A5210 resumes here

But going back to Problem 191, Mr. Ferguson has again demolished analytically a problem that was thought to be solvable only by computer. One of our goals is to seek out problem situations for which the computer is the best (if not the only) tool for solution. Perhaps the Z-Sequence, introduced in our issue No. 42, is a good example of what we seek.

The Z-Sequence is defined by:

A5210

$$a_n = \left|2a_{n-2} + a_{n-1} - n\right|$$

with $a_1 = a_2 = 1$. This sequence has interesting properties:

1. **Successive terms seem unpredictable; that is, the values jump around, but not too wildly.**

2. **In the long run, its terms tend toward n/3, where n is the term number. Thus, the sum of the first 8002 terms is 11245744, whereas the sum of n/3 for n = 1, 2, 3,...,8002 is 11213501, a difference of only 32233.**

3. **The sequence is readily generated in any language on any machine, using the simplest of operations, and dealing only with positive integers. The accompanying table gives restart values at various points in the sequence.**

4. **No term after the first is ever as large as n. The circled numbers in the original table (see PC42-13) show the appearances of new larger values. Such new larger values are seen to occur 68 times in the first 5081 terms.**

A51202

5. **Zero appears at terms 3, 7, 11, 28, 31, 140, 239, 600, and 6476, but does not seem to appear after that. Up to n = 5000, only 67 terms have a value less than 5.** False!

A256962

6. **Successive equal terms appear at n = 90 and n = 98, but this phenomenon does not seem to occur again.** False!

7. If item 2 above holds indefinitely, then the sum of the series should be approximately the sum of $n/3$. The sum of each term divided by n should be approximately $n/3$ itself. The sum of each term divided by the square of the term number should be approximately 1/3 the sum of the harmonic series (see PC9-14). The sum of each term divided by the cube of the term number, however, should approximate 1/3 the sum of $(1/n^2)$, which should converge. It appears to converge to 1.2346...

8. Almost every integer appears as a term in the sequence sooner or later, except for a few numbers, such as 245, 449, 569, 575, and 903.

A great deal of information has now accumulated about the Z-sequence. Table B shows the frequency, for every 1000 terms, of terms that are less than 1000. Although the trend is for terms to approximate $n/3$, where n is the term number, it is seen that small values persist with remarkable stability.

Zero terms occur well beyond the point (term 6476) first indicated. The following terms are zero:

3, 7, 11, 28, 31, 140, 239, 600, 6476, A51202
33172, 64375, 65287, 79051, 97864, 105099.

and successive equal terms occur again at 5518 and 40722. A256962

Table C shows the appearance of each integer from 000 to 499 for the first time in the sequence. Only 22 numbers are yet unaccounted for, from 000 to 999. A51203

Table D shows restart values for the sequence, up to term 100,000. For each entry in the table, the last number is the term number (e.g., 10,000) and the other three are the term values at that term and the two that precede it. Thus, at 10,000, the table shows

term 9998: 4823
term 9999: 3634
term 10000: 3280

Table B

1000	1000	26000	73	51000	31	76000	25
2000	883	27000	81	52000	35	77000	27
3000	642	28000	50	53000	42	78000	22
4000	480	29000	33	54000	25	79000	18
5000	396	30000	60	55000	34	80000	17
6000	327	31000	61	56000	30	81000	18
7000	267	32000	57	57000	35	82000	29
8000	262	33000	63	58000	37	83000	22
9000	212	34000	56	59000	32	84000	24
10000	213	35000	49	60000	37	85000	26
11000	195	36000	62	61000	34	86000	24
12000	178	37000	46	62000	25	87000	30
13000	168	38000	53	63000	36	88000	20
14000	146	39000	73	64000	36	89000	16
15000	136	40000	53	65000	35	90000	15
16000	129	41000	41	66000	33	91000	23
17000	115	42000	47	67000	16	92000	21
18000	124	43000	41	68000	22	93000	21
19000	114	44000	45	69000	23	94000	17
20000	108	45000	35	70000	20	95000	30
21000	93	46000	40	71000	21	96000	21
22000	95	47000	38	72000	30	97000	16
23000	94	48000	36	73000	25	98000	14
24000	92	49000	40	74000	31	99000	15
25000	75	50000	39	75000	25	100000	24

