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Numerical bounds preprint

One sequence

Numerical ~~values~~ bounds for the Arnol'd "meander" ~~problem~~ constant

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1. Introduction

Let a_n denote the number of topologically distinct ways in which an oriented line and a Jordan curve in the plane can cross at $2n$ points.* Call such figures "Arnol'd figures" of size n. Other objects counted by a_n include: the "planar permutations" of $2n$ letters (Rosenstiehl & Tarjan), the "rooted plane graphs with unique bicycles" with $2n$ edges (Rosenstiehl), the "simple alternating transit mazes" of depth $2n$ (Phillips), the "oriented folds" of loops of $2n$ postage stamps (Phillips's generalization of Koehler). A *two colored tour* of $2n$ given points in the plane is a polygon whose vertices are the given points, whose sides are colored alternately red and blue, such that no two red sides cross and no two blue sides cross. The number of two colored tours of $2n$ given distinct points on the circumference of a circle is a_n. Finally, suppose the real axis in the complex plane is a river, with bridges at $z=1,2,...2n-1$. A road (that is, a curve) leads from $-i\infty$ to $+i\infty$, which (1) does not cross itself, (2) crosses the river only at bridges, and (3) crosses the river exactly once at each bridge. We call such a path a "meander". The road necessarily visits all the bridges in some order, a permutation of $\{1,2, \cdots ,2n-1\}$. The number of such river crossing permutations which arise from meanders is also a_n. Figures 1, 2, and 3 illustrate some of these trivial equivalences in the cases a_1, a_2, and a_3, and Table 1 lists all known numerical values for a_n.

Fig 1 goes about here: $a_1=1$, $a_2=2$, $a_3=8$
Jordan curves + line.

Fig 2 goes about here: $a_1=1$, $a_2=2$, $a_3=8$
2 colored tours
(solid line = red, dashed line = blue)

Fig 3 goes about here: $a_1=1$, $a_2=2$
river & road

* The quantity a_n has been independently discovered and studied by several people, including Rosenstiehl, Tarjan, Phillips, and Arnol'd. It has been recently most intensely studied ~~in Moscow~~ by Arnol'd, Lando, and Zvonkin; we learned about a_n from V. I. Arnol'd by a personal communication.

n	a_n
1	1
2	2
3	8
4	42
5	262
6	1828
7	13820
8	110954
9	933458
10	8152860
11	73424650
12	678390116
13	6405031050
14	61606881612

Unfortunately, although a_n has a simple definition there is no simple formula for computing a_n or for estimating its numerical order of magnitude. This paper presents a new recipe for calculating numerical values of a_n (which we used to calculate the values in Table 1) and bounds on the asymptotic growth rate of a_n.

It is easy to see that a_n is submultiplicative, and that

$$c_n \leq a_n \leq c_n^{\,2}$$

where c_n denotes the n-th Catalan number, $c_n = {}_{2n}C_n \;/\; (n+1)$. Hence the limit

$$a = \lim_{n \to \infty} a_n^{\,1/n}$$

exists, and since $\lim_{n \to \infty} c_n^{\,1/n} = 4$, we see that $4 \leq a \leq 16$. Our main effort is aimed at determining better upper and lower bounds for a. Our current best result is that $8.8 \leq a \leq 13.01$.

It is tempting to conjecture (from the known numerical values) that the ratios a_n / a_{n-1} increase as n increases. This would then imply that $a = \lim_{n \to \infty} a_n / a_{n-1} \geq a_{14} / a_{13} = 9.6185$, but we have been unable to prove the monotonicity of a_n / a_{n-1}.

Lando [ref] has derived number theoretical properties of a_n such as the fact that for prime p, $a_p \equiv 2$ (modulo p), and even, for $q = p^k$, $a_q \equiv 2$ (modulo p). These results may be obtained by studying actions of the $2n$-th cyclic and dihedral groups on the two colored tours of $2n$ points, but are not of interest to us in this paper.

2. Submultiplicativity

By composing river and road diagrams one can derive various inequalities. These include:

$$a_m a_n \leq a_{m+n-1}$$

which is the same as saying a_{n+1} is submultiplicative, and

$$2 a_m a_n \leq a_{m+n}$$

which is the same as saying $2a_n$ is submultiplicative. These imply that for each n, we have $a \geq a_{n+1}^{1/n}$ and $a \geq (2a_n)^{1/n}$, respectively. Taking the best known values for a_n these yield $a \geq 6.760$ and $a \geq 6.197$ respectively.

(If m and n are restricted to be greater than or equal to 2 a more complex argument gives the stronger inequality

$$\frac{9}{2} a_m a_n \leq a_{m+n}$$

but this seems useless.)

3. Catalan and Simple Upper bound

Recall that the Catalan numbers c_n count *balanced parenthesis expressions* of size n, that is, sequences of n left parentheses and n right parentheses ordered such that each prefix sequence has at least as many left parentheses as right parentheses. Thus, ()()() and (()()) are balanced parenthesis expressions of size 3 and)(()() and ())(() are not. (All such expressions can be uniquely obtained from the trivial (size 0) expression and from repeated combinations $(x)y$ of simpler expressions x and y; hence the generating function for the Catalan numbers obeys the equation $C(z) = 1+z\,C(z)^2$.)

