SCAN A5169
H W Gould
R K Guy and
NJAS, Correspondence,
10 pages 1987
add to A^5 seqs
(there is a reference in A259879, the new sequence)

West Virginia University

Department of Mathematics
College of Arts and Sciences
Morgantown, West Virginia
26506

A5169
A47998
A1524
A259879
A259880

29 March 1987

Professor Richard K. Guy,
Dept. of Mathematics & Statistics,
University of Calgary,
CALGARY, ALBERTA,
CANADA T2N 1N4

Dear Richard,

I thank you profusely for all the goodies you have been sending my way that bear on Catalan numbers and their congeners! But most especially the n coins in a fountain is fascinating.

The fact is that I have approached the matter from another viewpoint and it is first of all not surprising that the Catalan numbers arise.

The matter can be viewed from paths in a lattice diagram.

Since Wilf et al like to use the fountain bubbling up, I will set out here how to place the Catalan case into one-to-one correspondence with a well-known equivalent problem that you can find on pp. 71-73 of the 1957 Second Edition of Vol. 1 of Willy Feller's charming "Introduction to Probability Theory and its Applications". (You probably know Feller wrote three versions of Volume 1 before he even did one version of Vol.2....I recall we joked him about this. By the way I met him at UNC (1957) as he gave a fascinating lecture about Sparre Andersen's work on the inverse arcsine law.)

To get the correspondence we need really a basement layer of n + 1 coins. I think all this will be self-evident to you if I just enclose a couple of pages I drew up a week or so ago to show what is involved. The fountains are analogous to just enumerating the zig-zag lattice paths.

QUERY - QUERY - QUERY : Did you ever get a copy from me of the 1977 master's thesis by my student Mike Kuchinski? I don't have my records at hand to tell. But you may wish to have a copy so I am taking the liberty of sending you a copy to have on hand...we sell these productions at $10 a copy. The references are keyed to my standard Catalan-Bell Bibliography. Riordan and others have liked the thesis. I am hoping a student now will do the Bell numbers the same way so that we can then put all of this plus extensions together into a book.

Anyway, the relevant Feller Catalan problem involving Catalan numbers manifests itself as Structure S17 - Diagonal Segmented Paths - pp.27-28 of the thesis. It is really also easily seen to be isomorphic to Structure S3 - well-known Staircase Arrays, pp.9-10.

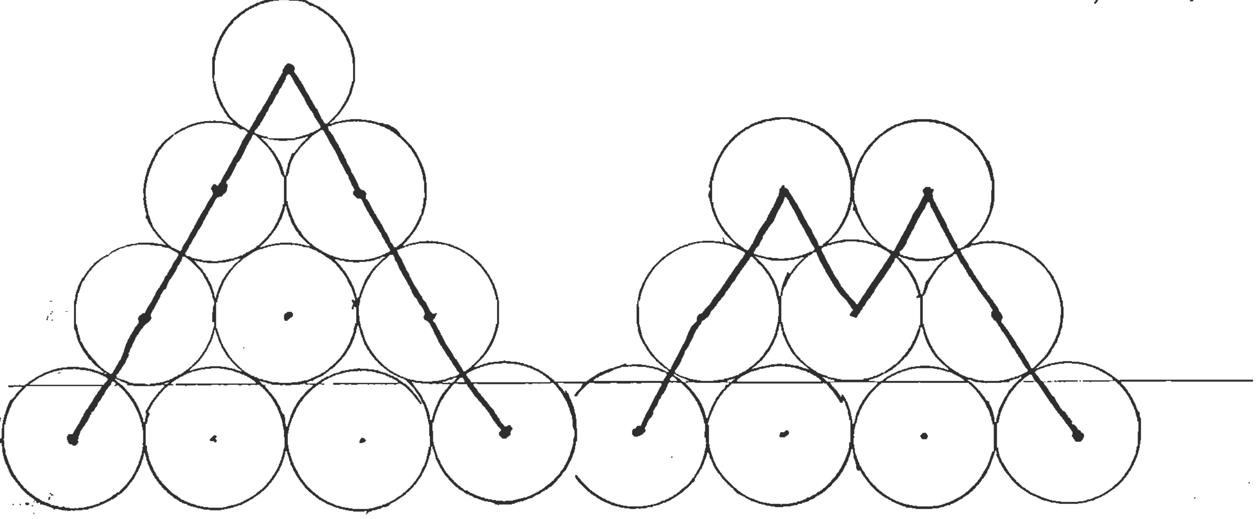
Can you see how to extend all of this to include the Bell or Partition numbers? After all, Catalan is a subset of Bell....all of which flow from some 'fountain'.

