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dedicated to Anton Kotzig

SETS OF INTEGERS WHOSE SUBSETS HAVE DISTINCT SUMS

Richard K. Guy North Holland

For Anton Kotzig, founder of the Bratislava school of graph theory.

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1. Introduction.

The subsets of the set of integers $\{2^i : 0 \le i \le k\}$ all have distinct sums. P. Erdős [6-12] has asked for the maximum number, m, of positive integers

$$a_1 < a_2 < \dots < a_m \le x$$

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so that all the 2^m sums (it is convenient to include 0 as the empty sum)

$$a_{i_1} + a_{i_2} + \dots + a_{i_j}, \qquad 0 \le j \le m$$

are different. In particular, is it possible to have $m = k+2$, when $x = 2^k$? We answer this last question affirmatively.

Erdős notes that all the sums are less than mx, so that

$$2^m - 1 \le mx \tag{1}$$

and it follows that

$$m \le \log x + \log\log x + 1 \qquad (x \ge 2) \tag{2}$$

where the logarithms, here and in Theorem 3, are to base 2.

Erdős and Leo Moser [6] give a result in which the second term in (2) is reduced by half. A proof of this is incorporated as Theorems 2 and 3. We are unable to decide whether the set of numbers defined by (7) and (10) is best possible in any sense, but we conjecture that the answer is affirmative.

Erdős and Turán [13] considered the corresponding problem in which sums of *pairs* of the a_i, with repetitions allowed, are required to be distinct, i.e. a_i+a_j $(i \le j)$ all distinct. If $a_h \le n \le a_{h+1}$ where $h = \phi(n)$ and $\Phi(n)$ is the maximum $\phi(n)$ for a given n, they show that

(3) $$(2^{-\frac{1}{2}}-\varepsilon)\,n < \Phi(n) < n + O(n^{\frac{1}{4}}) \quad .$$

In an addendum [14], Erdős notes that the Singer [17] difference sets show that the upper bound in (3) is the right result, except perhaps for the error term. Lindström [15,16] has improved this so that the right member of (2) may be replaced by $n^{\frac{1}{2}}+n^{\frac{1}{4}}+1$.

2. An upper bound for m.

Theorem 1. *If $a_1 < a_2 < \dots < a_m$ are positive integers whose subsets have distinct sums, then*

$$\sum_{i=1}^{m} a_i \ge 2^m-1 \quad .$$

Proof: The 2^m-1 non-empty subsets of the a_i have distinct positive integer sums, so the largest total is at least 2^m-1. Equality obtains just if $a_i = 2^{i-1}$.

Theorem 2. (Leo Moser) *Under the hypothesis of Theorem* 1,

$$\sum_{i=1}^{m} a_i^2 \ge \frac{1}{3}(4^m-1) \quad ,$$

with equality again holding just if $a_i = 2^{i-1}$.

Proof: Consider the sum of the squares of the 2^m quantities

(4) $$\pm a_1 \pm a_2 \pm \cdots \pm a_m .$$

The product terms cancel in pairs, so that

$$\Sigma(\pm a_1 \pm a_2 \pm \cdots \pm a_m)^2 = 2^m \sum_{i=1}^{m} a_i^2 \quad .$$

On the other hand, the quantities (4) are distinct, different from zero, and of the same parity, so that the sum is at least

$$1^2 + (-1)^2 + 3^2 + (-3)^2 + \ldots + (2^m-1)^2 + (1-2^m)^2 = \frac{2}{3} \cdot 2^{m-1}(2^{2m}-1) \quad ,$$

and the result follows.

Comparison of Theorems 1 and 2 tempts one to conjecture

(5) $$\sum_{i=1}^{m} a_i^4 \geq \frac{1}{15}(16^m-1) \ ?$$

but its falsity is shown by the set of six numbers, 11, 17, 20, 22, 23, 24, whose subsets have distinct sums. The sum of their fourth powers is 1104035, but the right member of (5) with $n = 6$ is 1118481.

Theorem 3. *With the notation of the Introduction,*

$$m < \log x + \tfrac{1}{2} \log\log x + 1.3 \qquad (x \geq 2)$$

where the logarithms are to base 2.

Proof: Since the a_i are distinct

$$\sum_{i=1}^{m} a_i^2 \leq x^2 + (x-1)^2 + \ldots + (x-m+1)^2 \quad ,$$

so Theorem 2 gives

$$\frac{1}{3}(4^m-1) \le mx^2$$

and

$$4^m < 3mx^2 \quad .$$

Take logs to base 2, and note that (1) implies $m \le \log m + \log x$, and, since $m \le x$, we have $m \le 2\log x$ and

$$2m < 2\log x + \log 3m \le 2\log x + \log\log x + \log 6$$

and the result follows since $\log 6 < 2.6$.

