the difficulty of reproducing bubbles places a high premium on single interconnections between logic modules [2]. In particular, Hayes [3] considered fanout-free networks of AND's, OR's, and inverters; Chakrabarti and Kolp [4], Butler and Breeding [5], and Marouka and Honda [6], [7] have considered networks of AND's, OR's, exclusive OR's, and inverters; Kodandapani and Seth [8] have considered networks which also include the majority function. A special case of the fanout-free network, the cascade, has received considerable attention (viz. Maitra [9] and Mukhopadhyay [10]). Cascades are fanout-free networks in which each gate connects to at least one net input.

One way to measure the relative merits of various circuits is to compare the number of $n$-variable functions realized. However, as we show, this can be deceptive since the relative number of functions realized by two networks for small $n$ may be quite different than for large $n$. As a basis of comparison, six different networks are considered. Let $F_T^{CM}(n)$ be the number of $n$-variable functions realized by networks whose topology $T$ is a cascade ($C$) or unrestricted fanout free ($FF$), and whose component modules $CM$ are two-input AND gates ($A$), two-input OR gates ($O$), two-input exclusive OR gates ($E$), three-input majority gates ($M$), and inverters ($N$).

The particular networks of interest are

1) unate cascades [10] —$F_C^{AON}(n)$
2) Maitra cascades [9] —$F_C^{AOEN}(n)$
3) fanout-free #1 [1], [3]—$F_{FF}^{AON}(n)$
4) fanout-free #2 [8] —$F_{FF}^{AOMN}(n)$
5) fanout-free #3 [4]-[7] —$F_{FF}^{AOEN}(n)$
6) fanout-free #4 [8] —$F_{FF}^{AOEMN}(n)$.

For all but one of the networks listed,[1] recursive relations have been derived and have been evaluated at least up to $n = 7$. However, computation time and the large values involved preclude computer evaluation much beyond this. For each network, we give an asymptotic approximation, a closed-form expression which approximates the function's behavior for large $n$. From this, it is a straightforward process to determine the relative merits of various gates and topologies. Although the approximations hold for large $n$, they are reasonably accurate in the range of $n$ where computer evaluation of recursive relations becomes cumbersome.

We use the following conventions.

$$F(n) \sim G(n) \text{ means } \lim_{n\to\infty} \frac{F(n)}{G(n)} = 1$$

$$F(n) = o(G(n)) \text{ means } \lim_{n\to\infty} \frac{F(n)}{G(n)} = 0.$$

### II. Cascade

Butler [11] has shown that the number of $n$-variable functions which are Maitra cascade realizable for $n > 1$ is given as

$$F_C^{AOEN}(n) = \sum_{p=1}^{n-1} (-1)^{p+1} \binom{n}{p} (2^{p+1} + 1) F_C^{AOEN}(n-p) - (-1)^n (2^n + 1) F_C^{AOEN}(1) \quad (1)$$

where $F_C^{AOEN}(1) = 2$. In order to derive an asymptotic approximation, we will make use of a result in Bender [12], which requires the generating function of $F_C^{AOEN}(n)$. Let $F_C^{AOEN}(x)$ be the exponential generating function of $F_C^{AOEN}(n)$. Thus,

$$F_C^{AOEN}(x) = \sum_{n=2}^{\infty} \frac{F_C^{AOEN}(n)}{n!} x^n + 2x. \quad (2)$$

Substituting (1) into (2) and rearranging yields

$$F_C^{AOEN}(x) = \sum_{n=2}^{\infty} \sum_{p=1}^{n-1} \frac{(-1)^{p+1}(2^{p+1}+1)x^p}{p!} \cdot \frac{F_C^{AOEN}(n-p)x^{n-p}}{(n-p)!} - 2 \sum_{n=2}^{\infty} \frac{(-1)^n (2^n + 1)}{n!} x^n + 2x. \quad (3)$$