Further, could you kindly check my fledgling table of f(n,k) ?

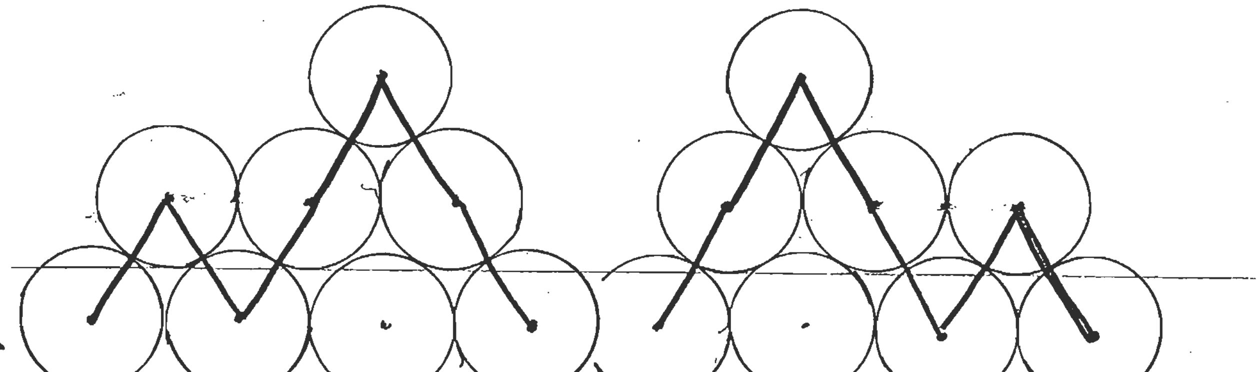
n \ k	0	1	2	3	4	5	6	7	8	9	10	11	12		f(n)
0	1														1
1		1													1
2			1												1
3			1	1											2
4				2	1										3
5				1	3	1									5
6				1	3	4	1								9
7					3	6	5	1							15
8					2	7	10	6	1						26
9					1	~~7~~ 5	14	15	7	1					43
10					1	5	17	25	21	8	1				78
11						5	16	37	41	28	9	1			134?
12						3 5	[illegible]	[illegible]	[illegible]	63	36	10	1		[illegible]
13						2	14	[illegible]	[illegible]	[illegible]	92	45	11	1	[illegible]
14						1	[illegible]	44	[illegible]	167	182	129	55	12	[illegible]
15						1	[illegible]	[illegible]	[illegible]	[illegible]	301	282	175	66	[illegible]
16							7	37	[illegible]	266	[illegible]	512	[illegible]	231	[illegible]
17							5	32	[illegible]	303	574	806	631	655	[illegible]
18							3	28	102	326	704	1143	1413	1292	[illegible]
19							2	22	136				2174	2369	1355
20							1	18	96					3321	[illegible]
21							1	13	85						
h(k)	1	1	2	5	14	42	132	429	1430	etc.					