3. The Conway-Guy sequence [4,5]

To find a set of $k+2$ numbers not exceeding 2^k whose subsets have distinct sums, define an auxiliary sequence by $u_0 = 0$, $u_1 = 1$ and

(7) $$u_{n+1} = 2u_n - u_{n-r}, \qquad n \ge 1,$$

where $r = \langle\sqrt{2n}\rangle$, the nearest integer to $\sqrt{2n}$. Table 1 shows the values of this auxiliary sequence for $1 \le n \le 45$. To obtain successive values of u_n (column 2), double the preceding value and subtract the corresponding entry, u_{n-r} (column 3). Note that these entries are repeated whenever $n-r$ (column 4) is a triangular number. This occurs when n is also triangular.

(1) n	(2) u_n	(3) u_{n-r}	(4) $n-r$	(5) r
1	1	0	0	1
2	2	0	0	2
3	4	1	1	2
4	7	1	1	3
5	13	2	2	3
6	24	4	3	3
7	44	4	3	4
8	84	7	4	4
9	161	13	5	4
10	309	24	6	4
11	594	24	6	5
12	1164	44	7	5
13	2284	84	8	5
14	4484	161	9	5
15	8807	309	10	5
16	17305	309	10	6
17	34301	594	11	6
18	68008	1164	12	6
19	134852	2284	13	6
20	267420	4484	14	6
21	530356	8807	15	6
22	1 051905	8807	15	7
23	2 095003	17305	16	7
24	4 172701	34301	17	7
25	8 311101	68008	18	7
26	16 554194	134852	19	7
27	32 973536	267420	20	7
28	65 679652	530356	21	7
29	130 828948	530356	21	8
30	261 127540	1 051905	22	8
31	521 203175	2 095003	23	8
32	1040 311347	4 172701	24	8
33	2076 449993	8 311101	25	8
34	4144 588885	16 554194	26	8
35	8272 623576	32 973536	27	8
36	16512 273616	65 679652	28	8
37	32958 867580	65 679652	28	9
38	65852 055508	130 828948	29	9
39	131573 282068	261 127540	30	9
40	262885 436596	521 203175	31	9
41	525249 670017	1040 311347	32	9
42	1 049459 028687	2076 449993	33	9
43	2 096841 607381	4144 588885	34	9
44	4 189538 625877	8272 623576	35	9
45	8 370804 628178	16512 273616	36	9

Table 1. The auxiliary sequence u_n.

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We prove a series of lemmas which lead to the existence and value of the limit of $u_n/2^n$.

Lemma 1. *u_n is strictly increasing with n.*

Proof: $u_1 > u_0$ and (7) may be written $u_{n+1}-u_n = u_n-u_{n-r}$ $(n \geq 1)$ and since $u_n > u_{n-r}$ for $n = 1$, the result follows by induction.

Lemma 2. $0 \leq u_n \leq 2^{n-1}$ $(n \geq 0)$.

Proof: $u_0 = 0$, so $u_n \geq 0$ by Lemma 1. From (7) $u_{n+1} \leq 2u_n$ $(n \geq 1)$ and since $u_n \leq 2^{n-1}$ for $n = 0$ and 1, the result follows by induction.

Lemma 3. *$u_n/2^n$ is a decreasing function of n for $n \geq 1$; strictly decreasing for $n \geq 4$.*

Proof: $u_n/2^n = \frac{1}{2}$ for $n = 0,1,2$. From (7), $\frac{u_{n+1}}{2^{n+1}} = \frac{u_n}{2^n} - \frac{u_{n-r}}{2^{n+1}}$, and since $u_{n-r} > 0$ for $n \geq 3$, it follows that $u_{n+1}/2^{n+1} < u_n/2^n$.

Theorem 4. *As $n \to \infty$, $u_n/2^n$ tends to a limit, α, where $0 \leq \alpha < \frac{1}{2}$.*

Proof: From Lemma 3.

In order to calculate α, write $u_n/2^n = \alpha_n$. In the range

(8) $$\tfrac{1}{2} m(m+1) + 1 \leq n \leq \tfrac{1}{2} (m+1)(m+2)$$

we have $r = m+1$, and division of (7) by 2^{n+1} gives

$$\alpha_{n+1} = \alpha_n - 2^{-(m+2)} \alpha_{n-m-1} .$$

Sum this over the range (8) and we have

$$\alpha_{\frac12(m+1)(m+2)+1} = \alpha_{\frac12 m(m+1)+1} - 2^{-(m+2)} \sum_{i=\frac12 m(m-1)}^{\frac12 m(m+1)} \alpha_i .$$

Write $m+j-1$ for m and sum this from $j = 1$ to $j = p$,

$$(9) \quad \alpha_{\frac12(m+p)(m+p+1)+1} = \alpha_{\frac12 m(m+1)+1} - \sum_{j=1}^{p} 2^{-(m+j+1)} \sum_{i=\frac12(m+j-1)(m+j-2)}^{\frac12(m+j)(m+j-1)} \alpha_i .$$

Since $\alpha_{23} = 2095003\times 2^{-23} < \frac14$, Lemma 3 implies that $\alpha < \alpha_n < \frac14$ for $n \geq 23$. Hence, for $m \geq 8$, the inner sum in (9) lies strictly between $\alpha(m+j)$ and $\frac14(m+j)$, and the last term itself lies between

$$2^{-m-1}\alpha(m+2-(m+p+2)2^{-p}) \quad \text{and} \quad 2^{-m-3}(m+2-(m+p+2)2^{-p}) .$$

Keep m fixed and let $p \to \infty$, and $\alpha = \alpha_{\frac12 m(m+1)+1} - \beta$ where β lies between $\alpha(m+2)2^{-m-1}$ and $(m+2)2^{-m-3}$.

