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The Number of Perfect Matchings in a Hypercube

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ABSTRACT

A perfect matching or a 1-factor of a graph G is a spanning subgraph that is regular of degree one. Hence a perfect matching is a set of independent edges which matches all the nodes of G in pairs. Thus in a hypercube parallel processor, the number of perfect matchings evaluates the number of different ways that all the processors can pairwise exchange information in parallel. Making use of matrices and their permanents one can write a straightforward formula which we evaluate for $n \leq 5$.

A *perfect matching* or a *1-factor* of a graph G is a regular spanning subgraph of degree one. In other words a perfect matching is a set of independent edges in $E(G)$ that spans $V(G)$. Define $f_1(G)$ as the number of 1-factors of G.

The *bipartite adjacency matrix* (*ba-matrix*) $B = B(G)$ of a bipartite graph $G = (V,E)$ where $V = U \cup W$, $|U| = m$, $|W| = n$, is the $m \times n$ matrix that indicates the presence or absence of an edge between each (u,w) pair of nodes by a one or zero, respectively. The following theorem was found independently by both Fisher [1] and Kasteleyn [3].

Theorem A. The number $f_1(G)$ of perfect matchings in an $n \times n$ bipartite graph G is $perB$, the permanent of the ba-adjacency matrix B of graph G.

$$f_1(G) = perB(G) \tag{1}$$

Proof. The permanent of a square binary matrix is simply the number of ways of choosing exactly one 1 from each row and each column. Hence, there exists a one-to-one correspondence between perfect matchings and the unit contributions to this permanent. []

The hypercube Q_n may be recursively defined [2,p.23] in terms of cartesian product:

$$Q_n = \begin{cases} K_2 & n = 1 \\ Q_{n-1} \times K_2 & n \geq 2 \end{cases} \tag{2}$$

Using this definition, the ba-matrix B_n of a hypercube Q_n may be conveniently written recursively, with I denoting the identity matrix of order 2^{n-1}:

$$B_1 = [1]\,, \quad B_{n+1} = \begin{bmatrix} B_n & I \\ I & B_n \end{bmatrix} \tag{3}$$

By Theorem A the value of $perB_n$ is the number of perfect matchings in Q_n.

A *submatrix* X of a matrix A is the matrix formed by choosing a subset of the rows of A and a subset of the columns of A. It is convenient to give an expression for counting the perfect matchings of Q_{n+1}.

Theorem 1. The number of perfect matchings of Q_{n+1} is given by

$$f_1(Q_{n+1}) = perB_{n+1} = \sum_{X \subset B_n} (perX)^2. \tag{4}$$

Proof. Consider any $k \times k$ matrix X in the upper left B_n in B_{n+1} as in (1) such that $perX \neq 0$. The permanent of X is the number of perfect matchings in the subgraph of Q_n induced by the nodes corresponding to the rows and columns of X. Call this induced subgraph G_X. Now match all nodes of $V(Q_n) - V(G_X)$ with their neighbors in the other copy of Q_n. Clearly, the unmatched nodes in the second copy of Q_n induce a graph isomorphic to G_X and its permanent is $perX$. Thus, for each square submatrix X in B_n there are $(perX)^2$ perfect matchings in B_{n+1}. []

For example $B_2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ so by (1), $B_3 = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$ and

$$f_1(Q_3) = perB_3 = 1^2 + 4per[1]^2 + per\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}^2 = 9$$

Note that the empty matrix with unit permanent is an admissible submatrix of B_n and contributes 1 to the above sum in the 1^2 term.

It is convenient to introduce some additional notation. For any square matrix A, let

$$\begin{bmatrix} A \\ k \end{bmatrix} = \sum_{X \subset A} (perX)^2 \tag{5}$$

where the summation is over all $k \times k$ submatrices X of A. Thus (1) may be rewritten

$$f_1(Q_{n+1}) = perB_{n+1} = \sum_{k=0}^{2^{n-1}} \begin{bmatrix} B_n \\ k \end{bmatrix} \tag{6}$$

Obviously, $\begin{bmatrix} B_n \\ 0 \end{bmatrix} = 1$ and $\begin{bmatrix} B_n \\ 2^{n-1} \end{bmatrix} = (perB_n)^2 = f_1{}^2(Q_n)$. The number of ones in B_n is $\begin{bmatrix} B_n \\ 1 \end{bmatrix}$ which is the number of edges in Q_n, so $\begin{bmatrix} B_n \\ 1 \end{bmatrix} = n2^{n-1}$.

