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Quasi-Polynomials: A Case Study in Experimental Combinatorics

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Quasi-Polynomials: A Case Study in Experimental Combinatorics

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March 30, 1993

Abstract

We prove that certain interesting combinatorial quantities (typically depending on two parameters) possess compact closed forms when one of the parameters becomes fixed. The examples include necklaces, 0, 1-matrices, bipartite graphs, multigraphs and polygon dissections. A subset of the examples can be treated in a uniform way which resides in the generalization of restricted partitions by means of finite group action. The theory given in this paper bases on empiric results of other authors and serves as a case study of the experimental methods in enumerative combinatorics. A computer algebra package for manipulating quasi-polynomials is shortly introduced in the last section.

Much effort was put into answering the question if a given sequence has a generating function within a specific domain or not. In the present paper

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we are able to prove that certain interesting combinatorial quantities possess *very nice* generating functions that give rise to simple closed forms for these quantities.

In the first section we recall some concepts from enumerative combinatorics. In particular, we define quasi-polynomials as a natural generalization of polynomials and show the form of their generating functions. This is done in a concise style but pointers to basic textbooks are provided on many places. Later on, we explain how the *experimental* methods can be used to conjecture quasi-polynomiality of various sequences. This approach has a twofold effect: *(i)* One can collect together more examples of quasi-polynomials comparing to what the textbooks normally present. *(ii)* One can try proving quasi-polynomiality of other sequences, which is the main objective of this paper.

It turns out that a certain subset of examples can be treated in a uniform way by generalizing the concept of so-called "restricted partitions" by instruments of finite group action. Section 2 is devoted to this topic. However, there still remain sequences that probably cannot be handled this way, and different methods must be taken to prove that they have generating functions of the desired form. Two examples of such situations are in section 3.

In the last section we briefly introduce a computer algebra package for efficient computations in the domain of quasi-polynomials. This tool may be used to convert generating functions of quasi-polynomial sequences into corresponding closed forms. Consequently, it may produce the closed form for any sequence studied in this paper. Only few samples are included in order to keep the modest size of this section. However, it should be noted that long tables of generating functions and/or quasi-polynomial closed forms could be manufactured in a routine way.

1 Definitions

Let $\mathbf{N}$ denote the set of nonnegative integers, let $\mathbf{Z}$ be the ring of integers and let $\mathbf{Q}$ denote the field of rational numbers. For a polynomial $P(x)$, let $[x^s]P(x)$ be the coefficient at x^s in P. This notation naturally generalizes to univariate and bivariate formal power series. Further, we will write $|X|$ for the cardinality of a finite set X. The symmetric group of all permutations of X will be denoted by S_X. The subgroup relation will be denoted by $\leq$.

1.1 Quasi-Polynomials

In this paper we study a natural generalization of polynomials, namely so-called *quasi-polynomials*.

Let $(a_n)_{n\geq 0}$ (or simply (a_n)) be an integral sequence, $a_n \in \mathbf{Z}$ for all $n \in \mathbf{N}$. We say that (a_n) is *quasi-polynomial* if and only if there are integers $p \geq 1$, $n_0 \geq 0$ and polynomials $P_0(n), P_1(n), \ldots, P_{p-1}(n) \in \mathbf{Q}[n]$ such that for each $n \geq n_0$

$$a_n = P_k(n) \qquad \text{where } k = n \bmod p. \tag{1}$$

There are two differences of this definition from the usual one ([Sta], p. 210): In our setting it suffices that the polynomials P_k determine the sequence's values only from some point onwards. Later the reader may recognize why this comes useful: Consider, for example, the sequences of section 3.1. The second difference is that we define the quasi-polynomiality only for sequences with integral entries. The reason for this limitation is that in this paper we deal exclusively with sequences that count combinatorial objects.

The number p will be called the *quasi-period* of the sequence. The polynomials $P_0, \ldots, P_{p-1}$ will be called the *class polynomials* of the sequence because they determine its entries on residue classes of the index.

In the sequel, we will be using the word quasi-polynomial both as a noun and as an adjective. Let D be the maximum degree amongst polynomials P_k and suppose $P_k = \sum_{l=0}^{D} c_{k,l} n^l$. Instead of (1) one usually writes

$$a_n = [c_{0,D}, c_{1,D}, \ldots, c_{p-1,D}]n^D + \ldots + [c_{0,0}, c_{1,0}, \ldots, c_{p-1,0}].$$

Further abbreviation is achieved by writing $[c_0, \ldots, c_t]$ instead of $[c_0, \ldots, c_t, c_0, \ldots, c_t, \ldots, c_0, \ldots, c_t]$ and c instead of $[c, c, \ldots, c]$. We will use such notation in section 4.2.

Let $\lfloor x \rfloor$ denote the floor function. The reader may verify that also each equation of the following form

$$a_n = \lfloor P(n) \rfloor, \qquad P(n) \in \mathbf{Q}[n] \tag{2}$$

defines a quasi-polynomial sequence. Unfortunately, the converse does not hold in general.

We start with two warm-up examples: The sequence $(2,1,2,1,1,4,1,1,6,1,\ldots)$ of terms in the continued fraction expansion of the well-known number $e = 2 + 1/(1 + 1/(2 + 1/(1 + 1/(1 + 1/(4 + 1/\ldots)))))$ is quasi-polynomial with $p = 3$, $n_0 = 1$ and $P_1 = 1$, $P_2 = 2/3(n+1)$, $P_0 = 1$.