Put $m = 26$ and we have

$$\alpha_{352} - 28\times 2^{-29} < \alpha < \frac{\alpha_{352}}{1+28\times 2^{-27}} < \alpha_{352} - 26\times 2^{-29} ,$$

so that $\alpha = \alpha_{352} - 27\times 2^{-29}$ with an error of less than 2^{-29}. For $n \geq 352$, α_n does not differ from α in the first seven decimal places. Hence the upper estimate for α, based on $\alpha_n > \alpha$, is much closer than the lower, based on $\alpha_n < \frac14$. So $\alpha = \alpha_{352}/(1+28\times 2^{-27})$, with an error of less than 2^{-48}. A computer calculation gave $\alpha_{352} = 0.235125333862141\ldots$, so that

$$\alpha = 0.23512524581118\ldots .$$

The fact that $\alpha < \frac14$ enables us to find $k+2$ numbers less than 2^k whose subsets have distinct sums. Since $\alpha > \frac18$ it is *not* possible to find $k+3$ such numbers by this method. We conjecture that it is not possible by any method.

4. Sets of numbers with distinct sums of subsets.

Define

$$a_i = u_{k+2} - u_{k+2-i} \quad . \tag{10}$$

We will show that distinct subsets of the set $\{a_i : 1 \le i \le k+2\}$ always have distinct sums, and we have seen in the proof of Theorem 4 that for $k \ge 21$, $u_{k+2} < 2^k$ so that

$$a_i < 2^k \qquad (k \ge 21) \quad .$$

Even when we had only established distinctness for comparatively small values of k (< 40, say), by actual computation, this still implies that we can find such sets with arbitrarily large cardinality, because given any set S of $k+2$ numbers less than 2^k whose subsets have distinct sets, we can find a larger set with this property, e.g. the $k+l+2$ numbers in the set

$$2^l S \cup \{2^i : 0 \le i \le l-1\}$$

(where $2^l S$ is the set S with its members each multiplied by 2^l) are each less than 2^{k+l}.

To establish the distinctness of sums of subsets of $\{a_i\}$ we prove some further lemmas about the auxiliary sequence $\{u_i\}$.

Lemma 4. *For* $n \ge 1$, $u_{n+1} > u_n + u_{n-1}$.

Proof: By (7), $u_{n+1} = u_n + (u_n - u_{n-r})$ which is greater than $u_n + (u_n - u_{n-2})$ provided $r > 2$, i.e. provided $n \ge 4$. So, for $n \ge 4$, $u_n > u_{n-1} + u_{n-2}$ implies that $u_{n+1} > u_n + u_{n-1}$, so the result follows by induction, since Table 1 shows that it is true for $1 \le n \le 4$.

Lemma 5. *For* $n \ge 4$, $u_{n+1} < \sum_{i=0}^{n} u_i \le u_{n+1} + u_{n-2}$.

Proof: It is true for $n = 4,5$ and 6. Suppose it is true for $n = k \ge 6$. Then, on the one hand

$$\sum_{i=0}^{k+1} u_i = u_{k+1} + \sum_{i=0}^{k} u_i > u_{k+1} + u_{k+1} \text{ by the induction hypothesis,}$$

$$> 2u_{k+1} - u_q = u_{k+2}, \text{ where } q = k+1-\langle\sqrt{(2k+2)}\rangle \text{ is positive for } k \ge 6.$$

On the other hand, the inductive hypothesis implies that the sum is less than $u_{k+1} + u_{k+1} + u_{k-2}$, which is equal to $u_{k+2} + u_q + u_{k-2} \le u_{k+2} + u_{k-1}$ by lemma 4, provided $q < k-2$, which is true for $k \ge 6$.

Theorem 5. *If two subsets of* $\{a_i\}$ *have equal sums, then there are two subsets of* $\{u_i\}$ *with equal sums and equal cardinalities. Conversely, if there are two subsets of* $\{u_i\}$ *with equal sums and cardinalities, then there are two subsets of* $\{a_i\}$ *with equal sums.*

Proof: Suppose $(u_{k+2}-u_{i_1}) + (u_{k+2}-u_{i_2}) + \dots + (u_{k+2}-u_{i_s}) = (u_{k+2}-u_{j_1}) + (u_{k+}-u_{j_2}) + \dots + (u_{k+}-u_{j_t})$. We may assume (i) that the two sets are disjoint, else we could cancel common terms; (ii) that $i_1 < i_2 < \dots < i_s$, $j_1 < j_2 < \dots < j_t$; and (iii) that $s \ge t$. Then

$$(s-t)u_{k+2} = u_{i_1} + u_{i_2} + \dots + u_{i_s} - (u_{j_1} + u_{j_2} + \dots + u_{j_t}) \quad . \tag{11}$$

The right member of (11) is less than $\sum_{i=0}^{k+1} u_i$, which is less than $u_{k+2} + u_{k-1}$ by Lemma 5, and so $s-t < 2$. If $s = t$, then (11) gives subsets of $\{u_i\}$ with equal sums and equal cardinality $s = t$, while if

$s = t+1$, (11) again gives such subsets, each of cardinality $s = t+1$.