(((a b c ≡ ((a b)c).d)

((a b c ≡ ((a(b c))d)



BASEMENT



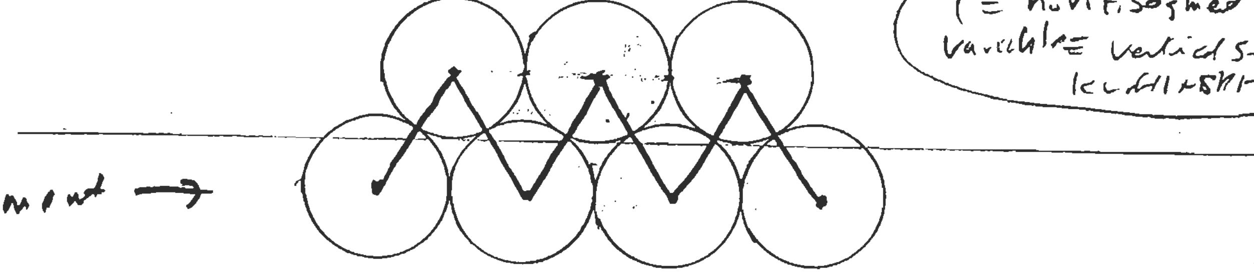
Basement

(a((b c ≡ (a((b c)d))

((a b c ≡ ((a b)(c d))

(a(b(c ≡ (a(b(c d)))

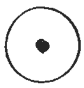
S2 ⟺ S3
(≡ horiz. segment
variable ≡ vertical segment
Kurchinski p. 52



Basement →

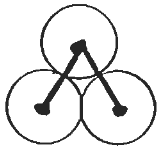
1–1 correspondence between "fountains"
with 3 coins in the bottom floor (4 in the basement)
& S17 (Diagonal Segmented Paths) C124
or S3 (Staircase Arrays) C57, 60, 94, 127, 416, 417

S3 ⟺ S17: Horiz. segment ≡ pos.-sloped segment
Vertical segment ≡ neg.-sloped segment) Kurchinski, p. 63



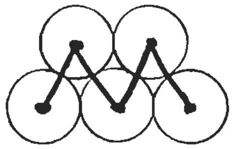
(1,1)

0,0



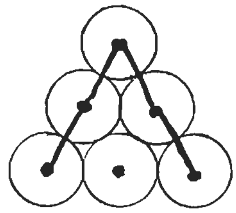
(3,2)

1,1



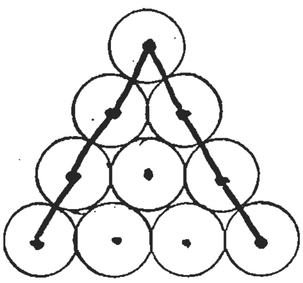
(5,3)

2,2



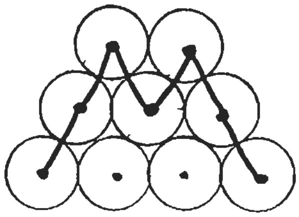
(6,3)

3,2



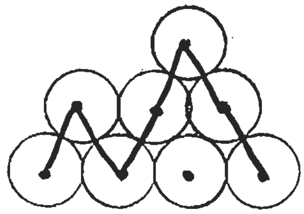
(10, 4)

6,3



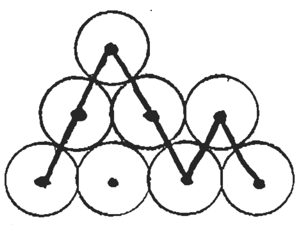
(9, 4)

5,3



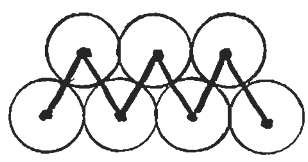
(8, 4)

4,3



(8 , 4)

4,3

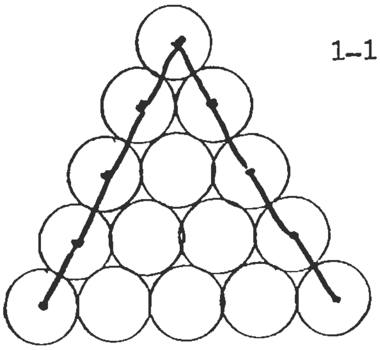


(7, 4)