The (sorted) sequence of numbers k such that $\lfloor\sqrt{k}\rfloor$ divides k (AMM problem E 2491) is the sequence of numbers of the form m^2, m^2+m, m^2+2m with $m \geq 1$. Indexing these items by $0, 1, \ldots$ we get a quasi-polynomial sequence with $p = 3$, $n_0 = 0$, $P_0 = (n/3+1)^2$, $P_1 = ((n+2)/3)^2 + (n+2)/3$ and $P_2 = ((n+1)/3)^2 + 2(n+1)/3$.

For an introduction on quasi-polynomials we recommend [Ehr] or [Sta].

1.2 Generating Functions

It follows from the theory of linear recurrence sequences that the integral sequence (a_n) is quasi-polynomial (in the sense of the preceding section) if and only if the following two conditions hold.

(QP1) The generating function of (a_n) is rational, $\sum_{n\geq 0} a_n x^n = P(x)/Q(x)$ with $P(x), Q(x) \in \mathbf{Z}[x]$, $\gcd(P(x), Q(x)) = 1$.

(QP2) All roots of the polynomial $Q(x)$ are roots of unity (not necessarily with same primitive periods). This can be rephrased by saying that all irreducible factors of $Q(x)$ are *cyclotomic polynomials*, see [Lan], p. 316.

The proof of this statement relies on Proposition 4.4.1 in [Sta] which in our setting reads as follows: "The integral sequence (a_n) is quasi-polynomial with $n_0 = 0$ if and only if (QP1) and (QP2) hold and, moreover, $\deg P < \deg Q$." In the general case (no restriction on degrees of P and Q) we can always find polynomials $R, S \in \mathbf{Z}[x]$ such that $P/Q = R + S/Q$ with $\deg S < \deg Q$. Let $r := \deg R$ and $S/Q = \sum_{n\geq 0} a'_n x^n$. Then $a_n = [x^n]R(x) + a'_n$ for $n \leq r$ while $a_n = a'_n$ for $n \geq r+1$. Hence, (a_n) is quasi-polynomial (in our sense) with $n_0 = r+1$.

We will say that a generating function has the *(QP)-form* if it fulfills both conditions (QP1) and (QP2). Quasi-polynomials are closed under addition, convolution and indefinite summation. They are also closed under multiplication, as may be seen by a direct argument (without consideration of generating functions).

1.3 Closed Forms

Mathematicians like to see things in "closed forms". For example, the following three equations (equation systems) define the same sequence (a_n):

$$\begin{aligned} a_n &= (-1)^m (3m)!/m!^3 \quad \text{for } n \text{ even, } n = 2m \\ a_n &= 0 \quad \text{for } n \text{ odd} \end{aligned} \tag{3}$$

$$a_n = \sum_{k=0}^{n} \binom{n}{k}^3 (-1)^k \tag{4}$$

$$(n+2)^2 a_{n+2} + 3(3n+4)(3n+2)a_n = 0, \quad a_0 = 1, \ a_1 = 0 \tag{5}$$

For many reasons (computation complexity, getting more mathematical insight, asymptotic analysis, or even aesthetic reasons etc.) we prefer the definition (3) to the other two cases. No fixed agreement used to be on the set of operations allowed to appear in a "closed form expression". Typically, closed forms may include addition, multiplication, exponentiation and factorials. In the present paper we adopt quasi-polynomials as "closed forms", since they very well meet all demands listed at the beginning of the paragraph.

1.4 Experimental Mathematics

Experiment has always been, and increasingly is, an important method of mathematical discovery. The main questions to be answered are *how one uses the computer:* To build intuition? To generate hypotheses? To discover nontrivial examples and counter-examples?

1.4.1 Method of Our Paper

One of the frequent approaches in experimental mathematics is to build large lists of examples and search for a specific pattern to occur.

The first subtask may be of great interest on its own, and resulting catalogs or programs are often distributed amongst the mathematical community. Recently, Bergeron and Plouffe [BePl] invented a Maple program which guesses the generating function of a series from its initial terms. Using this program and a similar one (by Salvy and Zimmermann), Plouffe has computed an amazingly large catalog [Plo] of more than one thousand conjectured generating functions. His work will be incorporated into the second edition of the famous *Handbook of Integer Sequences* by N.J. Sloane.

In the present paper, we use Plouffe's list as the catalog of guessed generating functions, and the pattern to be sought is the specific form of generating function quoted in section 1.2. Interestingly enough, more than 60 (non-polynomial) entries of [Plo] match this pattern. This fact enabled us

(i) to collect more examples of quasi-polynomials than is usually listed in the textbooks, i.e. to organize the knowledge

(ii) to discover that certain sequences are quasi-polynomial, a fact that has not been noticed by authors introducing these sequences.

It is a pleasure for us to note that in the course of our work we did not meet any wrong guess, i.e. that each sequence which we picked for a detailed study turned out to be an instance for task *(i)* or *(ii)*. Moreover, some of these items were generalized by showing not only the quasi-polynomiality of the particular sequence appearing in [Plo] but also the quasi-polynomiality of any other sequence of that kind. This was the case in sections 2.2, 2.3 and 3.1.

Since we lacked an effective tool for computation with quasi-polynomials, a bunch of Maple procedures was written which handles all known examples in a reasonable time, cf. section 4.