Define

$$S(x) = \sum_{n=2}^{\infty} \frac{(-1)^n (2^n + 1)}{n!} x^n = e^{-2x} + e^{-x} + 3x - 2. \quad (4)$$

Comparing (3) and (4) yields

$$F_C^{AOEN}(x) = S'(x) F_C^{AOEN}(x) - 2S(x) + 2x \quad (5)$$

where $S'(x)$ is the derivative of $S(x)$. Solving for $F_C^{AOEN}(x)$ and substituting (4) yields

$$F_C^{AOEN}(x) = 2\,\frac{2 - 2x - e^{-x} - e^{-2x}}{2e^{-2x} + e^{-x} - 2}. \quad (6)$$

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[1] As far as known, a recursive expression has not been published for $F_C^{AON}(n)$. A derivation is shown in the next section.

To find the asymptotic approximation corresponding to (6), we make use of the following result.

*Theorem 1 (Bender [12, p. 498]):* Suppose $A(x) = \sum_{n=0}^{\infty} a_n x^n$ is analytic near 0 and can be written in the form

$$A(x) = h(x) + \left(1 - \frac{x}{\alpha}\right)^{-w} g(x)$$

where $\alpha$ is the only singularity of $A(x)$ such that $|\alpha| < |\beta|$ for all other singularities $\beta$ of $A(x)$. Further, if $g$ and $h$ are analytic near $\alpha$, $w \neq 0, -1, -2, \cdots$, and $g(x) \neq 0$, then

$$a_n \sim \frac{g(\alpha) n^{w-1}}{\Gamma(w)\alpha^n} \tag{7}$$

where $\Gamma(w)$ is the gamma function.

The singularities of $F_C^{AOEN}(x)$ are poles at $x = \beta$ where

$$2e^{-2\beta} + e^{-\beta} - 2 = 0. \tag{8}$$

Solving (8) for $\beta$ yields

$$\beta = \ln \frac{4}{-1 \pm \sqrt{17}}.$$

There is a unique $\beta$ of least modulus, namely, the real number

$$\alpha = \ln \frac{4}{-1 + \sqrt{17}}. \tag{9}$$

The pole at $x = \alpha$ is simple, and so $w = 1$. We need $g(x)$ where

$$F_C^{AOEN}(x) = \left(1 - \frac{x}{\alpha}\right)^{-1} g(x) \tag{10}$$

Solve (10) for $g(x)$, and let $x \to \alpha$. Using l'Hopital's rule to evaluate the limit yields

$$g(\alpha) = 2\,\frac{1 - \alpha(e^{\alpha} + 2)}{\alpha(e^{\alpha} + 4)} = 0.288. \tag{11}$$

Since $a_n = F_C^{AOEN}(n)/n!$, we have from (7)

$$F_C^{AOEN}(n) \sim n!\, g(\alpha)\left(\frac{1}{\alpha}\right)^n = n!\, 0.288(4.04)^n. \tag{12}$$

This verifies a conjecture by Butler [11]. The degree to which (12) approximates the *exact* number of functions is very good. The accuracy is 6.0, 0.01, 0.02, 0.0002, 0.0001, and 0.000003 percent for $n = 2, 3, 4, 5,$ and 7, respectively.

A closed-form approximation for the number of unate cascade functions is obtained in a similar manner. The recursion relation for $F_C^{AON}(n)$ can be obtained (see [11]) by observing that the contribution of the exclusive OR gates in the term $2^{p+1} + 1$ of (1) is $+1$ (while AND or OR gates each contribute $2^p$).