To derive a closed form for $\begin{bmatrix} B_n \\ 2 \end{bmatrix}$ it is convenient to identify all dissimilar pairs of columns of B_n. These correspond to all dissimilar pairs of nodes in Q_n. Any pair of nodes may be completely specified by their distance because of symmetry, and the number of pairs at distance $2k$ is

$$\frac{1}{2}\binom{2^{n-1}}{1}\binom{n}{2k} = 2^{n-2}\binom{n}{2k}$$

When $k = 1$ any two nodes at distance 2 are mutually adjacent to exactly two nodes and are each individually adjacent to $n-1$ other nodes. Therefore, any pair of columns in B_n corresponding to nodes at distance 2 consist of two rows of the form 1 1 and $n-1$ rows of the form 1 0 and $n-1$ rows of the form 0 1 and the other $2^{n-1} - 2n$ rows 0 0.

Thus, the sum of the squares of all permanents formed by selecting two nodes at distance 2 is

$$per\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}^2 + 2(n-2)per\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^2 + 2(n-2)per\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^2 + (n-2)^2 per\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^2 = n^2$$

Any two nodes at distance greater than two obviously have disjoint neighborhoods and their corresponding pair of columns in B_n contain n copies of 1 0 and of 0 1 in their rows, the other rows consisting of 0 0 entries. Thus the permanents of the 2×2 matrices formed in these columns contribute a factor of n^2, giving

$$\begin{bmatrix} B_n \\ 2 \end{bmatrix} = n^2 2^{n-2}\binom{n}{2} + n^2 2^{n-2}\sum_{k=2}^{\lfloor n/2 \rfloor}\binom{n}{2k}, \text{ but}$$

$$\sum_{k=0}^{\lfloor n/2 \rfloor}\binom{n}{2k} = 2^{n-1}, \quad \text{hence} \quad \begin{bmatrix} B_n \\ 2 \end{bmatrix} = n^2 2^{n-2}(2^{n-1} - 1)$$

$$\text{To compute } per B_4 = per\begin{bmatrix} 1110 & 1000 \\ 1101 & 0100 \\ 1011 & 0010 \\ 0111 & 0001 \\ & \\ 1000 & 1110 \\ 0100 & 1101 \\ 0010 & 1011 \\ 0001 & 0111 \end{bmatrix}$$

we first evaluate $\begin{bmatrix} B_3 \\ 3 \end{bmatrix}$. Since all sets of three nodes of even weight in Q_3 are similar, any three columns of B_3 may be chosen. Within any three columns there are two dissimilar (with respect to the automorphism group of Q_3) 3×3 submatrices giving

$$\begin{bmatrix} B_3 \\ 3 \end{bmatrix} = \binom{4}{3}\left\{ per\begin{bmatrix} 1\,1\,0 \\ 1\,0\,1 \\ 0\,1\,1 \end{bmatrix}^2 + 3\, per\begin{bmatrix} 1\,1\,0 \\ 1\,0\,1 \\ 1\,1\,1 \end{bmatrix}^2 \right\} = 124, \quad \text{so}$$

$$f_1(Q_4) = \sum_{i=0}^{i=4} \begin{bmatrix} B_3 \\ i \end{bmatrix} = 1 + 3.2^2 + 3^2.2(2^2-1) + 124 + 9^2 = 272$$

We conclude by only mentioning the result that

$$f_1(Q_5) = 589{,}185$$

which was similarly calculated by a computer program.

An even more difficult unsolved problem in graphical enumeration is the exact determination of the number $f_1^*(Q_n)$ of equivalence classes of perfect matchings in hypercube Q_n with respect to its automorphism group $\Gamma(Q_n)$. It has been shown that $\Gamma(Q_n) = [S_2]^{S_n}$, the exponentiation group [2,p.177] of the two symmetric groups S_2 raised to the power S_n. It is also known that, in principle, the number $f_1^*(Q_n)$ of these similarity classes can be calculated from the group of the graph Q_n with respect to the group of the subgraph $2^{n-1}K_2$ (which is a perfect matching of Q_n). But this approach has not yet proved helpful. Obviously $f_1^*(Q_1) = f_1^*(Q_2) = 1$, $f_1^*(Q_3) = 2$ and we have also found that $f_1^*(Q_4) = 8$.

References

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