1.5 Examples of Quasi-Polynomials

The abovementioned process allowed us to gather more examples of quasi-polynomials than is usual in combinatorial textbooks. We merely quote the results, references may be found by tracking down the appropriate entries in [Plo].

All of the following quantities are quasi-polynomials:

The number of *partitions* of a given positive integer n into parts of possible sizes $s_1, \ldots, s_k$ is the *denumerant* $D(n; s_1, \ldots, s_k)$, being quasi-polynomial in n. (See [Com], chapter 2.6 for details.) It is common to use the notation $p_k(n)$ for $D(n; 1, 2, \ldots, k)$ and we will do so in section 4. Computing with denumerants covers the famous money changing problem, and many others.

The number of *distinct integer-sized triangles* with a given perimeter n is $\lfloor (n^2+6)/12 \rfloor - \lfloor n/4 \rfloor \lfloor (n+2)/4 \rfloor$. (For much more examples on counting lattice points in multi-dimensional polyhedra, see [Ehr].)

The maximal number of *triangles that may be packed in the clique on m* vertices (using each edge of the clique at most t times) is quasi-polynomial in m.

The postage stamp problem: The maximal integer $n = n(h, 2)$ such that all integer postage values from 1 to n can be made up by at most h stamps (with only 2 stamp denominations allowed) is $n(h,2) = \lfloor (h^2 + 6h + 1)/4 \rfloor$.

The *crossing number* of the complete graph on n nodes is conjectured to be $1/4 \lfloor n/2 \rfloor \lfloor (n-1)/2 \rfloor \lfloor (n-2)/2 \rfloor \lfloor (n-3)/2 \rfloor$.

The number of non-isomorphic graphs on n vertices having *exactly 2 cliques* is $\lfloor n^2/4 \rfloor$. The number of non-isomorphic graphs on n vertices having *exactly 3 cliques* is $\lfloor (n+3)(6n^4 - 18n^3 + 34n^2 - 42n + 105 + 45(-1)^n)/1440 \rfloor$.

We are coming to the main part of the paper. In the following two sections we will show that several combinatorial quantities are quasi-polynomial.

2 Restricted Partitions

As has been mentioned in the beginning, some examples can be gathered together by viewing them as restricted partitions under an action of a permutation group.

Let G be a finite group acting on a finite set X. This yields a permutation representation $\bar{G}$ of G, where $\bar{G} \leq S_X$. Consider the induced action $\bar{G} \times \mathbf{N}^X \longrightarrow \mathbf{N}^X$ defined by $(\pi f)(x) := f(\pi^{-1}(x))$ for $\pi \in \bar{G}$, $f \in \mathbf{N}^X$.

For an $f \in \mathbf{N}^X$, let

$$c_f := \sum_{x \in X} f(x)$$

be the *content* of f and let

$$\bar{G}(f) := \{\pi f \mid \pi \in \bar{G}\}$$

be the orbit of f under $\bar{G}$. We say that $\bar{G}(f)$ is a *$\bar{G}$-partition of the number* c_f. The number partition defined this way should not be confused with the set partition of X (or Y^X) into G-orbits in the action of G on X (or Y^X).

Informally, given a natural number c, a permutation group $\bar{G}$ of degree l arising from the action of a group G on a set X with cardinality l, then a $\bar{G}$-partition of the number c is a set T of l-tuples over $\mathbf{N}$ where each tuple sums up to c, and with each $t \in T$, T contains all l-tuples obtained by permuting entries of t by permutations from $\bar{G}$. (We think of the l-tuples as indexed by the elements of X.) Since the length of tuples must be equal to the cardinality of X, we speak about restricted partitions. We note that zero parts are allowed as well, a fact that fits the combinatorial applications and ensures that the set of $\bar{G}$-partitions of c is always nonempty.

From the *Pólya's Theorem* ([Ker], p. 71) it follows that the number of distinct $\bar{G}$-partitions of c is the coefficient at x^c in

$$C\left(G, X \,\middle|\, \frac{1}{1-x}, \frac{1}{1-x^2}, \ldots, \frac{1}{1-x^{|X|}}\right). \tag{6}$$

The expression (6) denotes the Pólya's substitution of $1 + x + x^2 + \ldots = \frac{1}{1-x}$ into $C(G, X \mid s_1, s_2, \ldots, s_{|X|})$, the cycle index of G's action on X. The cycle index is a multivariate polynomial in all its variables $s_1, s_2, \ldots, s_{|X|}$.

Hence. (6) meets the conditions (QP1) and (QP2). We arrive at the following theorem:

Theorem 1. For each permutation group $\bar{G}$, the number $P_{\bar{G}}(c)$ of $\bar{G}$-partitions of c is quasi-polynomial in c.

By taking suitable groups $\bar{G}$, we can prove quasi-polynomiality of various interesting combinatorial quantities. All of the following examples were treated elsewhere but (with exception of section 2.1) only occasionally the quasi-polynomial closed forms for some special cases were recognized. Here we aim at a unifying treatment of all situations.