$$F_C^{AON}(n) = \sum_{p=1}^{n-1} (-1)^{p+1} \binom{n}{p} 2^{p+1} F_C^{AON}(n-p) - (-1)^n 2^n F_C^{AON}(1) \tag{13}$$

where $F_C^{AON}(1) = 2$. The generating function for (13) is

$$F_C^{AON}(x) = \frac{2[x + e^{-2x} - 1]}{1 - 2e^{-2x}}. \tag{14}$$

As before, Theorem 1 can be applied, yielding

$$F_C^{AON}(n) \sim n!\left(\frac{1 - \alpha e^{-2\alpha}}{2\alpha e^{-2\alpha}}\right)\left(\frac{1}{\alpha}\right)^n = n!\, 0.443(2.89)^n \tag{15}$$

for $\alpha = \ln\sqrt{2}$. This approximation is also very good. For $n = 2, 3, 4, 5, 6,$ and 7 the accuracy is respectively 7.9, 3.0, 0.06, 0.005, 0.0005, and 0.00007 percent. Comparing (15) with (12) shows that the fraction of Maitra cascade realizable functions which are unate cascade realizable becomes vanishingly small as $n$ approaches $\infty$.

## III. Fanout-Free Networks

The asymptotic approximations for one case only, $F_{FF}^{AOEMN}(n)$, will be derived. Expressions for the other cases follow in a like manner. A function realized by a fanout-free network consisting of $A$, $O$, $E$, $M$, and $N$ gates can be classified according to the output gate used in a canonical realization of that function. This realization excludes $N$ as an output gate except for the case where the number of inputs $n = 1$. Thus, for $n > 1$, there are four sources of functions, $A$, $O$, $E$, and $M$. Let $A(n)$, $O(n)$, $E(n)$, and $M(n)$ be the number of fanout-free functions on $n$ variables in which the output gate is the AND, OR, exclusive OR, and majority gate, respectively. From Kodandapani and Seth [8], we have

$$F_{FF}^{AOEMN}(n) = \sum_{X \in \{A,O,E,M\}} X(n) \qquad n > 1 \tag{16}$$

where

$$X(n) = \sum_T X(n, T). \tag{17}$$

The sum in (17) is over all partitions (additive factorizations) $T$ of $n$ except the trivial partition $00 \cdots 0n$. $X(n, T)$ is the number of functions on $n$ variables realized with output gate $X$ associated with $T$.

For convenience, $T$ represents repeated parts with exponents: for example, partition 111244 of 13 is written as $1^3 2^1 3^0 4^2$. Let $p$ be the number of nonzero parts of $T$. In this example, $p = 3$. We use a somewhat modified version of Kodandapani and Seth's equations for $X(n, T)$:

$$A(n, T) = n! \prod_{i=0}^{n} \left(\frac{O(i) + E(i) + M(i) + N(i)}{i!}\right)^{k_i} \frac{1}{k_i!} \tag{18}$$

$$O(n, T) = n! \prod_{i=0}^{n} \left(\frac{A(i) + E(i) + M(i) + N(i)}{i!}\right)^{k_i} \frac{1}{k_i!} \tag{19}$$

$$E(n, T) = 2n! \prod_{i=0}^{n} \left(\frac{A(i) + O(i) + M(i) + N(i)}{i!}\right)^{k_i} \frac{1}{2^{k_i} k_i!} \tag{20}$$

$$\begin{aligned} M(n, T) &= n! \prod_{i=0}^{n} \left(\frac{A(i) + O(i) + E(i) + M(i) + N(i)}{i!}\right)^{k_i} \frac{1}{k_i!} \qquad p = 3 \\ &= 0 \qquad \text{otherwise} \end{aligned} \tag{21}$$

where the partition $T$ is given as $1^{k_1} 2^{k_2} \cdots n^{k_n} (k_0 = 0)$ and where $X(0) = X(1) = 0$ for $X \neq N$ and $N(0) = 1$, $N(1) = 2$. Substituting (18) into (17) and adding $O(n) + E(n) + M(n) + N(n)$ to both sides yields

$$F_{FF}^{AOEMN}(n) = n! \sum_{\substack{k_1, k_2, \cdots, k_n \geq 0 \\ 1k_1 + 2k_2 + \cdots + nk_n = n}} \prod_{i=0}^{n} \left(\frac{O(i) + E(i) + M(i) + N(i)}{i!}\right)^{k_i} \frac{1}{k_i!}.$$