Conversely, if $u_{i_1}+u_{i_2}+\ldots+u_{i_s} = u_{j_1}+u_{j_2}+\ldots+u_{j_s}$, then, for any $n > \max(i_s, j_s)$, $(u_n-u_{i_1})+(u_n-u_{i_2})+\ldots+(u_n-u_{i_s}) = (u_n-u_{j_1})+(u_n-u_{j_2})+\ldots+(u_n-u_{j_s})$.

Since we appealed to Lemma 5, our proof is only valid for $k+1 \geq 4$. However, the theorem is vacuously true for small cardinalities. By Lemmas 1 and 4, there are no equal singletons and no pairs with equal sums. Lemmas 6 and 7 concern triples and quadruples.

Lemma 6. *There are no distinct triples of the u_i with equal sums.*

Proof: The result will follow if we show that

$$(12) \qquad u_{n+1} \geq u_n + u_{n-1} + u_{n-2} \qquad (n \geq 2).$$

We can verify this from Table 1, for $2 \leq n \leq 7$ (with equality for $3 \leq n \leq 6$). For $n > 7$, $r > 3$, $n-r < n-3$ and (7) with Lemma 1 gives $u_{n+1} > 2u_n - u_{n-3}$. The inequality (12) follows inductively on adding this to $u_n \geq u_{n-1}+u_{n-2}+u_{n-3}$.

Lemma 7. *There are no distinct quadruples of the u_i with equal sums.*

Proof: This can be checked from Table 1 if no value of i is greater than 11, while for $n \geq 11$, the result follows from

$$(13) \qquad u_{n+1} > u_n + u_{n-1} + u_{n-2} + u_{n-3} \qquad (n \geq 11).$$

This is proved as in Lemma 6, since $n \geq 11$ implies $r > 4$, $n-r < n-4$ and $u_{n+1} > 2u_n - u_{n-4}$.

These results can clearly be extended, but at the cost of increasing amounts of computation. However, we will show how to avoid this.

Theorem 6. *If* $T_s = \frac{1}{2}s(s+1)$, $s \geq 0$ *and* $0 \leq t \leq s+2$, *then*

$$(14) \qquad \sum_{i=T_s+t}^{T_{s+1}+t} u_i = u_{T_{s+1}+t+1} + \sum_{i=2}^{s} u_{T_i} .$$

(If $s = 1$ *or* 0, *interpret the empty or "less than empty" sum on the right as* 0 *or* -1 *respectively.)*

Proof: The theorem may be verified from Table 1 for $s = 0, 1, 2$ and $0 \leq t \leq s+2$. Note that the result for $t = s+2$ is the same as that for $t = 0$ and $s+1$ in place of s, if we add $u_{T_{s+1}}$ to each side, since $T_{s+1}+s+2 = T_{s+2}$ and $T_s+s+2 = T_{s+1}+1$. So we assume inductively that (14) is true for some $s \geq 2$ and some t, $0 \leq t \leq s+1$ and deduce its truth for the same s and for $t+1$ in place of t. To replace t by $t+1$ in the left member of (14), add $u_{T_{s+1}+t+1} - u_{T_s+t}$. Combine this with $u_{T_{s+1}+t+1}$ on the right, and we have

$$2u_{T_{s+1}+t+1} - u_{T_s+t} = u_{T_{s+1}+t+2} \qquad (0 \leq t \leq s+1)$$

by (7), which, for $n = T_{s+1}+t$, may be written

$$(15) \qquad u_{T_{s+1}+t+1} = 2u_{T_{s+1}+t} - u_{T_s+t-1} \qquad (1 \leq t \leq s+2).$$

This completes the proof.

Lemma 8. *If* $s \geq 0$, *and with the convention of Theorem* 6,

$$\sum_{i=2}^{s} u_{T_i} < \tfrac{1}{2}(u_{T_s+1} + u_{T_{s-1}+2}).$$

Proof: $s = 0$, $-1 < \frac{1}{2}(1+2)$. $s = 1$, $0 < \frac{1}{2}(2+2)$. $s = 2$, $4 < \frac{1}{2}(7+4)$. $s = 3$, $4+24 < \frac{1}{2}(44+13)$. Assume that the theorem is true for $s = v \geq 3$.

Then

$$\sum_{i=0}^{v+1} u_{T_i} = u_{T_{v+1}} + \sum_{i=0}^{v} u_{T_i} < u_{T_{v+1}} + \tfrac{1}{2}(u_{T_v+1}+i_{T_{v-1}+2})$$

$$= \tfrac{1}{2}(2u_{T_{v+1}}+u_{T_v+1}+u_{T_{v-1}+2})$$

$$= \tfrac{1}{2}(u_{T_{v+1}+1}+u_{T_v}+u_{T_v+1}+u_{T_{v-1}+2}) \text{ by (7) with } n = T_{v+1}$$

$$\le \tfrac{1}{2}(u_{T_{v+1}+1}+u_{T_v+2}) \text{ by (12) with } n = T_v+1, \text{ provided}$$

$T_{v-1}+2 \le T_v-1$, which is true for $v \ge 3$,

so the result follows by induction.