2.1 Ordinary Partitions

Let $\bar{G}$ be the full symmetric group of degree n acting on $\underline{n}$, i.e. $\bar{G} = S_{\underline{n}}$. This is the context in which partitions are usually studied, and the number of these "ordinary" partitions into n parts (with zero parts allowed) is well-known to have the generating function $1/((1-x)(1-x^2)\dots(1-x^n))$. On the other hand, this series also results from the substitution (6) into the cycle index of $S_{\underline{n}}$ which is derived in [Ker], p. 72. Comparing both, we obtain

$$\sum_{a\vdash n}\prod_k \frac{1}{a_k!}\left(\frac{1}{k(1-x^k)}\right)^{a_k} = \prod_{k=1}^{n}\frac{1}{1-x^k} \tag{7}$$

where the sum extends over all $a = (a_1, a_2, \dots)$ such that $a_1 \cdot 1 + a_2 \cdot 2 + \dots = n$. The identity (7), due to MacMahon ([MacMah], Vol. II, p. 62), is normally used as the first step in a partial fraction decomposition of its right-hand side. This identity is typically derived by means of symmetric functions, even in the textbooks which introduce Pólya's counting theory ([Rio], pp. 118–119).

2.2 Necklaces and Bracelets

Taking $\bar{G} = C_{\underline{n}}$ to be the cyclic group acting on $\underline{n}$, the $C_{\underline{n}}$-partitions of a number m are models for *two-colored necklaces* (black and white, say) with the fixed number n of black beads and a varying number m of white beads. The bijection between these two sets is as follows: For a given $C_{\underline{n}}$-partition of the number m with a representative $(m_1, m_2, \dots, m_n)$ we construct a necklace with n black beads and n blocks of white beads (of sizes $m_1, m_2, \ldots, m_n$, respectively) by inserting one white block between each consecutive pair of black beads, keeping the cyclic order of m_i's unchanged. Thus the black beads provide the "marks" between consecutive $C_{\underline{n}}$-parts.

Let $N_n(m)$ denote the number of necklaces with n black and m white beads. Theorem 1 implies that $N_n(m)$ is quasi-polynomial in m. Similarly, when the dihedral group $D_{\underline{n}}$ is acting, Theorem 1 proves the quasi-polynomiality of $B_n(m)$, the number of *two-colored bracelets* with a fixed number n of black beads. The latter values are of interest in diverse applications, see [HoPe] or [Eth].

Taking $n = 4$, the substitution into the cycle index of D_4 yields the generating function for bracelets with 4 black and m white beads:

$$\begin{aligned}\sum_{m=0}^{\infty} B_4(m)x^m &= \frac{1}{8}\frac{1}{(1-x)^4} + \frac{3}{8}\frac{1}{(1-x^2)^2} + \frac{1}{4}\frac{1}{1-x^4} + \frac{1}{4}\frac{1}{(1-x)^2(1-x^2)} \\ &= \frac{x^2-x+1}{(1-x)^2(1-x^2)(1-x^4)} = 1 + x + 3x^2 + 4x^3 + 8x^4 + \ldots\end{aligned}$$

This is the sequence **A5232** in [Plo].

In the case of $C_{\underline{n}}$- and $D_{\underline{n}}$-partitions, the class polynomials can be expressed explicitly by binomial sums, providing this way an *alternative* proof of the quasi-polynomiality of the quantities under examination. This approach is less elegant (comparing to the argument using generating functions). Pólya's Theorem is now applied for the cyclic and dihedral group of order $m+n$, respectively.

Next we show these computations for the case of $C_{\underline{n}}$-partitions. The number of necklaces with n black and m white beads is the coefficient at x^m in

$$\frac{1}{m+n}\sum_{d|(m+n)} \phi(d)(x^d+1)^{(m+n)/d}$$

which by the binomial theorem turns out to be

$$\frac{1}{m+n}\sum_{d|\gcd(m,n)} \phi(d)\binom{(m+n)/d}{n/d}. \tag{8}$$

The basic property of greatest common divisor

$$\gcd(m,n) = \gcd(m \bmod n, n)$$

allows us to change the description of summation range in a convenient way: It is now clear that the summation range depends just on the value of m mod n, i.e. on the residue class of m, and that (8) is polynomial in m on each such residue class. This means that $N_n(m)$ is quasi-polynomial with quasi-period n.

The case when m and n are relatively prime is particularly nice. Think of a sequence T of m white and n black beads and let σ denote the cyclic shift. The smallest p such that $\sigma^p(T) = T$ will be called the *primitive period* of T. Clearly, T is composed of $(m+n)/p =: b$ identical blocks B. Let B contain m' white and n' black beads. Then $bm' = m$, $bn' = n$ and since we assume m and n to be relatively prime, $b = 1$, i.e. $p = m + n$. This means that for any such T, the (rotationally equivalent) sequences $\sigma(T), \sigma^2(T), \ldots, \sigma^{m+n}(T)$ are pairwise different. Since there are $\binom{m+n}{n}$ sequences in total and they form equally sized orbits of cardinality $m + n$ each, it follows as a special case of (8) that for relatively prime numbers m and n, the number of necklaces with m white and n black beads is

$$\frac{1}{m+n}\binom{m+n}{n}.$$

The probability that two randomly picked integers are relatively prime is $6/\pi^2$. ([Knu], p. 324.) Hence, in the case of two colors about 60% of the necklace problem is covered by an easy formula.

2.3 0, 1-Matrices and Bipartite Graphs

Many problems in the switching theory can be recast as problems involving 0, 1-matrices. In [Har73], the author develops methods for finding number of equivalence classes of 0, 1-matrices with m rows and n columns under two definitions of equivalence:

(E1) equivalent matrices are obtained by row and column permutations;

(E2) equivalent matrices are obtained by row permutations together with column permutations and/or complementations.