Associated with each Arnol'd figure of size n is a distinct pair of balanced parenthesis expressions of size n, as illustrated in Fig 4. This implies $a_n \le c_n^2$. It is not hard to see that for any given expression T there is at least one other expression of same size, B, such that the pair T, B corresponds to some Arnol'd figure; hence $c_n \le a_n$.

Suppose a given Arnol'd figure of size n corresponds to expression pair T, B, where $T=t_1 t_2 \cdots t_{2n}$ and $B=b_1 b_2 \cdots b_{2n}$, where each of t_i and b_i is either the symbol (or the symbol). Then there is no i for which $t_i=b_i=($ and $t_{i+1}=b_{i+1}=)$. Let $X=x_1 x_2 \cdots x_{2n}$ represent T and B merged, so each x_i is an element of the four letter alphabet $\{a, b, c, d\}$, where $a = {(\atop (}$, $b = {(\atop)}$, $c = {) \atop (}$, and $d = {) \atop)}$, so that $x_i = {t_i \atop b_i}$. Then the symbol sequence X contains no instance of a followed by d. Thus a_n is bounded by the number of all sequences of $2n$ letters from the alphabet $\{a, b, c, d\}$ in which the word ad is forbidden. Elementary means show that the number of such sequences is given by an expression of form $C_1 \lambda_1^{2n} + C_2 \lambda_2^{2n}$ where the λ_i are the roots of the polynomial x^2-4x+2 and the C_i are non zero. Hence the number of such sequences is roughly $(2+\sqrt{3})^{2n}$ and hence the exponential growth rate for the a_n obeys $a \le (2+\sqrt{3})^2 \approx 13.9282 \cdots$.

4. Tightest Upper bound

The upper bound in the previous section can be strengthened, using an improvement of an idea of Lehmann's. The starting point is to consider the set of all c_n^2 topologically distinct ways in which an oriented line and a finite number of Jordan curves in the plane can cross at $2n$ points, each Jordan curve crossing the line at least once. Call such an arrangement a "Lehmann" figure of size n; they are in one to one correspondence with pairs of arbitrary balanced parenthesis expressions of size n. In each Lehmann figure there is a distinguished Jordan curve, namely the one which crosses the oriented line See Fig 5 for an example. The distinguished Jordan curve and the oriented line thus form an embedded Arnol'd figure of size k where $k \le n$; any remaining Jordan curves in the Lehmann figure are each either *inside* or *outside* the distinguished curve. Thus we have an equation

? first

$$c_n^2 = \sum_{k=1}^{n} \sum_{\gamma_k} \sum_{r+s=n-k} I_r(\gamma_k) O_s(\gamma_k)$$

where the sum extends over all possible distinguished Arnol'd figures γ_k of all possible sizes k, where the notation $I_r(\gamma)$ denotes the number of ways to place Jordan curves *inside* γ with exactly r crossings and where the notation $O_s(\gamma)$ denotes the number of ways to place Jordan curves *outside* γ with exactly s crossings.

5. McIlroy's Lower bound

Lower bounds on a_n or on a may be obtained by precise counting of particular systematically constructed subsets of Arnol'd figures. The largest subset we have been able to count precisely is due to M. D. McIlroy. We count "river crossing" permutations of the first $2n-1$ integers. Given any such permutation $p = p_1 p_2 \cdots p_{2n-1}$ an *odd block* is a sequence of an odd number of consecutive subscripts $i, i+1, \cdots j=i+2k$ such that the set of numbers $\{p_i, p_{i+1}, \cdots, p_j\}$ is a set of consecutive integers, although possibly in mixed order. (For instance, if $n=3$ the permutation $1,2,7,6,5,4,3$ has an odd block of size 3, with $i=3$ and $j=5$, because the set $\{7,6,5\}$ is a set of consecutive integers.) Given any such odd block form a new permutation q by flipping the odd block, as follows: $q = p_1 p_2 \cdots p_{i-1} p_j p_{j-1} \cdots p_{i+1} p_i p_{j+1} p_{j+2} \cdots p_{2n-1}$. That is, $q = q_1 q_2 \cdots q_{2n-1}$ where $q_t=p_t$ if

$t<i$ or $t>j$ and $q_t=p_{i+j-t}$ if $i \le t \le j$. (For example, 1, 2, 7, 6, 5, 4, 3 can be flipped this way to 1, 2, 5, 6, 7, 4, 3.)

McIlroy observes that any odd block flip of a river crossing permutation is also a river crossing permutation, so the number of distinct permutations derivable from $1,2,3, \cdots ,2n-1$ by repeated application of odd block flips is a lower bound on a_n. A key observation here is that in any such permuation, the odd blocks used for flips nest, and the resulting permutation is completely determined by the odd blocks which were used for an odd number of flips. This in turn is related to the fact that each of the odd block flips is an example of a ''braid'', and that the flips corresponding to nested or disjoint blocks commute in the braid group $B(2n-1)$.

6. Stamp folding

7. Computational formula