Lemma 9. *If* $v > T_{s+1}$, *then* $\sum_{i=v-s}^{v} u_i < u_{v+1}$.

Proof: The cases $s = 0,1,2,3$ are Lemmas 1, 4 and inequalities (12), (13). If $v = T_{s+1}+1$, Theorem 6 with $t = 1$ and Lemma 8 give

$$\sum_{i=T_s+2}^{T_{s+1}+1} u_i = \sum_{i=T_s+1}^{T_{s+1}+1} u_i - u_{T_s+1} = u_{T_{s+1}+2} + \sum_{i=2}^{s} u_{T_u} - u_{T_s+1}$$

$$< u_{T_{s+1}+2} + \tfrac{1}{2}(u_{T_s+1}+u_{T_{s-1}+2}) - u_{T_s+1}$$

$$\le u_{T_{s+1}+2} \text{ provided } T_{s-1}+2 \le T_s+1, \text{ which is true for } s \ge 1.$$

Now suppose the lemma holds for $v = w > T_{s+1}$. Then

$$\sum_{i=w-s+1}^{w+1} u_i = u_{w+1} - u_{w-s} + \sum_{i=w-s}^{w} u_i < 2u_{w+1} - u_{w-s} < u_{w+2} \text{ by (7), since}$$

$w > T_{s+1}$, and the result follows by induction.

Theorem 7. *If* $s \geq 0$ *and* $1 \leq t \leq s+2$, *then, with the same convention as in Theorem* 6,

$$(16) \qquad u_{T_s+t+1} > \sum_{i=0}^{T_s+t-1} u_i + \sum_{i=2}^{s} u_{T_i} .$$

Proof: The theorem may be verified from Table 1 for $0 \leq s \leq 2$ and $1 \leq t \leq s+2$. If the theorem is true for given values of s and t, then it is true for the same s and for $t+1$ in place of t, since, to increase t by one we add $u_{T_s+t+2} - u_{T_s+t+1}$ to the left member of (16), and we add u_{T_s+t} to the right member, and the latter is less than the former by Lemma 4.

If the theorem is true for some $s \geq 2$ and for $t = 1$, then it is true for $s+1$ and $t = 1$, since, to increase s by one when $t = 1$ we add $u_{T_{s+1}+2} - u_{T_s+2}$ to the left member of (16), while we increase the right member by an amount

$$\sum_{i=T_s+1}^{T_{s+1}} u_i + u_{T_{s+1}} = u_{T_{s+1}+2} + \sum_{i=2}^{s} u_{T_i} - u_{T_{s+1}+1} + u_{T_{s+1}}$$

by Theorem 6, i.e. by an amount $u_{T_{s+1}+2} - (u_{T_{s+1}+1} - u_{T_{s+1}}) + \sum_{i=2}^{s} u_{T_i}$,

which is less than

$$u_{T_{s+1}+2} - (u_{T_{s+1}+1} - u_{T_{s+1}}) + \tfrac{1}{2}(u_{T_s+1} + u_{T_{s-1}+2}) \text{ by Lemma 8}$$

$$= u_{T_{s+1}+2} - u_{T_s+2} - \tfrac{1}{2}(2u_{T_{s+1}+1} - u_{T_s+1} - 2u_{T_s+2} - u_{T_{s+1}} - u_{T_{s-1}+2})$$

$$\leq u_{T_{s+1}+2} - u_{T_s+2} - \tfrac{1}{2}(u_{T_{s+1}} + 2u_{T_{s+1}-1} + 2u_{T_{s+1}-2} - 2u_{T_s+2} - u_{T_s+1} - u_{T_{s-1}+2})$$

by (12), $s \geq 1$

$< u_{T_{s+1}+2} - u_{T_s+2}$ provided $T_{s+1}-1 \geq T_s+2$, i.e. provided $s \geq 2$, so the theorem is proved.

Theorem 8. *If there are two sets of the u_i with equal sums, and the largest member of either set is $u_{T_{s+1}+t+1}$ where $1 \leq t \leq s+2$, then the other set contains at least $s+2$ members, including the $s+1$ members u_i, $T_s+t+1 \leq i \leq T_{s+1}+t$.*

Proof: If the other set did not contain these $s+1$ members, then its sum would be at most

$$\sum_{i=T_s+t}^{T_{s+1}+t} u_i - u_{T_s+t+1} + \sum_{i=0}^{T_s+t-1} u_i = u_{T_{s+1}+t+1} + \sum_{i=2}^{s} u_{T_i} - u_{T_s+t+1} + \sum_{i=0}^{T_s+t-1} u_i$$

by Theorem 6, and this is $< u_{T_{s+1}+t+1}$ by Theorem 7. Also by Theorem 6, the sum of the $s+1$ members

$$\sum_{i=T_s+t+1}^{T_{s+1}+t} u_i = u_{T_{s+1}+t+1} + \sum_{i=2}^{s} u_{T_i} - u_{T_s+t} < u_{T_{s+1}+t+1} - \tfrac{1}{2}(2u_{T_s+t} - u_{T_s+1} - u_{T_{s-1}+2})$$

by Lemma 8, and since this is less than $u_{T_{s+1}+t+1}$, the set must contain at least one more member if there is to be equality of sums. This completes the proof of the theorem.