For the equivalence (E1), the number $s_{m,n}$ of classes of $m \times n$ matrices may be determined as follows: Consider the action of $S_{\underline{n}}$ on matrix rows.

This yields a permutation group S'_n of degree 2^n, acting on $\{0,1\}^n$ and being a permutation representation of $S_{\underline{n}}$. Then each E1-equivalence class of $m \times n$ binary matrices is an S'_n-partition of the number m. The formula for cycle indices of S'_n appears in [Har65]. For example, $C(S'_2, \{0,1\}^2) = 1/2(s_1^4 + s_1^2 s_2)$ and so the generating function for E1-classes of $m \times 2$ matrices is

$$\sum_{m \geq 0} s_{m,2} x^m = \frac{1}{2}\left((\frac{1}{1-x})^4 + (\frac{1}{1-x})^2 \frac{1}{1-x^2} \right) \tag{9}$$
$$= \frac{1}{(1-x)^3(1-x^2)} = 1 + 3x + 7x^2 + 13x^3 + \ldots$$

This is the sequence **A2623** in [Plo].

In order to find $t_{m,n}$, the number of classes under (E2), [Har73] proceeds in a similar way, arriving at the group S''_n of degree 2^n which is the permutation representation of the exponentiation group $S_{\{0,1\}}{}^{S_{\underline{n}}}$.

Taking the group $\bar{G}$ in Theorem 1 to be S'_n and S''_n, respectively, we conclude that for a fixed number of columns n, the numbers $s_{m,n}$ and $t_{m,n}$ of E1,E2-equivalence classes of $m \times n$ binary matrices are quasi-polynomial in m. It is worth mentioning that for $m \neq n$, $s_{m,n}$ gives also the number of bipartite graphs with vertex set partition (m, n). The bijection is achieved by viewing 0,1-matrices of the shape $m \times n$ as a special kind of incidence matrices for bipartite graphs with m and n vertices. Thus (9) tells us that we have thirteen non-isomorphic bipartite graphs with vertex partition (3,2). They are drawn in [HaPa], p. 95 as an illustration for their enumeration via the cycle index of $S_{\underline{m}} \times S_{\underline{n}}$. The case $m = n$ needs different treatment, see [HaPa], pp. 97–99.

2.4 Multigraphs

The famous pair group (or graph group) $S_n^{(2)}$ is the permutation representation of the symmetric group $S_{\underline{n}}$ acting on pairs from $\underline{n}$. Considering these pairs as unordered (ordered), we can enumerate unoriented (oriented) graphs on n points. In this instance, Theorem 1 implies that the number of (unoriented, oriented) multigraphs on n points with e edges is quasi-polynomial in e. Pólya's substitution of $\frac{1}{1-x}$ into the cycle index of $S_n^{(2)}$ is (without any further comments) mentioned in [HaPa], p. 88.

3 Other Quasi-Polynomial Quantities

There are some examples which cannot be adjusted into the framework of generalized restricted partitions. Instead, we must use different methods. In the first example. we can compute the generating functions directly. In the latter example. we prove the quasi-polynomiality by other (albeit elementary) arguments. We determine the quasi-period and the degree of the resulting quasi-polynomial and then the actual computation of the closed form is done by interpolation from the initial values. (The interpolation idea is used e.g. in [Com], p. 114. In the general case. this approach may be cumbersome since it requires a lot of non-trivial data. We tried to avoid it in the present paper by using generating functions whenever it was possible.)

3.1 Polygon Dissections

By a *polygon dissection* we mean each subdivision of the interior of the convex n-gon into smaller polygons by means of nonintersecting diagonals. The enumeration of dissections was treated by many authors. In the special case when all parts happen to be triangles, the number of *triangulations* of the n-gon is well-known to be the Catalan number C_{n-2} ([GKP], exercise 7.22). Up to now. no symmetries are considered so that. for example, the two possible triangulations of the square are regarded as distinct.

Restricting the attention to regular n-gons. one can make the problem more natural by viewing two dissections as identical if one can be obtained from the other by rotating and/or reflecting the n-gon. Depending on whether the reflection is or is not allowed as a possible symmetry, we come to two different problems. and the counted objects will be called "dissections with reflection" and "dissections without reflection", respectively.

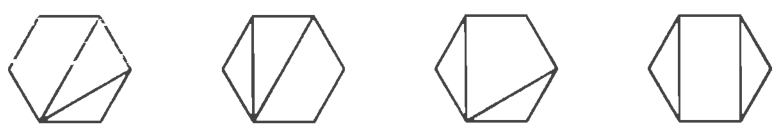
In the general setting when the regular s-gon is to be divided into r polygons. the symmetry classes of dissections were enumerated by R.C. Read in [Rea]. We will extend the results of this article a little bit by showing that for fixed r. all arising quantities happen to be quasi-polynomials in the variable s. First of all. we introduce Read's notation and illustrate it on a simple example. Since the general idea of [Rea] is to turn the dissection problem into a cell-growth problem. we will use the term *cells* for the polygons arising in the dissection. With each pair (r, s) we associate five numbers counting different kinds of dividing up the s-gon into r polygons:

$V_{r,s}$	number of dissections without reflection rooted at an edge
$F_{r,s}$	number of dissections without reflection rooted at a cell
$H_{r,s}$	number of unrooted dissections without reflection
$f_{r,s}$	number of dissections with reflection rooted at a cell
$h_{r,s}$	number of unrooted dissections with reflection

Obviously, these values are nonzero exactly if $s \geq r+2$. Additionally, we set $V_{0,1} := 1$.

