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Number of rhyme schemes

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ociating this factor with each node immediately above P. The result of this is t every x_h in F will be replaced by

$$z^2 \frac{1-t^h}{1-t}(1-zxt^h)^{-1}(1-zxt^{h-1})^{-1}$$

le x_0 is replaced by $(1-zx)^{-1}$.

It follows from this that $K_{r,p}(t)$ is the coefficient of $z^p x^t$ in the continued tion,

$$\cfrac{(1-zx)^{-1}}{1-\cfrac{z^2(1-zxt)^{-1}(1-zx)^{-1}}{1-\cfrac{z^2(1+t)(1-zxt^2)^{-1}(1-zxt)^{-1}}{1-\cfrac{z^2(1+t+t^2)(1-zxt^3)^{-1}(1-zxt^2)^{-1}}{1-\cdots}}}}$$

ch, after some manipulation, becomes

$$\cfrac{z^{-1}}{z^{-1}-x-\cfrac{1}{z^{-1}-xt-\cfrac{1+t}{z^{-1}-xt^2-\cfrac{1+t+t^2}{z^{-1}-xt^3-\cdots}}}}$$

is another J-function, but more general than that of (5). If we put $x = 0$, thus wing no uprights, this reduces to (5), as we would expect.

Stieltje's expansion, applied to this J-function will give a family of functions ogous to the $k_{i,j}$ that we obtained before, but they will now be functions of x, ell as t. Are these functions solutions to an even more general problem? It seems likely, but I have not yet explored this possibility.

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A BUDGET OF RHYME SCHEME COUNTS*

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INTRODUCTION

A rhyme scheme for a stanza of n verses is a sequence of rhymes numbered in order of appearance without restriction on number of appearances of any rhyme or of the number of rhymes; the first rhyme scheme is $(1, 1, \ldots, 1)$, the last $(1, 2, \ldots, n)$. Thus, a rhyme scheme is a (number) sequence $(r_1, r_2, \ldots, r_n)$, in which r_j may be any of the numbers $1, 2, \ldots, d_{j-1} + 1$, with d_j the number of distinct numbers among the $r_1, \ldots, r_j$. The fifteen rhyme schemes for $r = 4$ are:

1111, 1112; 1121 1122 1123; 1211, 1212, 1213;
1221, 1222, 1223; 1231, 1232, 1233, 1234

The mapping to set-partitions $(b_1, b_2, \ldots)$, given by Netto [3] with b_k the kth block, is $b_k = \{j;\, r_j = k\}$. For $n = 3$, the mapping is,

111	112	121	122	123
(123)	(12)(3)	(13)(2)	(1)(23)	(1)(2)(3)

The mapping is clearly bijective.

An alternate rhyme scheme, suggested by the level codes for lattice paths in the Catalan domain, is obtained by replacing r_j by $j + 1 - r_j$, as illustrated by

111	112	121	122	123
123	122	113	112	111

The alternate sequences $(R_1, R_2, \ldots, R_n)$ in rising order follow the rule: $R_{j+1} = d_j 1(1)\overline{j+1}$ with d_j as above and of course the tablemakers range convention. These schemes afford new enumerations, some of minor significance and others more surprising.

The enumerator $d_n(x) = \sum d_{n,k} x^k$ of rhyme schemes by number of distinct elements (rhyme numbers) is an immediate consequence of their definition, which implies,

$$d_{n+1,k} = k d_{n,k} + d_{n,k-1} \tag{1}$$

which is the recurrence for Stirling numbers of the second kind. With the convention $d_0(x) = 1$, it follows that,

$$d_n(x) = S_n(x) = \sum S(n,k) x^k \tag{2}$$

and, of course, the total number of rhyme schemes is $d_n(1) = S_n(1) = B_n$, the *Bell number*.

Next any rhyme scheme sequence has a specification $s_1, s_2, \ldots, s_n$, where s_j is the number of appearances of rhyme j; for $n = 3$, the rhyme schemes and their specifications are,

111	112	121	122	123
300	210	210	120	111

Replacing j by x_j in the specification arrays leads to (ignoring order) the polynomial,

$$x_3 + 3x_2 x_1 + x_1{}^3$$

which is the Bell multivariable polynomial $Y_3(x_1, x_2, x_3)$.

