The Square of a Directed Graph

and

At least one vertex doubles it's out-degree

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Abstract

I begin with a brief examination of the concepts directed graphs use, and develop a definition of the squaring process. Next I demonstrate that this process always at least doubles the out-degree of at least one vertex, thus proving a conjecture at: http://dimacs.rutgers.edu/~hochberg/undopen/graphtheory/graphtheory.html

Directed Graphs

A directed graph is a simple graph (no loops or multiple edges) with each edge assigned a direction. Given vertices u and v, this direction can be any of u->v, v->u or both, i.e. u<->v. This is similar to a one-way traffic system. The edge is now called an arc.

The out-degree is the number of arcs that are leaving a given vertex u and the in-degree is the number of arcs coming into u.

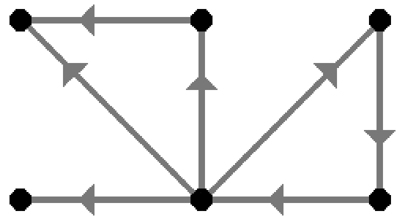
To square a directed graph, we consider vertices u, v and w. If u->v and v->w then we add a new arc (assuming it is not already there) of (u,w). Note that if u<->v then we create 2 loops, namely (u,u) and (v,v). If this is the case, the resulting directed graph is no longer simple.

An adjacency matrix for a graph consists of a table with each (labelled) vertex forming the co-ordinates. An entry (i,j) is 1 if i->j and 0 otherwise. An undirected graph always has a symmetric adjacency matrix, this is not always the case with a directed graph.

I have introduced the terms input graph as the initial directed graph which we are about to square, and output graph as the resultant graph of squaring the input graph.

Squaring a Directed Graph

To begin with we examine an input graph and develop it's adjacency matrix.

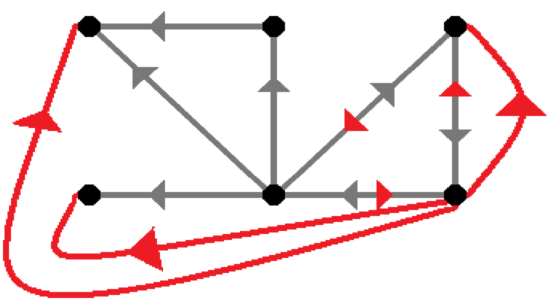


If we label the vertices 1 to 6 (top three are 1, 2 and 3, bottom three from left to right are 4, 5 and 6), we get the following adjacency matrix:

	1	**2**	**3**	**4**	**5**	**6**
1	0	0	0	0	0	0
2	1	0	0	0	0	0
3	0	0	0	0	0	1
4	0	0	0	0	0	0
5	1	1	1	1	0	0
6	0	0	0	0	1	0

The main diagonal is all 0's as this is part of the criteria for the input graph – no loops.

The square of this graph is:



This graph has the following adjacency matrix:

	1	2	3	4	5	6
1	0	0	0	0	0	0
2	1	0	0	0	0	0
3	0	0	0	0	1	1
4	0	0	0	0	0	0
5	1	1	1	1	0	1
6	1	1	1	1	1	0

The arc (5,1) is not doubled up because it already exists.

Generating the adjacency matrix for the output graph

There is a simple rule for generating the adjacency matrix for the square of a given input graph.

Let r_i be the contents of the i-th row of the input adjacency matrix.
Let r_{ij} be the contents of the i-th row and j-th column of the input adjacency matrix.
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Then we have:

$R_i = r_i + r_j$ [if $r_{ij} = 1$], with each R_{ij} being limited to 1.

Proof of the conjecture

If a row consists of entirely zeroes, the vertex in question has an out-degree of 0. Double of 0 is 0, and so we have trivially proved the conjecture if this is the case.

This means that the input graph must have every vertex with an out-degree of 1 or more. Consider an input graph with at least one vertex with out-degree exactly 1.

We have two situations to consider. If we call our vertex u and its arc-neighbour v, then either u->v->w where w is not u, or u->v->u. In the first case we now have u->w and because the out-degree of u is 1, this is a new arc and so the new out-degree of u is 2. In the second case we now have u->u, a loop, and the out-degree of u is also now 2.

So we can conclude that the input graph must have every vertex with out-degree at least 2, and we consider an input graph with at least one vertex with out-degree exactly 2.

The argument that follows will prove that we need an input graph with every vertex of at least out-degree 3, and more specifically at least one vertex of exactly out-degree 3. And we can use the same argument to extend 3 to 4 to 5 and so on, thus proving the conjecture.

If a vertex u has out-degree 2, then we may assume it points to vertices v and w. In the best case, v and w are not connected, so u has at least 3 new arcs, to vertices x, y and z, where v and w both point to x. If v points to w (or vice versa), it must at least point to x say, and w at worst can point to x and y, giving u two new arcs (u,x) and (u,y). So the out-degree must be at least 3.

We can now see that u points to k vertices, and even in the worst case of each vertex pointing to as many 'in common' vertices as possible, u will always double its out-degree.

QED.