The terms $O(n) + E(n) + M(n) + N(n)$ on the right side have been absorbed in the sum, which is now over all partitions. The exponential generating function of $F_{FF}^{AOEMN}(n)$ is then

TABLE I
GENERATING FUNCTIONS AND ASYMPTOTIC APPROXIMATIONS FOR SIX FANOUT-FREE NETWORKS

| FUNCTION F = | GENERATING FUNCTION | ALGEBRAIC ASYMPTOTIC EXPRESSION | NUMERIC ASYMPTOTIC EXPRESSION |
|---|---|---|---|
| $F_{C}^{AON}(n)$ | $F = \frac{2(x+e^{-2x}-1)}{1-2e^{-2x}}$ | $n!\left(\frac{1-e^{-2\alpha}-\alpha}{2\alpha e^{-2\alpha}}\right)\left(\frac{1}{\alpha}\right)^n;\ \alpha = \ln\sqrt{2}$ | $n!(.433)(2.89)^n$ |
| $F_{C}^{AOEN}(n)$ | $F=2\frac{2-2x-e^{-x}-e^{-2x}}{e^{-x}+2e^{-2x}-2}$ | $n!\left(2\frac{1-\alpha\,(e^{\alpha}+2)}{\alpha(e^{\alpha}+4)}\right)\left(\frac{1}{\alpha}\right)^n;\ \alpha = \ln\frac{1+\sqrt{17}}{4}$ | $n!(.288)(4.04)^n$ |
| $F_{FF}^{AON}(n)$ | $F^2 = e^{F-1+2z}$ | $n!\left(\frac{2\alpha}{\pi}\right)^{\frac{1}{2}}\frac{1}{n^{3/2}}\left(\frac{1}{\alpha}\right)^n;\ \alpha = \ln 2-\frac{1}{2}$ | $n!(.351)\frac{1}{n^{3/2}}(5.18)^n$ |
| $F_{FF}^{AOMN}(n)$ | $F^2 = e^{F-1+2z+\frac{1}{6}(F-1)^3}$ | | $n!(.187)\frac{1}{n^{3/2}}(6.54)^n$ |
| $F_{FF}^{AOEN}(n)$ | $F(F+1)=2e^{F-1+z}$ | $n!\left(\frac{\alpha(2+\sqrt{5})}{2+\sqrt{5}}\right)^{1/2}\frac{1}{n^{3/2}}\left(\frac{1}{\alpha}\right)^n;\ \alpha=\ln\left(\frac{\sqrt{5}+2}{2}\right)-\frac{\sqrt{5}-1}{2}$ | $n!(.200)\frac{1}{n^{3/2}}(7.55)^n$ |
| $F_{FF}^{AOEMN}(n)$ | $F(F+1)=2e^{F-1+z+\frac{1}{12}(F-1)^3}$ | | $n!(.148)\frac{1}{n^{3/2}}(8.46)^n$ |

$$F_{FF}^{AOEMN}(z) = \sum_{n=0}^{\infty} \frac{F_{FF}^{AOEMN}(n)}{n!} z^n$$
$$= \sum_{n=0}^{\infty} \sum_{\substack{k_1,k_2,\cdots,k_n \geq 0 \\ 1k_1+2k_2+\cdots+nk_n=n}} \cdot \prod_{i=0}^{n} \left(\frac{O(i)+E(i)+M(i)+N(i)}{i!}\right)^{k_i} \frac{1}{k_i!} z^n.$$

The right side can be rewritten as follows (Knuth [13, p. 92]):

$$F_{FF}^{AOEMN}(z) = \exp\left(\sum_{k=0}^{\infty} \frac{O(k)}{k!} z^k + \sum_{k=0}^{\infty} \frac{E(k)}{k!} z^k + \sum_{k=0}^{\infty} \frac{M(k)}{k!} z^k + \sum_{k=0}^{\infty} \frac{N(k)}{k!} z^k - 1\right).$$