In general, the polynomial $Y_n(x_1, \ldots, x_n)$ is associated with the specifications for schemes of n verses, ignoring order. Hence, as is well-known, $Y_n(x_1, \ldots, x_n)$ is the enumerator of set-partitions by block size specification; of course $Y_n(x, \ldots, x) = S_n(x)$.

When order is not ignored the polynomial for $r = 3$ is replaced by

$$x_3 + 2x_2 x_1 + x_1 x_2 + x_1{}^3$$

For $n = 4$, 5 the polynomials are,

$$x_4 + 3x_3 x_1 + x_1 x_3 + 3x_2{}^2 + 3x_2 x_1{}^2 + 2x_1 x_2 x_1 + x_1{}^2 x_2 + x_1{}^3$$

and

$$\begin{aligned} &x_5 + 4x_4 x_1 + x_1 x_4 + 6x_3 x_2 + 4x_2 x_3 + 6x_3 x_1{}^2 + 3x_1 x_3 x_1 + x_1{}^2 x_3 \\ &\quad + 8x_2{}^2 x_1 + 4x_2 x_1 x_2 + 3x_1{}^2 x_2 + 4x_2 x_1{}^3 + 3x_1 x_2 x_1{}^2 + 2x_1{}^2 x_2 x_1 + x_1{}^3 x_2 + x_1{}^4 \end{aligned}$$

These are the refined Bell polynomials, new to me but possibly noticed earlier only to be forgotten. A brief look at their structure appears later.

As number sequences, whose literature is extensive, rhyme schemes have enumeration possibilities without hope of exhaustion here. We are limited to the following:

1) the enumeration by elements in the last position with enumerator $r_n(x; n)$; its structure is so transparent as to invite the extension to elements in the jth position. This leads naturally to the enumeration by fixed points associated with the mapping to forests (trivially mapped to trees), also leading to the enumeration by total appearances of elements (in the complete collection of schemes for given n).
2) The enumeration by doubles (pairs of like consecutive rhymes) whose enumerator $d_n(x)$ turns out to be $(B + x)^{n-1}$, $B^j \equiv B_j$, $n > 0$; closely related is the enumerator by units (number of appearances of the first rhyme), which is $x(B + x)^{n-1}$. This is extended to determine $\rho_n(x; j)$, the enumerator by number of appearances of rhyme j.
3) Brief notice is given to restricted rhyme schemes; first the (known) case in which each rhyme appears two or more times, then three or more times.

ENUMERATION BY ELEMENT AND POSITION

Consider first the enumeration by elements in the nth position with enumerator $\omega_n(x) = \sum \omega_{n,k} x^k$ with $\omega_{n,k}$ the number of appearances of element k. Since the last elements are fixed by the number of distinct elements in the first $n - 1$ positions enumerated by $S_{n-1}(x)$, it follows at once that,

$$\omega_n(x) = \sum S(n-1, j)(x + x^2 + \cdots + x^{j+1}) \tag{3}$$

Hence,

$$(1 - x)\omega_n(x) = \sum S(n-1, j)(x - x^{j+2}) = xB_{n-1} - x^2 S_{n-1}(x) \tag{3a}$$

It follows from (3) that (the prime denotes a derivative),

$$\omega_n(1) = \sum S(n-1, j)(j + 1) = S_{n-1}(1) + S'_{n-1}(1) = S_n(1)$$

since,

$$S_n(x) = x[S_{n-1}(x) + S'_{n-1}(x)]$$

The first few values for $\omega_n(x)$ are x, $x + x^2$, $2x + 2x^2 + x^3$, $5x + 5x^2 + 4x^3 + x^4$.

Turn now to enumeration by element in the jth position: enumerator $r_n(x; j)$, $j = 1(1)n[r_n(x; n) = \omega_n(x)]$. It is clear that $r_n(x; 1) = xB_n$, since all rhyme schemes start with rhyme 1. For $j = n - 1$, the first few values for $r_n(x; n-1)$ are $2x$, $2x + 3x^2$, $5x + 6x^2 + 4x^3$, $15x + 16x^2 + 16x^3 + 5x^4$, the last of which is obtained from

$$\begin{aligned} &r_5(x; 4) = \\ &\quad S(3, 1)(2x + 3x^2) + S(3, 2)(3x + 3x^2 + 4x^3) + S(3, 3)(4x + 4x^2 + 4x^3 + 5x^4). \end{aligned}$$