To enlighten the definition of these sequences, let us have a look at their values for $(r,s) = (3,6)$. The following picture will help us:

A B C D



Since each dissection of the hexagon into 3 parts is rotationally equivalent to one of A, B, C, D, we have $H_{3,6} = 4$. Under reflection, A and B fall into one class so that $h_{3,6} = 3$. If the reflection is not allowed, we may root each of A, B and C at 3 cells and D at 2 cells, which together gives $F_{3,6} = 11$. Allowing the reflection, we must give up one rooting of C and all rootings of B, hence $f_{3,6} = 7$. Finally, under rotational symmetry we may root A, B, C at any outer edge and D at half of its outer edges which implies $V_{3,6} = 21$.

Since we deal with double-indexed sequences, their generating functions are bivariate:

$$V(x,y) := \sum_r \sum_s V_{r,s} x^r y^s$$

and similarly for F, H, f and h. We recall the formulas derived by Read, for detailed proofs see [Rea]:

$$V_{r,s} = \frac{1}{r}\binom{s-2}{r-1}\binom{r+s-1}{s} \qquad (r \geq 1)$$

$$\begin{aligned}
F(x,y) &= x\sum_{k\geq 3} C(C_k|V(x,y)) & (10)\\
H(x,y) &= F(x,y) - \frac{1}{2}\left(U^2(x,y) - U(x^2,y^2)\right) & (11)\\
f(x,y) &= \frac{1}{2}F(x,y) + \frac{1}{4}\left(2T(x,y) + V(x^2,y^2) + T^2(x,y)\right) R(x,y) & (12)\\
h(x,y) &= f(x,y) - \frac{1}{4}\left(U^2(x,y) - 2U(x^2,y^2) + W^2(x,y)\right) & (13)
\end{aligned}$$

where $U(x,y) = V(x,y) - y$, $xR(x,y) = (1+x^2)V(x^2,y^2) - y^2$ and $W(x,y) = T(x,y) - y$ with

$$T(x,y) = \frac{y + R(x,y)}{1 - R(x,y)}. \qquad (14)$$

We show that for each fixed r, all five enumerating sequences V, F, H, f, h are quasi-polynomial in s. To this end, for each r we introduce five univariate generating functions V_r, F_r, H_r, f_r and h_r in the variable y:

$$V(x,y) =: \sum_r V_r(y)x^r$$

and analogously for the other sequences. Clearly, our goal is to show that for any r, each of these five univariate functions meet the conditions (QP1), (QP2), see section 1.2. This statement is trivial for the functions V_r because for $r \geq 1$, $V_{r,s}$ is polynomial in s of degree $2r-2$, i.e.

$$V_r(y) = \frac{P_r(y)}{(1-y)^{2r-1}} \qquad (r \geq 1) \qquad (15)$$

for some polynomial P_r ([Sta], p. 202), while for $r = 0$ we have

$$V_0(y) = y. \qquad (16)$$

The simple form of V_0 will consequently play a notable role in our computations.

The series F_r is obtained by a rearrangement of (10):

$$\begin{aligned}
F_r(y) &= [x^r]\, F(x,y) = [x^{r-1}] \sum_{k\geq 3}\sum_{d|k} \frac{\phi(d)}{k} V^{k/d}(x^d,y^d)\\
&= [x^{r-1}] \sum_{d'|(r-1)} \sum_{md'\geq 3} \frac{\phi(d')}{md'} V^m(x^{d'},y^{d'}).
\end{aligned}$$

By multinomial theorem and another rearrangement we find

$$F_r(y) = \sum_{d'|(r-1)} \sum_{(a_1,a_2,\ldots)\vdash\frac{r-1}{d'}} \sum_{m=\left\lceil\frac{max(3,\sum a_i)}{d'}\right\rceil}^{\infty}$$
$$\frac{1}{m}\frac{m!}{a_1!a_2!\ldots(m-\sum a_i)!} V_0^{m-\sum a_i}(y^{d'})\cdot V_1^{a_1}(y^{d'})\cdot V_2^{a_2}(y^{d'})\cdot\ldots$$

where the second sums extends over all $(a_1, a_2, \ldots)$ such that $a_1\cdot 1+a_2\cdot 2+\ldots = (r-1)/d'$. $\lceil x\rceil$ is the ceiling of x and $\sum a_i = a_1 + a_2 + \ldots$. The multinomial coefficient multiplied by $1/m$ is a polynomial in m of degree $(\sum a_i) - 1$, let us call it $Q(m)$. Due to (16), the innermost sum is

$$\sum_{m=m_0}^{\infty} Q(m)\cdot y^{d'm-(d'\sum a_i)}\cdot V_1^{a_1}(y^{d'})\cdot V_2^{a_2}(y^{d'})\cdot\ldots$$

where the summation bound was replaced by a symbol. The factors independent of m may be put apart, which gives