Table D

167	799	1545	1953	2253	767	2517	7807
194	462	240	1404	340	4654	3260	22
472	60	330	1310	154	188	1294	7636
1000	2000	3000	4000	5000	6000	7000	8000
1291	4823	1655	5753	4447	8309	6187	3625
7422	3634	6818	2080	3546	4884	2778	9188
1004	3280	872	1586	560	7502	152	438
9000	10000	11000	12000	13000	14000	15000	16000
7847	2659	3833	12913	14039	6417	2669	15729
7050	6330	1024	5768	1806	11276	3968	3752
5744	6352	10310	11594	8884	2110	13694	11210
17000	18000	19000	20000	21000	22000	23000	24000
13021	16787	663	7277	9099	1991	3871	5747
7712	7294	17854	2348	6174	2686	16290	4698
8754	14868	7820	11098	4628	23332	6968	15808
25000	26000	27000	28000	29000	30000	31000	32000
15177	4655	23449	7149	3915	26897	6115	12507
7532	25758	3480	26576	32954	7328	7274	21946
4886	1068	15378	4874	3784	23122	19496	6960
33000	34000	35000	36000	37000	38000	39000	40000

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Table C

A51203	0	1	2	3	4	5	6	7	8	9
00	3	6	4	5	35	10	8	26	15	38
01	20	13	55	77	27	70	56	53	36	282
02	44	73	75	69	64	34	32	585	51	30
03	139	165	72	121	535	97	83	253	67	469
04	168	61	147	146	59	93	123	286	815	1398
05	112	294	119	129	347	138	124	81	144	194
06	256	142	295	190	79	101	271	109	136	445
07	163	529	127	145	107	162	200	174	460	478
08	143	117	167	270	223	2979	203	678	316	157
09	408	293	115	274	188	218	228	730	551	134
10	423	186	1028	177	336	229	132	290	155	433
11	287	3188	1348	201	160	213	252	446	315	2252
12	320	1521	219	3609	307	193	667	153	2604	249
13	208	690	296	1078	359	257	356	577	236	321
14	592	489	184	314	980	313	268	393	247	2538
15	416	1074	407	182	291	226	216	537	732	426
16	484	266	455	962	1607	206	255	277	487	1537
17	319	341	3975	326	639	285	892	1573	399	1093
18	367	798	656	402	412	733	351	505	875	986
19	960	562	243	245	999	301	532	741	324	1441
20	392	834	7512	697	1135	942	364	477	243	3229
21	264	241	968	1285	283	410	1552	794	339	334
22	1223	654	467	637	380	262	1212	1597	1071	337
23	723	310	1215	734	672	4082	1340	533	1160	354
24	260	1893	619	538	4680		895	773	332	1566
25	695	346	1979	2193	1891	397	1128	362	1036	1030
26	2208	693	560	1022	419	2857	1155	518	5088	782
27	424	454	584	750	904	389	452	1221	432	482
28	2808	2590	515	1650	4252	885	1932	441	344	430
29	599	1018	1260	497	475	882	680	422	883	378
30	803	761	771	470	748	698	387	510	480	4101
31	439	742	536	1741	2279	549	1556	657	916	793
32	376	3997	1396	1433	1496	5450	996	4890	2239	1290
33	1812	1561	1159	1226	675	737	1600	554	596	530
34	707	581	952	374	2135	1166	792	578	372	525
35	1791	3218	751	858	1279	2178	1456	513	851	370
36	1908	649	508	450	1316	754	1251	1218	523	437
37	4135	2382	4444	1026	992	797	628	670	547	1458
38	731	1850	1419	1126	495	1921	1356	1137	907	846
39	3267	2277	1191	545	576	1393	543	1814	1332	1174
40	696	3510	2496	1382	3247	574	591	2501	1288	1849
41	1559	1261	1068	617	687	749	552	493	8747	865
42	647	721	804	694	612	1125	1528	1162	640	2085
43	747	1258	2264	922	816	2201	2180	1142	1852	826
44	1696	521	4643	953	2927	1581	1432	685	2299	
45	1184	785	1167	1053	1720	2406	2484	1354	1016	2674
46	1575	2821	1335	1670	1104	1678	2008	1861	716	541
47	5588	1005	1000	1517	823	1797	1015	3381	764	1278
48	3591		1324	1037	3139	3785	8891	589	668	969
49	1380	7410	2611	10082	2603	890	2867	2094	3347	610