The terms of this expression are explained as follows: $S(3, 1)(2x + 3x^2)$ corresponds to triad 111 followed by 11, 12 (factor $2x$) or 21, 22, 23, (factor $3x^2$); $S(3, 2)$ indicates the initial triads 112, 121, 122 which are followed by 11, 12, 13 or 21, 22, 23 or 31 32 33 34, the factors $3x$, $3x^2$, $4x^3$; for $S(3, 3)$ the initial triad is 123 and its followers should be transparent. The general result turns out to be,

$$r_n(x; n-1) = \sum S(n-2, i)[(i + 1)(x + x^2 + \cdots + x^i) + (i + 2)x^{i+1}] \tag{4}$$

which implies,

$$\begin{aligned} (1 - x)r_n(x; n-1) &= \sum S(n-2, i)[(i + 1)(x - x^{i+1}) + (i + 2)(x^{i+1} - x^{i+2})] \\ &= xB_{n-1} + (x - 2x^2)S_{n-2}(x) - x^3 S'_{n-2}(x) \end{aligned} \tag{5}$$

Again the prime denotes a derivative.

It is worth noting that $i + 1 = B(B)_i$, $i + 2 = B(B)_{i+1}$ with $B(B)_i = B^2(B - 1) \cdots (B - i + 1)$ and $B^j \equiv B_j$; while for $r_n(x; n)$ the units appearing match $(B)_i = 1$. We do not take space to show that the corresponding factors for $r_n(x; j)$ are $B^{n-j}(B)_i$ and $B^{n-j}(B)_{i+1}$, so that,

$$r_n(x; j) = \sum S(j - 1, i)[B^{n-j}(B)_i(x + \cdots + x^i) + B^{n-j}(B)_{i+1} x^{i+1}] \tag{6}$$

Instances of (6) are,

$$\begin{aligned} r_n(x; 2) &= B_{n-1}x + (B_n - B_{n-1})x^2 \\ r_n(x; 3) &= B_{n-1}x + 2(B_{n-1} - B_{n-2})x^2 + B^{n-3}(B)_3 x^2 \\ r_n(x; 4) &= B_{n-1}x + (B_{n-1} + B_{n-2} - 2B_{n-3})x^2 + 4B^{n-4}(B)_3 x^3 + B^{n-4}(B)_4 x^4 \end{aligned}$$

Note that,

$$r_{nj}(j) = B^{n-j}(B)_j,\ r_{n,j-1}(j) = \left[1 + \binom{j-1}{2}\right] B^{n-j}(B)_{j-1}$$

where $r_n(x; j) = \sum r_{nk}(j)x^k$.

Numerical results are illustrated by the following array for $n = 7$; $r_{7k}(j)$ is the number in row k and column j

k/j	1	2	3	4	5	6	7
1	877	203	203	203	203	203	203
2		674	302	225	208	204	203
3			372	308	247	216	202
4				141	182	181	171
5					37	66	81
6						7	16
7							1

ENUMERATION BY FIXED POINTS

If $r_j = j$, j is a fixed point. For $n = 3$, the rhyme schemes arranged by number of fixed points are 111 112; 121 122; 123. Note that the schemes with j fixed points have initial segment $12 \cdots j$.

The fixed points are roots of forests obtained by a familiar mapping, illustrated for $n = 3$ by,

Hence, in the enumerator for fixed points $F_n(x)$, F may be read as fixed or forest. It is immediate that F_{n1}, the number of schemes with 1 fixed point, is $B_{n-1} \equiv B^{n-2}(B)_1$. For two fixed points, $F_{n2} = 2B^{n-3}(B)_2$, since the choices for the third position are 1, 2, and as in the enumerator above, the further choices are enumerated by $B^{n-3}(B)_2$. A similar argument shows that,

$$F_{nj} = jB^{n-1-j}(B)_j$$

and

$$F_n(x) = \sum_0^{n-1} jB^{n-1-j}(B)_j x^j + x^n, \; n > 0, \; F_0(x) = 1 \tag{7}$$

For $n = 1(1)5$, $F_n(x) = x(B + x)^{n-1}$, $B^j \equiv B_j$, a deception from which we were saved only by $F_6(x) = x(B + x) + x^2 - x^3$.