$$C(y)\cdot\sum_{m=m_0}^{\infty} Q(m)y^{d'm} \tag{17}$$

with $C(y)$ in the (QP)-form because of (15). Hence, the expression (17) meets (QP)-form and finally the generating function F_r, being a finite sum of expressions of the type (17), must also fit (QP)-form. Hence, we have proved that for any fixed r, the sequence $F_{r,s}$ is quasi-polynomial in s. Taking $r = 3$ as an example, we compute

$$\begin{aligned} F_3 &= \sum_{m\geq 3} y^{m-1}V_2(y) + \sum_{m\geq 3}\frac{m-1}{2}y^{m-2}V_1^2(y) + \sum_{m\geq 2}\frac{1}{2}(y^2)^{m-1}V_1(y^2) \\ &= V_2(y)\cdot\frac{y^2}{1-y} + V_1^2(y)\cdot\frac{y(2-y)}{2(y-1)^2} + V_1(y^2)\cdot\frac{y^2}{2(1-y^2)} \\ &= -\frac{(y^3+y^2-5y-3)y^5}{(1-y^2)^2(1-y)^2} = 3y^5+11y^6+24y^7+46y^8+75y^9+\ldots \end{aligned}$$

(cf. [Rea], Table 2.)

The functions H_r give little trouble since from (11) we directly obtain

$$H_r = F_r - \frac{1}{2}\left(\sum_{i=1}^{r-1} V_i(y)V_{r-i}(y) - \left\{V_{r/2}(y^2)\right\}\right)$$

where the term in curly brackets is (or is not) present depending on if r is even (odd). In both cases, the yet known forms of F_r and V_i's imply that H_r meet (QP)-form.

Next we must deal with the functions $f_r(y)$. (They should not be mixed up with the bivariate functions $f_k(x,y)$ in [Rea].) We will show that also the functions $R_r(y)$ and $T_r(y)$ happen to be in the (QP)-form which obviously will settle the problem for f_r, cf. (12).

We observe that $R_r = 0$ for r even and $R_r = V_{(r+1)/2}(y^2) + V_{(r-1)/2}(y^2)$ for r odd which again meets (QP)-form. The treatment of T_r needs a bit of rewriting of (14):

$$T_r(y) = y \cdot R'_r(y) + \sum_{k=0}^{r} R_k R'_{r-k}$$

where

$$\begin{aligned} R'_l(y) &:= [x^l] \sum_{i=0}^{\infty} R^i(x,y) \stackrel{(*)}{=} [x^l] \sum_{i=0}^{l} R^i(x,y) \\ &= \sum_{(a_1,a_2,\ldots)\vdash l} \frac{(a_1+a_2+\ldots)!}{a_1!a_2!\ldots} R_1^{a_1} R_2^{a_2} \ldots . \end{aligned}$$

The step $^{(*)}$ is possible due to $R_0 = 0$. We conclude that each R'_l is in (QP)-form. Thus, also T_r fulfills (QP)-form. Now, by similar means, f_r can be spelled out as a finite algebraic expression in F_r, $V_{r/2}$, T_i's and R_i's. Hence, f_r meets (QP)-form.

The last remaining series are h_r. Here, the argumentation is quite similar as in the case of their "big brothers" H_r. This concludes the proof that for any fixed r, each of the series V_r, F_r, H_r, f_r and h_r has the (QP)-form.

Finally we note that the reasoning on univariate generating functions not only proves the existence of certain closed forms for the studied sequences but also provides a practical method for computing terms of those sequences. As pointed in [Rea], univariate series are much easier to handle than the bivariate ones, and with the use of modern computer algebra systems it is quite easy to obtain the functions F_r, H_r, f_r and h_r for modest values of r along the preceding lines. Doing so, we had the pleasure to verify the huge amount of data contained in [Rea], Tables 2 to 5 with the exception of the three positions $f_{3,16}$, $f_{6,16}$ and $h_{8,15}$ that according to our results should hold the values 372, 624355 and 384035, respectively. For each of these entries, the value given in [Rea] and our value differ only in a single digit so the difference clearly should be accounted to a transcription mistake rather than to a mathematical error.

3.2 3×3 Matrices With Constant Row and Column Sum

This example shows that multiple sums with some simple functions (maximum,ceiling) as summation bounds may naturally lead to quasi-polynomials.

Let $P = [p_{i,j}]$ be a 3×3 matrix with $p_{ij} \in \{0, 1, 2, \ldots, \mu - 1\}$ such that $\sum_{i=1}^{3} p_{ik} = \sum_{j=1}^{3} p_{kj} = \mu$, for $k = 1, 2, 3$. Any matrix which can be obtained from P by permuting rows and columns of P or by taking the transpose of P is said to be equivalent to P. In [Mor], the number $n(\mu)$ of such inequivalent matrices for any $\mu \geq 2$, is given by a somewhat horrible looking triple sum

$$n(\mu) = \sum_{i=1}^{\mu-l} \sum_{m=\max\{0,2i-\mu\}}^{\lfloor i/2 \rfloor} \sum_{r=0}^{\lfloor i/2 \rfloor - m} c_{imr} \tag{18}$$

where $\mu = 3l$ or $\mu = 3l-1$ or $\mu = 3l-2$ for some $l \in \mathbf{N}$ and $c_{imr} = \mu - 2i + m + 1$ if $m + r \neq i/2$, and $c_{imr} = \lfloor 1/2(\mu - 2i + m + 2) \rfloor$ if $m + r = i/2$.