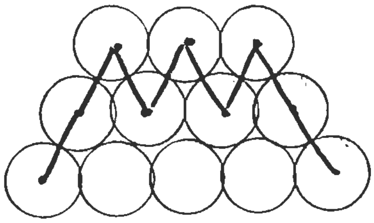
3,3

FOUNTAINS WITH BASEMENTS

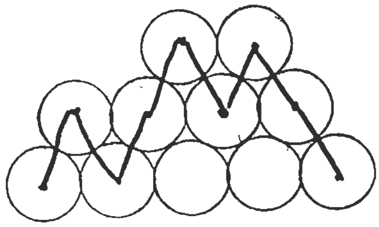
1-1 Correspondences of fountains with basements and Catalan structures called diagonal segmented paths



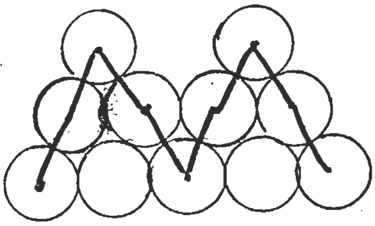
(15, 5)
10,4



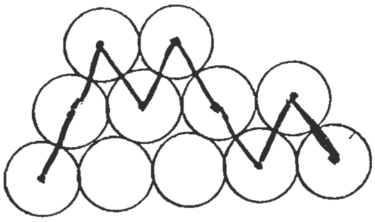
(12, 5)
(7, 4)