Also, the general formula is verified by the obvious results $F_{nn} = 1$, $F_{n,\,n-1} = n - 1$, and by $F_{n,\,n-2} = (n - 2)(n - 1)$, obtained with slight calculation. It follows from (7) that,

$$F_n(1) = B_n = \sum_0^{n-1} jB^{n-1-j}(B)_j + 1 \tag{8}$$

an identity readily verified for the first few values of n. For its proof by induction, it is convenient to recast it as,

$$B_n - \sum_0^{n-2} jB^{n-1-j}(B)_j = (n-1)(B)_{n-1} + (B)_n = n = B(B)_{n-1}$$

Then,

$$\begin{aligned} B(B)_n &= (B - n + 1)B(B)_{n-1} \\ &= (B - n + 1)\left[B^n - \sum_0^{n-2} jB^{n-1-j}(B)_j\right] \\ &= B_{n+1} - \sum_0^{n-2} jB^{n-j}(B)_j - (n-1)B(B)_{n-1} \\ &= B_{n+1} - \sum_0^{n-1} jB^{n-j}(B)_j \end{aligned}$$

Use of the recurrence,

$$(1 - j)F_{nj} + jF_{n,\,j-1} = j(j-1)F_{n-1,\,j-1}$$

leads to the relation,

$$\begin{aligned} (1 + x)F_n(x) - (x - x^2)DF_n(x) - 2x^2DF_{n-1}(x) - x^3D^2F_{n-1}(x) \\ = (n+1)x^{n+1} - (n-1)x^n \end{aligned} \tag{9}$$

where $D = d/dx$. An array for $F_n(x)$, $n = 6(1)9$, is as follows:

n/k	1	2	3	4	5	6	7	8	9
6	52	74	51	20	5	1			
7	203	302	231	104	30	6	1		
8	877	1348	1116	564	185	43	7	1	
9	4140	6526	5145	3196	1175	300	56	8	1

ENUMERATION BY DOUBLES AND BY UNITS

A double is a pair of like consecutive elements, a triple of the same kind is counted as two doubles, and so on. It is convenient to find the joint distribution of rhyme schemes by both doubles and distinct elements, denoted by,

$$d_n(x, y) = \sum d_{jk}(n)x^j y^k$$

where $d_{jk}(n)$ is the number of rhyme schemes on n with j doubles and k distinct elements. Omitting functional arguments, the first few values are: $d_1 = y$, $d_2 = y(y + x)$, $d_3 = y(y + y^2 + 2xy + x^2)$. Note that d_3 may also be written as $y[S(y) + x]^2$, $S^j(y) \equiv S_j(y)$, the polynomial for Stirling numbers of the second kind, leading to the guess: $d_n = y[S(y) + x]^{n-1}$.

Since the schemes corresponding to $d_{jk}(n)$ may be followed by elements $1, 2, \ldots, k + 1$, only one of which leads to a new double, the transition from n to $n + 1$ may be indicated by,

$$d_{jk}(n) \to (k-1)d_{jk}(n+1) + d_{j,\,k+1}(n+1) + d_{j+1,\,k}(n+1)$$

That is to say,

$$y^k \to (k-1)y^k + y^{k+1} + xy^k$$

An immediate consequence of this is,

$$S_k(y) \to S_{k+1}(y) + xS_k(y)$$

Use of this in $d_n = y(S(y)+x)^{n-1}$ proves the result by induction; hence,

$$d_n(x) = d_n(x, 1) = (B+x)^{n-1}$$
$$S_n(y) = d_n(1, y) = (S(y)+1)^{n-1} \qquad \textbf{(10)}$$

the latter a simple consequence of $\exp xS(y) = \exp y(e^x - 1)$.