Since the sums in the exponent are the exponential generating functions $O(z)$, $E(z)$, $M(z)$, and $N(z)$, we have

$$F_{FF}^{AOEMN}(z) = \exp\left(O(z) + E(z) + M(z) + N(z) - 1\right). \tag{22}$$

The generating functions corresponding to (19), (20), and (21) are obtained in a similar manner. Thus,

$$F_{FF}^{AOEMN}(z) = \exp\left(A(z) + E(z) + M(z) + N(z) - 1\right). \tag{23}$$

$$(F_{FF}^{AOEMN}(z) + 1)^2 = 4\exp\left(O(z) + A(z) + M(z) + N(z) - 1\right). \tag{24}$$

and

$$M(z) = \tfrac{1}{6}(F_{FF}^{AOEMN}(z) - 1)^3. \tag{25}$$

Also,

$$N(z) = 2z + 1. \tag{26}$$

Multiplying (22), (23), and (24) and substituting (25) and (26) yields

$$F_{FF}^{AOEMN}(z)(F_{FF}^{AOEMN}(z) + 1) = 2\exp\left(F_{FF}^{AOEMN}(z) - 1 + z + \tfrac{1}{12}(F_{FF}^{AOEMN}(z) - 1)^3\right) \tag{27}$$

We now use the following result.

*Theorem 2 (Bender [12, p. 502]):* Assume that the power series $w(z) = \sum_{n=0}^{\infty} a_n z^n$ with nonnegative coefficients satisfies $F(z, w) \equiv 0$. Suppose there exist real numbers $r > 0$ and $s > a_0$ such that

a) for some $\delta > 0$, $F(z, w)$ is analytic whenever $|z| < r + \delta$ and $|w| < s + \delta$

b) $F(r, s) = F_w(r, s) = 0$

c) $F_z(r, s) \neq 0$ and $F_{ww}(r, s) \neq 0$

d) if $|z| \leq r$, $|w| \leq s$, and $F(z, w) = F_w(z, w) = 0$, then $z = r$ and $w = s$.

Then

$$a_n \sim \left(\frac{rF_z}{2\pi F_{ww}}\right)^{1/2} \frac{1}{n^{3/2}} \left(\frac{1}{r}\right)^n \tag{28}$$

where the partial derivatives are evaluated at $z = r$ and $w = s$.

Let $w = F_{FF}^{AOEMN}(z)$ and set

$$F(z, w) = w(w+1) - 2\exp\left((w-1) + \tfrac{1}{12}(w-1)^3 + z\right) = 0.$$

Since $F(z, w)$ is analytic everywhere, a) of the theorem is satisfied. Satisfy b) by setting $F(r, s) = F_w(r, s) = 0$. Thus,

$$s^4 - s^3 + 3s^2 - 3s - 4 = 0 \tag{29}$$

and

$$r = \ln\left(\frac{(s_r^2 + s_r)}{2}\right) - s_r + 1 - \frac{(s_r - 1)^3}{12} \tag{30}$$

where $s_r$ is a root of (29). The roots of (29) were found by computer. One is positive real, one is negative real, and the other two are complex. Choosing the positive real root yields $s_r = 1.505$ and $r = 0.1182$. These values satisfy d) of the theorem. Condition c) is satisfied as follows: $F_z(r, s) = -3.77$ and $F_w(r, s) = -3.22$. All conditions of Theorem 2 are satisfied, and thus

$$F_{FF}^{AOEMN}(n) \sim n!\,0.148\,\frac{1}{n^{3/2}}\,8.46^n \tag{31}$$

The approximate values for small $n$ are not quite as accurate as in the case of cascades. For $n = 2, 3, 4, 5, 6$, and 7, (31) approximates the exact values to within 25, 15, 11, 8, 7, and 6 percent, respectively.

Table I[2] shows the generating functions and the asymptotic

[2] Certain algebraic expressions are not listed because the parameters of (28) were evaluated by computer. They involve the determination of the roots of a polynomial of order greater than 2.