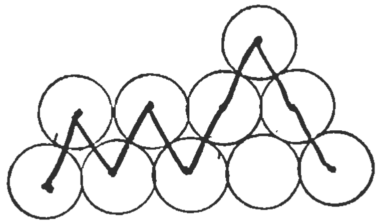
(11, 5)
6,4



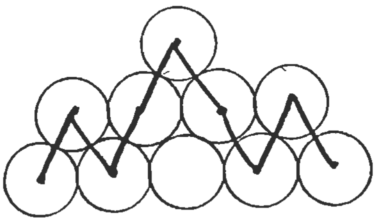
(11, 5)
6,4



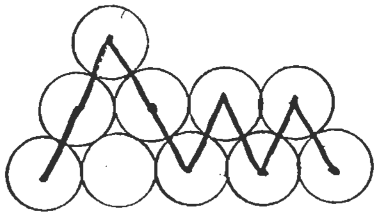
(11, 5)
6,4



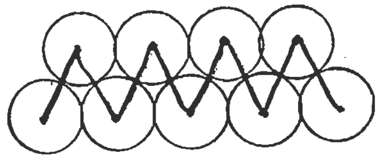
(10, 5)
5,4



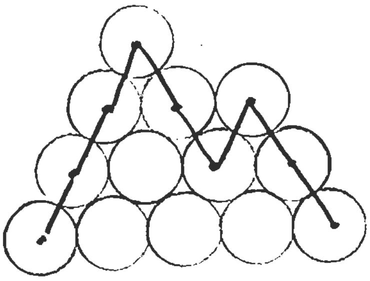
(10, 5) 5,4



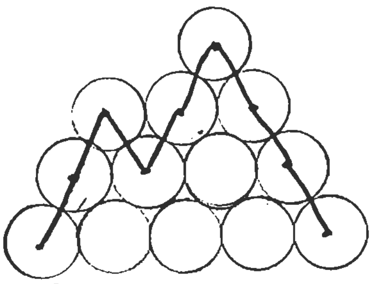
(10, 5) 5,4



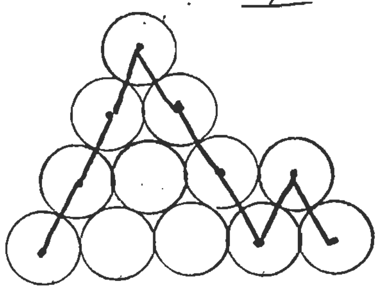
(9, 5) 4,4



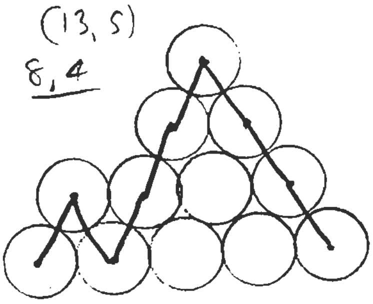
(13, 5)
8,4



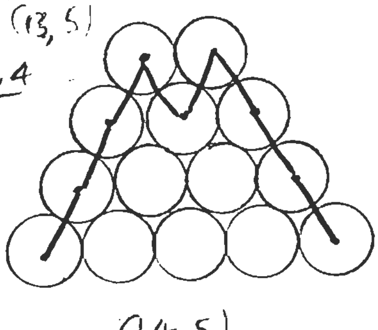
(13, 5)
8,4



(12, 5)
7,4



(12, 5)
7.4



(14, 5)
9.4

f91

4/6/87

~~1614~~

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A5169

87-04-06

A47998

A1524

Professor Henry W. Gould,
Department of Mathematics,
West Virginia University,
Morgantown,
West Virginia 26506.

Dear Henry Gould,

Thank you for your letter of 87-03-29. I may have mentioned that John Conway & I are writing *The Book of Numbers* for the Scientific American Library. The Catalan numbers get a mention, along with all other kinds of numbers. We diagrammatically show the correspondences between

frieze patterns (à la Coxeter & Conway; I don't think Kuchinski has these in his thesis, of which I have purchased a copy - if you've sent another I'll try to persuade a colleague to buy it),

parenthesizations,

triangulated polygons,

two kinds of trees,

"mountain (range)s", and

the binomial coefficient definition.

Your relation with "n coins in a fountain" (by adding a "basement") corresponds to what we called mountain ranges (= walks on a chessboard = ballots coming in = ...).

I have checked and extended your table of $f(n,k)$, the number of "fountains" with n coins, k of which are in the bottom row. There are 3 small errors: $f(9,5) = 7$, not 5; $f(12,5) = 3$, not 5 (these 2 errors cancel as far as the Catalan number $h(5) = \Sigma f(n,5) = 42$ is concerned); $f(11,7) = 35$, not 37. So $f(9) = 45$ and $f(11) = 135$, where $f(n) = \Sigma_k f(n,k)$. Other values, supplied by Jim Propp, are

n =	12	13	14	15	16	17	18
$f(n)$ =	234	406	704	1222	2120	3678	6368

A5169

I have been able to check all but the last two of these (which I get to be 3679 & 6385) by extending your table, including your list of polynomials, which I write in terms of k, rather than n. The array formed by their coefficients may be worth studying:

1							
1	-1						
1	-3	2					
1	-6	17	-18				
1	-10	59	-170	168			
1	-15	145	-765	1954	-1920		
1	-21	295	-2415	11224	-27084	25920	
1	-28	532	-6160	44359	-190372	438948	-418320

Δ A259879

A259880

Apart from the triangular numbers, and the Catalan numbers, none of the sequences found by reading diagonals in your table, or in the above array, seem to be in Sloane's *Handbook*. What *is* included is seq.253, "Auluck's penny partitions", in which *all* rows of the fountain consist of *contiguous* pennies. A1524

I will copy this letter to Propp, Wilf, Odlyzko,& Sloane, and take the liberty of copying your letter to them as well.

Best wishes,

Yours sincerely,

Richard K. Guy

Richard K. Guy.

RKG:jw

encl: table of $f(n,k)$
list of polynomials } back to back

φc: Herb Wilf
Jim Propp
Andy Odlyzko
Neil Sloane

A47998

$f(n,k)$ = # of "fountains" with n coins, k in the bottom row.