Note that Theorem 8 is *not* vacuous, since there *are* sets of the with equal sums, e.g. $7 = 4+2+1$, $13 = 7+4+2$, $24 = 13+7+4$ and $44 = 24+13+7$. These sets do not have equal cardinality, though after adding $u_0 = 0$ to the first member, the cardinalities differ only by one. More generally, Theorem 6 exhibits sets of the u_i with equal sums, whose cardinalities are $s+2$ and s or $s+1$. For example

$$1164 + 594 + 309 + 161 + 84 = 2284 + 24 + 4 \ (+0).$$

In Theorems 9 to 13, on the other hand, we will assume the following conditions, which include equal cardinality, so after Theorem 14 these theorems will be seen to be only vacuously true. Conditions C and D are of "minimal criminal" type.

A: There are two sets of the u_i with equal sums.

B: These two sets have the same cardinality, c.

C: Of such pairs of sets we choose one with the least possible greatest member, u_{n+1}, and write n in the form $T_{s+1}+t$, where $1 \le t \le s+2$. Note that if we stay in this interval, formula (15) holds good, but if we stray outside it, some care is needed. For example, if $t = 1$ or 2, $u_{T_{s+1}+t-1} = 2u_{T_{s+1}+t-2} - u_{T_s+t-2}$.

D: Among pairs of sets satisfying A to C, choose one with least value of c. This condition implies that the two sets are disjoint. Lemmas 1, 4, 6 and 7 imply that $c \ge 5$.

We call the set containing u_{n+1} the major set and the other the minor set. If we use subscripts j and i respectively for the two sets, then Theorem 8 implies that $\{i\}$ contains $\{i : T_s+t+1 \le i \le T_{s+1}+t\}$.

Theorem 9. *Under conditions A to D, u_{T_s+t-1} belongs to the minor set.*

Proof: By Theorem 8 we have

(17) $$u_{i_1}+u_{i_2}+\ldots+u_{T_s+t+1}+u_{T_s+t+2}+\ldots+u_{T_{s+1}+t} = u_{j_1}+u_{j_2}+\ldots+u_{T_{s+1}+t+1}.$$

Substitute for $u_{T_{s+1}+t+1}$ by formula (15), cancel $u_{T_{s+1}+t}$ from each side, and add u_{T_s+t-1} to each side and we will have produced two sets of with equal sums, but with a smaller largest member, $u_{T_{s+1}+t}$, than before, contrary to condition C, *unless* u_{T_s+t-1} was already present on the left of (17), i.e. in the minor set.

Theorem 10. *Under conditions A to D, the minor set does not contain all the $s+4$ members u_i, $T_s+t-2 \le i \le T_{s+1}+t$.*

Proof: If the minor set contained these $s+4$ members, its sum, by Theorem 6, would be at least

$$u_{T_{s+1}+t+1} + \sum_{i=2}^{s} u_{T_i} + u_{T_s+t-1} + u_{T_s+t-2},$$

while the sum of the major set would be at most $u_{T_{s+1}+t+1} + \sum_{i=0}^{T_s+t-3} u_i$, which, by Lemma 5, is at most $u_{T_{s+1}+t+1} + u_{T_s+t-2} + u_{T_s+t-5}$. This contradicts condition A.

Theorem 11. *Under conditions A to D, u_{T_s+t} belongs to the minor set.*

Proof: We first show that u_{T_s+t} does not belong to the major set. If it did, then the sum of the major set would exceed $u_{T_{s+1}+t+1} + u_{T_s+t}$, while the sum of the minor set is less than

$$\sum_{i=T_s+t+1}^{T_{s+1}+t} u_i + \sum_{i=0}^{T_s+t-1} u_i = \sum_{i=T_s+t}^{T_{s+1}+t} u_i - u_{T_s+t} + \sum_{i=0}^{T_s+t-1} u_i$$

$$= u_{T_{s+1}+t+1} + \sum_{i=2}^{s} u_{T_i} - u_{T_s+t} + \sum_{i=0}^{T_s+t-1} u_i \text{ by Theorem 6,}$$

$$< u_{T_{s+1}+t+1} + \tfrac{1}{2}(u_{T_s+1}+u_{T_{s-1}+2}) - u_{T_s+t} + u_{T_s+t} + u_{T_s+t-3}$$

by Lemmas 8 and 5, and $u_{T_s+t} > \frac{1}{2}(u_{T_s+1}+2u_{T_s+t-3}+u_{T_{s-1}+2})$, since

$$u_{T_s+t} - u_{T_s+t-3} \geq u_{T_s+t-1} + u_{T_s+t-2} \text{ by (12)}$$

$$\geq u_{T_s} + u_{T_s-1} \quad (t \geq 1)$$

$$> \tfrac{1}{2}(u_{T_s+1} + u_{T_{s-1}+2}) \text{ by (7).}$$

This contradicts condition A.