A unit is an appearance of the first rhyme. Again it is convenient to consider the joint enumeration of units and distinct numbers: $u_n(x, y)$. The initial results suggest,

$$u_n(x, y) = xy[S(y)+x]^{n-1},\ S^j(y) \equiv S_j(y)$$

A direct proof of this is as follows: A single unit must be in the first position and followed by the rhyme schemes on $(2, 3, \ldots, n)$ for which the enumerator, by distinct numbers, is $yS_{n-1}(y)$. With two units, the second unit may be in any of positions $2, 3, \ldots, n$ and the remaining positions are on $2, 3, \ldots, n-1$; the contribution is $(n-1)yS_{n-2}(y)$. By a similar argument, j units are associated with $\binom{n-1}{j-1}yS_{n-j}(y)$. Thus $u_n(x) = u_n(x, 1) = x(B+x)^{n-1}$. Of course, $u_n(x)$ is the enumerator of set-partitions by size of the set containing 1.

REFINED BELL POLYNOMIALS

It is convenient to use a compressed notation, illustrated for $n = 4$ by

$$Y_4 = x_4 + (3, 1)x_3x_1 + 3x_2^2 + (3, 2, 1)x_2x_1^2 + x_1^3$$

Thus, the sum $3x_2x_1x_1 + 2x_1x_2x_1 + x_1x_1x_2$ is compressed to $(3, 2, 1)x_2x_1^2$; when order is ignored the number sequence $(3, 2, 1)$ is replaced by its sum 6, the coefficient of $x_2x_1^2$ in the regular Bell polynomial. These coefficients may be examined by using the recurrence [6, p. 174]

$$Y_{n+1}(x_1, \ldots, x_{n+1}) = \sum_0 \binom{n}{k} x_{k+1} Y_{n-k}(x_1, \ldots, x_{n-k}) \qquad \textbf{(11)}$$

with proper attention to order. Writing, following [2],

$$Y_n = \sum B_{nj}(x_1, x_2, \ldots)$$

with B_{nj} the terms of degree j, **(11)** is equivalent to,

$$B_{n+1,j} = \sum_0 \binom{n}{k} x_{k+1} B_{n-k,\, j-1} \qquad \textbf{(12)}$$

Of course

$$B_{n+1,1} = \binom{n}{n} x_{n+1} B_{00} = x_{n+1}, \text{ and } B_{n+1,n+1} = x_1^{n+1}$$

satisfy **(12)**. For $j = 2$, **(12)** supplies,

$$B_{32} = x_1B_{21} + 2x_2B_{11} = x_1x_2 + 2x_2x_1 = (2, 1)x_2x_1$$
$$B_{42} = x_1B_{31} + 3x_2B_{21} + 3x_3B_{11} = x_1x_3 + 3x_3x_1 + 3x_2^2$$

and it may be shown by induction that $C(x_jx_k)$, the compressed coefficient of x_jx_k, $j > k$ is $\left[\binom{j+k-1}{k}, \binom{j+k-1}{j}\right]$; for $j = k$, the coefficient is $\binom{2j-1}{j}$ as for the unrefined Bell polynomial. For $j = 3$, the forms are more numerous. Typical results are:

$$C(x_j^2x_1) = \left[j\binom{2j}{j+1}, \binom{2j}{j+1}, \binom{2j-1}{j}\right]$$

$$C(x_jx_1^2) = \left[\binom{j+1}{2}, j, 1\right]$$

$$C(x_jx_2^2) = \left[3\binom{j+1}{4}, (j+3)\binom{j+1}{2}, (j+3)(j+1)\right]$$

$$C(x_jx_2x_1) = \left[2\binom{j+2}{3}, \binom{j+2}{3}, j(j+2), j+2, \binom{j+1}{2}, j+1\right]$$

$$C(x_jx_3x_1) = \left[3\binom{j+3}{4}, \binom{j+3}{4}, j\binom{j+3}{2}, \binom{j+3}{2}, \binom{j+2}{3}, \binom{j+2}{2}\right]$$

$$C(x_jx_1^k) = \left[\binom{j+k-1}{k}, \binom{j+k-2}{k-1}, \ldots, \binom{j+1}{2}, j, 1\right]$$

It is worth noting that the last result follows from the recurrence:

$$C(x_jx_1^k) = \binom{j+k-1}{k} x_j C(x_1^k) + x_1 C(x_jx_1^{k-1})$$

Similar recurrences may be found for forms of higher degree. We leave them to the interested reader.