n\k	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28			
0	1																															1
1		1																														1
2			1																													1
			1	1																												2
4				2	1																											3
5				1	3	1																										5
6				1	3	4	1																									9
7					3	6	5	1																								15
8					2	7	10	6	1																							26
9					1	7	14	15	7	1																						45
10					1	5	17	25	21	8	1																					78
11						5	16	35	41	28	9	1																				135
12						3	16	40	65	63	36	10	1																			234
13						2	14	43	86	112	92	45	11	1																		406
14						1	11	44	102	167	182	129	55	12	1																	704
15						1	9	40	115	219	301	282	175	66	13	1																1222
16							7	37	118	268	434	512	420	231	78	14	1															2120
17							5	32	118	303	574	806	831	605	298	91	15	1													[illegible]	3678
18							3	28	113	[illegible]	[illegible]	1149	1419	1297	847	377	105	16	1												[illegible]	6368
19							2	22	[illegible]	337	813	[illegible]	2174	2383	1958	1157	469	120	17	1											11081	
20							1	18	[illegible]		[illegible]	[illegible]	[illegible]	3921	3872	2872	1547	575	136	18	1											
21							1	13	[illegible]			2203	4032		6786	6073	4108	2030	696	153	19	1										
22								11	73				[illegible]	[illegible]		11330	9256	5747	2620	833	171	20	1									
23								7	[illegible]								18329	13755	7883	3332	987	190	21	1								
24								5	53									28834	19986	10624	4182	1159	210	22	1							
25								3	42										44242	28460	14093	5187	1350	231	23	1						
26								2	34											66579	39797	18429	6365	1561	253	24	1					
27								1	26												97596		23788	7735	1793	276	25	1				
28								1	20															30344	9317	2047	300	26	1			
29									15																38290	11132	2324	325	27	1		
30									11																	47839	13202	2625	351	28		
31									7																		59225	15550	2951	378		
32									5																			72704	18200	3303		
33									3																				88555	21177		
34									2																					107081		
35									1	40																						
36									1	28																						
										22																						
	1	1	2	5	14	42	132	429	1430	4862	16796	58786																				

$$f(k,k) = 1 \qquad (k \geq 0)$$

$$f(k+1,k) = k-1 \qquad (k \geq 1)$$

$$f(k+2,k) = \binom{k-1}{2} \qquad (k \geq 1)$$

$$f(k+3,k) = \binom{k-1}{3} + (k-2) \qquad (k \geq 2)$$

$$f(k+4,k) = \binom{k-1}{4} + 2\binom{k-2}{2} \qquad (k \geq 2)$$

$$f(k+5,k) = \binom{k-1}{5} + 3\binom{k-2}{3} + (k-3) \qquad (k \geq 3)$$

$$f(k+6,k) = \binom{k-1}{6} + 4\binom{k-2}{4} + 3\binom{k-3}{2} + (k-3) \qquad (k \geq 3)$$

$$f(k+7,k) = \binom{k-1}{7} + 5\binom{k-2}{5} + 6\binom{k-3}{3} + 2\binom{k-3}{2} + (k-4) \qquad (k \geq 4)$$

$$f(k+8,k) = \binom{k-1}{8} + 6\binom{k-2}{6} + 10\binom{k-3}{4} + 3\binom{k-3}{3} + 4\binom{k-4}{2} + 2(k-4) \qquad (k \geq 4)$$

$$f(k+9,k) = \binom{k-1}{9} + 7\binom{k-2}{7} + 15\binom{k-3}{5} + 4\binom{k-3}{4} + 10\binom{k-4}{3} + 6\binom{k-4}{2} + (k-4) + (k-5) \qquad (k \geq 5)$$

$$f(k+10,k) = \binom{k-1}{10} + 8\binom{k-2}{8} + 21\binom{k-3}{6} + 5\binom{k-3}{5} + 20\binom{k-4}{4} + 12\binom{k-4}{3} + 2\binom{k-4}{2} + 5\binom{k-5}{2} + (k-4) + 3(k-5)$$

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87-04-08

Neil J.A. Sloane,
AT&T Bell Laboratories, Room 2C-376
600 Mountain Avenue,
Murray Hill,
New Jersey 07974.

Dear Neil,

To save your having to sort through all this, the main sequences of interest are the diagonals (and totals in the right hand column) of the table for $f(n,k)$, which are the values of the polynomials listed on ~~a separate sheet~~, and perhaps the triangular array on page 2 of my letter to Gould. The table for $f(n,k)$ may contain errors: I don't agree with (my copying of) Jim Propp's totals on the right.

Best wishes,

Yours sincerely,

Richard

RKG:jw

Richard K. Guy.

encl: the back of the $f(n,k)$ table.