On the other hand, if u_{T_s+t} belongs to neither set, we may transfer u_{T_s+t-1} to the major set and balance this by adjoining u_{T_s+t} and $u_{T_{s-1}+t-2}$ ($u_{T_{s-1}}$ if $t = 1$) to the minor set*. We are then able to replace $u_{T_{s+1}+t+1}$ and u_{T_s+t-1} in the major set by $2u_{T_{s+1}+t}$. On cancelling $u_{T_{s+1}+t}$ from each set we have found a pair of sets with smaller largest member, contrary to condition C.

*An apparent difficulty arises if $u_{T_{s-1}+t-2}$ is already in the minor set (if it is in the major set, cancel it and reduce the cardinality by one). However we can then replace the resulting $2u_{T_{s-1}+t-2}$

by $u_{T_{s-1}+t-1}$ and u_q for suitable q ($q = T_{s-2}+t-2$ or $T_{s-2}+t-3$). This may in turn produce a duplicate, $2u_q$, but eventually the process will end (e.g. by filling the space left by the missing u_{T_s+t-1}).

We illustrate this reduction in Fig. 1, which shows two sets $\{u_j\}$, $\{u_i\}$ with equal sum 267443 (and cardinalities 6 and 10). To get from

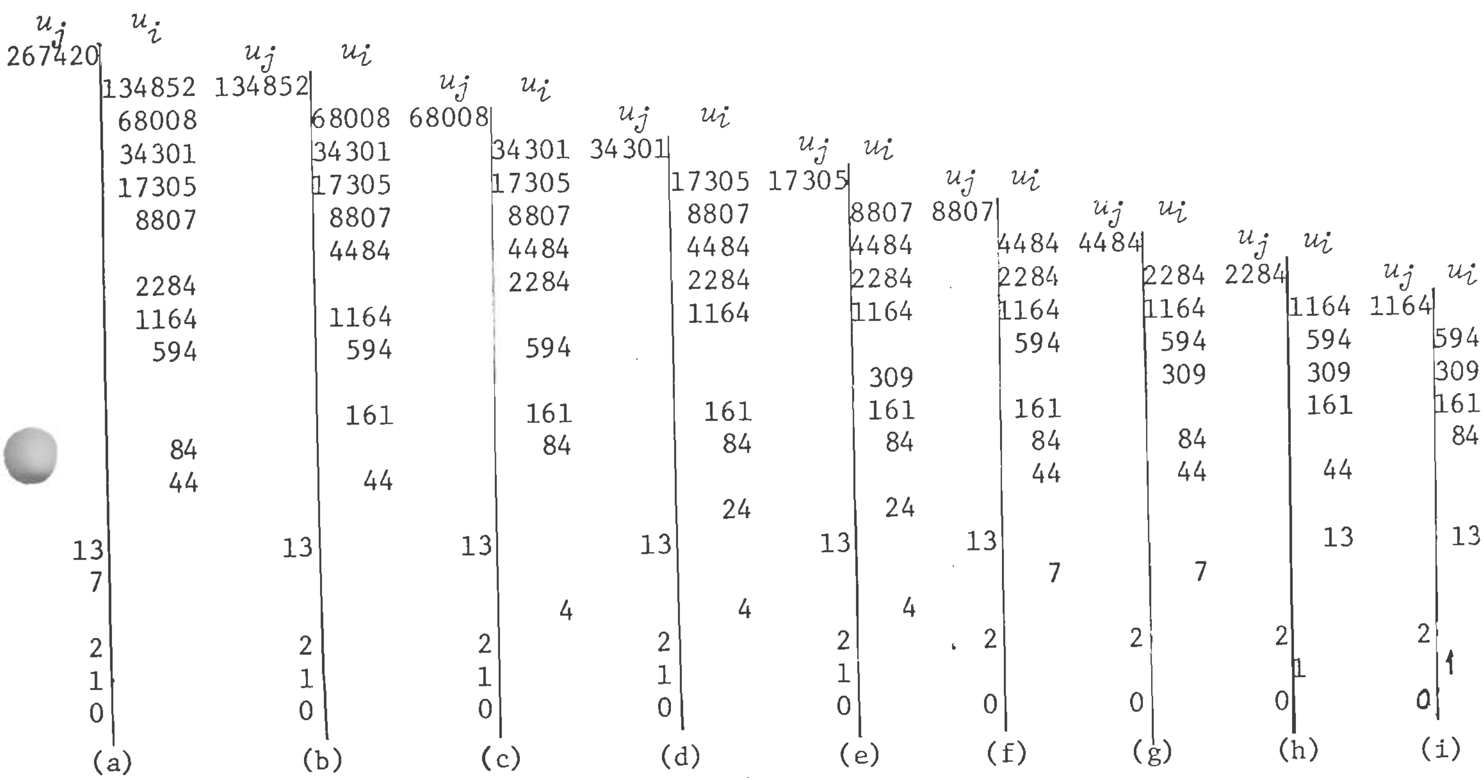

Fig. 1. Reduction of sets with equal sums.