THE ENUMERATOR $\rho_n(x; j)$

This is the enumerating generating function by number of appearances of the jth rhyme; the case $j = 1$ has already been given. First, it is immediate that $\rho_{n0}(j)$, the number of schemes in which rhyme j is absent, is given by,

$$\rho_{n0}(j) = S(n, 1) + S(n, 2) + \cdots + S(n, j-1)$$

with $S(n, k)$ as above. Since $S_n(1) = B_n$, $S(n, n) = 1$, it follows that $\rho_n(x;\ n) = B_n - 1 + x$.

The schemes for $\rho_{n1}(j)$ have an initial block of length $j-1$ and size dependent on the position of j, which has the range $j(1)n$. If j is in position n, the initial block is of size $S(n-1, j-1)$. If j is in position $n-1$, the size of the initial block is $S(n-2, j-1)$. The choices for the last position are $1, 2, \ldots, (j-1), j+1$; hence the contribution is $S(n-2, j-1)j = S(n-2, j-1)B(B)_{j-1}$. Similar arguments lead to,

$$\rho_{n1}(j) = \sum_0^{n-j} S(n-1-i, j-1)B^i(B)_{j-1}$$

Rather than continue the tedious detail for multiple appearances of j, we show the results for $n = 8$, $j = 4$, as follows:

	$B^4(B)_3$	$B^3(B)_3$	$B^2(B)_3$	$B(B)_3$	$(B)_3$
1	$S(3, 3)$	$S(4, 3)$	$S(5, 3)$	$S(6, 3)$	$S(7, 3)$
2		$4S(3, 3)$	$3S(4, 3)$	$2S(5, 3)$	$S(6, 3)$
3			$6S(3, 3)$	$3S(4, 3)$	$S(5, 3)$
4				$4S(3, 3)$	$S(4, 3)$
5					$S(3, 3)$

The column heads are the enumerators of final choices, multipliers of the column entries. The rows are labeled by number of appearances of rhyme $j(= 4)$. The binomial coefficients arise from enumerating positions for these appearances. In summary, the table may be written,

$$x(B)_3[S(3, 3)(B + x)^4 + S(4, 3)(B + x)^3 + S(5, 3)(B + x)^2 + S(6, 3)(B + x) + S(7, 3)]$$

Hence,

$$\rho_n(x; j) = \sum_0^{j-1} S(n, k) + x \sum_0 S(n - 1 - k, j - 1)(B + x)^k (B)_{j-1} \tag{13}$$

From the identity $B^j(B)_k = (B + k)^j$ (see Ref. 4), it follows that,

$$(B + x)^k (B)_{j-1} = \sum \binom{k}{i} B^i (B)_{j-1} x^{k-i},$$
$$= \sum \binom{k}{i} (B + j - 1)^i x^{k-i}$$
$$= (B + j - 1 + x)^k$$

Thus (**13**) may be written as

$$\rho_n(x; j) = \sum_0^{j-1} S(n, k) + x \sum_0^{n-j} S(n - 1 - k, j - 1)(B + j - 1 + x)^k \tag{13a}$$

Numerical results for $\rho_{5k}(j)$ are as follows:

j/k	0	1	2	3	4	5
1	0	15	20	12	4	1
2	1	23	20	7	1	
3	16	26	9	1		
4	41	10	1			
5	51	1				

The identity following from $\rho_n(1; j) = B_n$ may be cast in the form

$$S(n, j) + S(n, j + 1) + \cdots + S(n, n) = \sum_0^{n-j} (B + j)^i S(n - 1 - i, j - 1)$$

ENUMERATOR BY TOTAL APPEARANCES OF EACH RHYME

Take $T_n(x) = \sum T_{nk} x^k$ with T_{nk} the total appearances of rhyme k in the rhyme schemes for n. The first few values are $T_1(x) = x$, $T_2(x) = 3x + x^2$, $T_3(x) = 9x + 5x^2 + x^3$. The structure present appears for $n = 4$ classified by number of units (u) and total appearances of rhyme j, as in the array,

