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2 sequence

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Suppose now that

$$T_k(2t-1)=a_{k0}+a_{k1}t+\cdots+a_{kk}t^k$$

where $T_k(t)=\cos[k\cos^{-1}(t)]$ is the usual Chebyshev polynomial. Since $T_{n-1}(2t-1)$ alternates $n-1$ times on $[0,1]$ we may use Descartes' Rule of Signs [5, p. 43, # 36] to infer that each a_{k_i} is nonzero and that $t^{k-1}-T_{n-1}(2t-1)/a_{n-1,k-1}$ is the unique best approximation to t^{k-1} by a linear combination of $t^{i-1}, i\neq k, i=1,\ldots,n$. Thus,

$$\|S^{-1}\|=\max\{\|\mathbf{q}_k\|/\|H(\mathbf{q}_k,-)\|:k=1,\ldots,n\}$$
$$=\max\{|a_{n-1,k}|:k=0,1,\ldots,n-1\}.$$

Using a well-known expression [7, p. 32] for T_n together with the identity

$$T_n(2t-1)=T_{2n}(\sqrt{t}\,),\qquad 0\leqslant t\leqslant 1$$

it follows that

$$|a_{n,n-k}|=\frac{2n}{2n-k}\binom{2n-k}{k}2^{2n-2k-1}, n=1,2,\ldots, k=0,1,\ldots,n. \tag{3}$$

For a given $n\geqslant 1$ the moduli given in (3) increase monotonically from 2^{2n-1} when $k=0$ up to the desired maximum and then decrease monotonically to 1 when $k=n$, and so the largest modulus occurs for the smallest nonnegative integer k for which

$$|a_{n-1,n-k-1}/a_{n-1,n-k}|=\frac{(n-k-1)(2n-2k-3)}{2(k+1)(2n-k-3)}\leqslant 1$$

or equivalently for which

$$n-\frac{13}{8}-\frac{1}{8}\sqrt{32(n-1)^2-7}\leqslant k. \tag{4}$$

The rapid growth of $\|S^{-1}\|$ and hence the condition number C is seen in Table 1 where $\|S\|$, $\|S^{-1}\|$, and C are listed for $n=1,2,\ldots,10$. In particular, we can expect a relative error of order $10^7\cdot u$ when approximating by a ninth degree polynomial. More generally, Stirling's approximation for the factorial (cf. [1, p. 257]) can be used in conjunction with (4) to show that

$$C\sim\sqrt{n-1}\cdot(1+\sqrt{2}\,)^{2n-2}/2^{1/4}\cdot\sqrt{\pi}\qquad \text{as}\quad n\to+\infty$$

with this approximation having a relative error of less than 5% for $n\geqslant 10$.

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TABLE 1.
Local Conditioning for the Canonical Polynomial Parametrization

n	$\|S\|$	$\|S^{-1}\|$	C
1	1	1	1
2	2	2	4
3	3	8	24
4	4	48	192
5	5	256	1280
6	6	1280	7680
7	7	6912	48384
8	8	39424	315392
9	9	212992	1916928
10	10	1118208	11182080

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In contrast, we consider the parametrization

$$Y(\mathbf{p},t)=p_1T_0(2t-1)+\cdots+p_nT_{n-1}(2t-1)$$

(which can be evaluated almost as easily as the canonical parametrization, cf. [7, pp. 125–126]).