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TABLE II
EXACT NUMBER OF FUNCTIONS REALIZED BY FANOUT-FREE NETWORKS

| n | $F_C^{AON}(n)$ | $F_C^{AOEN}(n)$ [11] | $F_{FF}^{AON}(n)$ [3] | $F_{FF}^{AOMN}(n)$ [8] | $F_{FF}^{AOEN}(n)$ [11] | $F_{FF}^{AOEMN}(n)$ [8] |
|---|---|---|---|---|---|---|
| 2 | 8 | 10 | 8 | 8 | 10 | 10 |
| 3 | 64 | 114 | 64 | 72 | 114 | 122 |
| 4 | 736 | 1,842 | 832 | 1,152 | 2,154 | 2,554 |
| 5 | 10,624 | 37,226 | 15,104 | 26,304 | 56,946 | 75,386 |
| 6 | 183,936 | 902,570 | 352,256 | 773,376 | 1,935,210 | 2,865,370 |
| 7 | 3,715,072 | 25,530,658 | 10,037,248 | 27,792,384 | 80,371,122 | 133,191,386 |

approximations for other networks. The accuracy of the approximations for all four unrestricted fanout-free networks is comparable to the accuracy stated for $F_{FF}^{AOEMN}(n)$. Exact values for $F_{FF}^{AOEN}(n)$ are known out to $n = 15$ ([11]) where the accuracy improves to 3 percent.

## IV. CONCLUDING REMARKS

Consider now the relative number of functions for the various networks. The following conclusions can be made from Table I.

1) The fraction of $n$-variable fanout-free functions which are cascade realizable becomes arbitrarily close to 0 as $n$ approaches $\infty$ for two cases, i.e., $F_C^{AON}(n) = o(F_{FF}^{AON}(n))$ and $F_C^{AOEN}(n) = o(F_{FF}^{AOEN}(n))$.

2) From Table I, $F_{FF}^{AOMN}(n) = o(F_{FF}^{AOEN}(n))$ showing that the exclusive OR in combination with AND's, OR's, and inverters produces more functions than the majority gate. This is interesting in view of the fact that $F_{FF}^{EN}(n) = 2$ for all $n$, while $F_{FF}^{MN}(n) \to \infty$ as $n \to \infty$.

3) The number of strictly fanout-free functions $F_{FF}^{AON}(n)$ represents a vanishingly small fraction of the number of functions realized by $AOEN$ nets, verifying a conjecture by Kodandapani and Seth [8].

4) The relative number of functions for large $n$ can be quite different than for small $n$. In Table I, if we compare $AOEN$ cascades with $AON$ fanout-free networks, and with $AOMN$ fanout-free networks, we see that $F_C^{AOEN}(n) = o(F_{FF}^{AON}(n))$ and $F_C^{AOEN}(n) = o(F_{FF}^{AOMN}(n))$. However, for small $n$ a different situation exists. Table II lists the exact numbers of functions for the six networks considered.[3] It shows for $2 \le n < 6$ $AOEN$ cascades realize *more* functions than either $AON$ or $AOMN$ fanout-free networks.

The asymptotic approximations for the six networks suggest that the asymptotic behavior of cascades and fanout-free networks with *general* component module sets have the same characteristics of the specific examples shown here. If this is the case, the addition of a distinct module to the module set in general enlarges the function set by an arbitrarily large amount as the number of variables approaches infinity.

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[3] The data for Table II come from (13) above, [3], [8], and [11].

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# On the Generation of Permutations in Magnetic Bubble Memories

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*Abstract* — **A better algorithm is presented for the generation of an arbitrary permutation in a model of magnetic bubble memories that have been investigated previously.**

*Index Terms* — **Generation of permutations, magnetic bubble memory.**

## I. INTRODUCTION

In [1] Wong and Coppersmith investigated the problems of data accessing and permutation generating based on two models of magnetic bubble operations.

In this note we present a new algorithm for the generation of an arbitrary permutation for the second model in [1]. The algorithm

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