u/j	1	2	3	4
1	5	9	5	1
2	12	9	3	
3	9	3		
4	4			

It is apparent that,

$$T_4(x) = xT_{41} + x[T_3(x) + 3T_2(x) + 3T_1(x)]$$

whose extension is,

$$T_n(x) = xT_{n1} + x[(T(x) + 1)^{n-1} - 1] \tag{14}$$

where $T_{n1} = B_n + (n - 1)B_{n-1}$. An immediate consequence of (**14**) is,

$$T_{nk} = \sum_1^{n-1} \binom{n-1}{i} T_{i, k-1}$$

which may be used to show that

$$T_{n1} - T_{n2} = \beta_{n-1}(1) = B_0 + B_1 + \cdots + B_{n-1}$$

If $T_{nk} - T_{n, k-1} = \beta_{n-1}(k)$ it can be shown that,

$$\beta_n(k) = k\beta_{n-1}(k) + B_n - [S(n, 1) + \cdots + S(n, k - 2)]$$

The first few values of $\beta_n(k)$ are as follows

k/n	1	2	3	4	5	6	7	8	9
1	1	2	4	9	24	76	279	1156	5296
2		1	4	13	41	134	471	1819	7778
3			1	7	35	156	670	2886	12797
4				1	11	80	490	2777	15120

None of these (nor their successors) appears in Sloane [7].

RESTRICTED RHYME SCHEMES

This is an area scarcely broached; two small examples serve merely to whet the appetite. The first is that of schemes with every rhyme appearing at least twice; that is, set-partitions without 1-element blocks. The enumerator, following the notation in Riordan [5, p. 76], is,

$$b_n(x) = \sum b_{nk} x^k = Y_n(0, x, \ldots, x) \tag{15}$$

where b_{nk} is an "associate Stirling number of the second kind."

An immediate consequence of (**15**) is the exponential generating function,

$$\exp yb(x) = \exp x(e^y - 1 - y) = \exp y(S(x) - x) \tag{16}$$

so that,

$$b_n(x) = (S(x) - x)^n, S^j(x) \equiv S_j(x)$$

Also differentiation of (**16**) leads to,

$$b_{n+1}(x) = x\{[b(x) + 1]^n - b_n(x)\}, b^j(x) \equiv b_j(x)$$

and,

$$b(n + 1, k) = kb(n, k) + nb(n - 1, k - 1)$$

both of which are in Riordan [5].

Values of $b_n(1)$ appear in sequence 1387 of Sloane [7]. Of course,

$$b_n(1) = (B - 1)^n, B^j \equiv B_j \qquad \text{and} \qquad b_n(1) + b_{n+1}(1) = B_n$$

The first extension of this, to schemes in which every rhyme appears at least three times, goes as follows. Taking the enumerator as $\gamma_n(x)$, then,

$$\gamma_n(x) = \sum \gamma_{nk} x^k = Y_n(0, 0, x, \ldots, x) \tag{17}$$

and,

$$\exp y\gamma(x) = \exp x(e^y - 1 - y - y^2/2) \tag{18}$$

Note that $\gamma_0(x) = 1$ (convention), $\gamma_1(x) = \gamma_2(x) = 0$, $\gamma_3(x) = \gamma_4(x) = \gamma_5(x) = x$. The recurrence for $\gamma_n(x)$ is,

$$\begin{aligned}\gamma_{n+1}(x) &= x[(\gamma(x) + 1)^n - \gamma_n(x) - n\gamma_{n-1}(x)], \gamma^j(x) \equiv \gamma_j(x)\\ &= x \sum_0^{n-2} \binom{n}{i} \gamma_i(x)\end{aligned}$$

Since the numbers $\gamma_n(1)$ do not appear in Sloane [7], we have,

n	0	1	2	3	4	5	6	7	8	9	10
$\gamma_n(1)$	1	0	0	1	1	1	11	36	92	491	2537

SUMMARY

Rhyme schemes, perhaps the oldest combinatorial setting behind the Stirling and Bell numbers, are rescued from current neglect by this budget of enumerations. A rhyme scheme for a stanza of n verses is a (number) sequence $(r_1, r_2, \ldots, r_n)$ in which r_j may be any of 1, 2, …, $d_{j-1} + 1$, with d_j the number of distinct numbers among $r_1, \ldots, r_j$; the rhyme schemes for $n = 3$ are: 111, 112, 121, 122, 123. The total number of schemes for n verses is of course the Bell number B_n. They have a trivial mapping of set-partitions but some enumerations familiar in the study of sequences are odd (even idiotic) when rephrased as set-partitions. The budget displayed (by no means exhaustive) is selected for its interest in the study of Bell and Stirling numbers. One surprise is the appearance of refined Bell multivariable polynomials.

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