This is the sequence **A5045** in [Plo], and its conjectured generating function matches the form required in section 1.2 with denominator containing roots of unity with primitive periods 1, 2, 3 and 4. Thus we suspect $n(\mu)$ to be quasi-polynomial with quasi-period equal to 12, and we will prove this hypothesis now:

To make things easier for a short while, consider μ running over the congruence classes modulo 24 (instead of 12). The *Ansatz* $\mu = 24a + b$, with $a \in \mathbf{N}$ and a *fixed* $b \in \{0, \ldots, 23\}$, will make it possible to split (18) into several triple sums whose bounds are free of functions symbols "max" and "$\lfloor\ \rfloor$". To sketch how this is done, note that we know μ mod 3 and so we are able to express $\mu - l$ by μ and b. Also, we can split the first sum into four sums according to $i \equiv 0, 1, 2, 3 \pmod 4$. This allows not only get rid of the floor function in upper bounds but also determines the parity of $i/2$ if this happens to be integer. The latter information is needed to split the second sum into cases "m even" and "m odd" in order to be able to treat the values c_{imr} correctly. Up to now it would suffice to know the residue of μ modulo 12 but there is one more obstacle to overcome: the lower bound of the second sum needs to divide cases $i < \mu/2$ and $i \geq \mu/2$ which can be done by another split of the first sum by inserting a "middle bound". To do this, however, we need to express the integral values in the neighborhood of $\mu/2$ in all residue classes modulo 4, and this is why we need the information μ mod 8. There may be a way to get around this but the author does not see any at the moment. All together, we need to know μ mod 3 and μ mod 8, and this is exactly what b gives us. Once all the splits are done, we arrive at a couple of triple sums over expressions linear in all variables. Because of the form of the bounds, we get rid of one indeterminate at each summation sign, and finally we have to add a couple of quartic polynomials in μ. This means that $n(\mu)$ is a (quartic) polynomial on each residue class modulo 24, and so the whole sequence $(n(\mu))_{\mu\geq 0}$ is quasi-polynomial with quasi-period being a divisor of 24.

It would be of course too cumbersome to compute the 24 polynomials defining $(n(\mu))_{\mu\geq 0}$ by methods of the previous paragraph. (In this sense, our proof is *existential* rather then *constructive.*) Instead, we compute $24 \cdot 5$ initial values by (18) and do the polynomial interpolation on each residue class. The result turns out to have the quasi-period 12 indeed. The closed form of $n(\mu)$ is shown in section 4.2.

4 Computational Considerations

We shortly introduce the Maple package `QP` developed for easy computations with quasi-polynomials. The software may be obtained from the author on an e-mail request. (Please indicate your Maple version.)

This package, as each computer algebra product, is *not supposed to replace student's knowledge* (of working with generating functions, in this case). Rather, it should support tedious computations by saving time and human energy. For an interesting discussion on usage of such systems, see [Buc].

4.1 Functionality of the Package

We describe the package by specifications of its functions:

`gf2qp` takes a rational generating function and decides whether the underlying sequence is quasi-polynomial. If this is the case, the quasi-polynomial coefficients are computed.

`eval_qp` evaluates the given quasi-polynomial at a given integer.

`denumerant` computes the denumerant from given part sizes.

`p` computes $p_k(n)$.

A special message is printed if the result may be represented in terms of the floor function.

To give some feeling about the performance, we include three examples with their reference and CPU time needed to compute the quasi-polynomial closed form (i.e. the class polynomials) by our package on a DEC-5200 running Maple V Release 2.

problem	reference	CPU time
$D(n; 1, 2, 3)$	[Sta], p. 211	1.0 sec
$D(n; 1, 2, 3, 4, 5, 6)$	[Sta], p. 211	3.5 sec
$D(n; 1, 5, 10, 25, 50)$	[GKP], p. 331	37.9 sec

4.2 Examples of Closed Forms

Finally, we tabulate a couple of quasi-polynomial closed forms for some of the sequences treated in this paper. In front of each sequence we include the number of the section where it was introduced. The bracket and ceiling notations for quasi-polynomials were explained in section 1.1.

2.2	$B_4(m)$	$=$	$\frac{1}{48}m^3 - \frac{1}{16}m^2 + [\frac{1}{6}, -\frac{1}{48}, \frac{1}{6}, -\frac{1}{48}]m + [0, \frac{1}{16}, -\frac{1}{4}, \frac{1}{16}]$
2.3	$s_{m,2}$	$=$	$\lfloor \frac{1}{24}(m+2)(m+4)(2m+3) \rfloor$
3.1	$f_{3,s}$	$=$	$\frac{1}{8}s^3 - \frac{1}{2}s^2 + [-1, -\frac{9}{8}]s + [4, \frac{9}{2}] \quad (s \geq 5)$
3.2	$n(\mu)$	$=$	$\frac{1}{576}\mu^4 + \frac{1}{32}\mu^3 + \frac{59}{288}\mu^2 + [\frac{1}{8}, \frac{1}{32}]\mu + [0, -\frac{155}{576}, -\frac{25}{72}, \frac{5}{64}, -\frac{2}{9}, -\frac{155}{576}, -\frac{1}{8}, -\frac{83}{576}, -\frac{2}{9}, -\frac{3}{64}, -\frac{25}{72}, -\frac{83}{576}] \quad (\mu \geq 2)$

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