(a) to (b), transfer 2284 from the minor to the major set; balance by adjoining 4484 and 84 to the minor set; replace 267420 and 2284 in the major set by 2×134852; cancel 134852 from each set; replace 2×84 in the minor set by 161 and 7, and finally cancel 7. The sets now have a smaller greatest member, contradicting C (and the cardinalities are now 5, 9).

Whenever there is a term missing (4484 in (a), 2284 in (b)) we can continue this reduction algorithm. To get from (b) to (c), adjoin

1164 to each set, replace 134852+1164 in the major set by 2×68008; cancel 68008; replace 2×1164 by 2284 and 44; and replace 2×44 by 84 and 4. And so on.

Theorems 8 to 11 show that the minor set contains the whole subsequence of u_i, $T_s+t-1 \le i \le T_{s+1}+t$, of $s+3$ terms. Theorem 12 will show that t must take the particular value $s+2$.

Theorem 12. *Under conditions A to D, $t = s+2$.*

Proof: From theorems 8 to 11, the sum of the minor set is at least

$$\sum_{i=T_s+t-1}^{T_{s+1}+t} u_i = u_{T_{s+1}+t+1} + \sum_{i=2}^{s} u_{T_i} + u_{T_s+t-1},$$

by (14), while the sum of the major set is at most

$$u_{T_{s+1}+t+1} + \sum_{i=0}^{T_s+t-2} u_i = u_{T_{s+1}+t+1} + u_{T_s+t-1} + \sum_{i=2}^{s-1} u_{T_i} + \sum_{i=0}^{T_{s-1}+t-3} u_i,$$

and $u_{T_s} > \sum_{i=0}^{T_{s-1}+t-3} u_i$ by Lemma 5, unless $t = s+2$.

Theorem 13. *Under conditions A to D, the major set contains the $s+1$ members u_i, $T_s \le i \le T_s+s$.*

Proof: Suppose one of these u_i is not in the major set. Then even if this is the least, u_{T_s}, the major set has sum at most

$$u_{T_{s+1}+s+3} + \sum_{i=T_s+1}^{T_s+s} u_i + \sum_{i=0}^{T_s-1} u_i = u_{T_{s+1}+s+3} + \sum_{i=T_{s-1}+s}^{T_s+s} u_i - u_{T_s} + \sum_{i=0}^{T_s-1} u_i <$$

$$u_{T_{s+1}+s+3} + u_{T_s+s-1} + \sum_{i=2}^{s-1} u_{T_i} - u_{T_s} + u_{T_s} + u_{T_s-3},$$

by Lemma 5, while the minor set has sum at least

$$\sum_{i=T_s+s+1}^{T_{s+1}+s+2} u_i = u_{T_s+s+1} + u_{T_{s+1}+s+3} + \sum_{i=2}^{s} u_{T_i},$$

contradicting condition A.

Lemma 10. *If $s \geq 0$, then*

$$(18) \qquad \sum_{i=T_s+s+1}^{T_{s+1}+s+2} u_i = u_{T_{s+1}+s+3} + \sum_{i=T_s}^{T_s+s} u_i + u_{T_s} .$$

Proof: By (14), the left side $= u_{T_{s+1}+s+3} + \sum_{i=2}^{s} u_{T_i} + u_{T_s+s+1}$, while, since $T_s = T_{s-1}+s$, the right side $= u_{T_{s+1}+s+3} + u_{T_s+s+1} + \sum_{i=2}^{s-1} u_{T_i} + u_{T_s}$.

Theorems 8 to 12 tell us that the $s+3$ terms of the left sum belong to the minor set, and Theorem 13 that the first $1+(s+1)$ terms of the right side belong to the major set. Lemma 10 tells us that the sums of these differ by exactly u_{T_s}. In fact Lemma 10 exhibits two sets of the u_i with equal sum and with equal cardinality, $s+3$, but the right side contains two copies of u_{T_s}.

Theorem 14. *Distinct subsets of the a_i, defined by* (10), *always have distinct sums.*

Proof: After Theorem 5, it suffices to show that there are no pairs of sets of the u_i which satisfy conditions A and B. But we now have the essential step for a "method of descent" proof that there are no such pairs. If there were, choose one which also satisfies conditions C and D. Replace the $s+3$ terms now known to be in the minor set, and the $s+2$ terms known to be in the major set by the single term u_{T_s} in the minor set. This

(a) reduces the common sum by the same amount (either side of (18), less u_{T_s}),

(b) reduces the common cardinality by $s+2$,

(c) reduces the greatest member from $u_{T_{s+1}+s+3}$ to u_{T_s}, and

(d) interchanges the rôles of major and minor set.

So we have a new pair of sets contradicting the minimality conditions C and D. This proves the theorem.

Conclusion.

Many of the results of this paper have been known for twenty years, at least to John Conway and the author, and perhaps to others. It is only recently, however, that it has been possible to push through to the long-suspected result of Theorem 14. The more difficult problem, of showing that the sequence is the densest with the given property, remains unsolved.

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