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EITHER TOURNAMENTS OR ALGEBRAS?*

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The paper deals with tournaments (i.e., with trichotomic relations) and their homomorphisms. The study of tournaments by means of their homomorphisms is natural as tournaments are algebras of a special kind. We prove (1) theorems which relate combinatorial and algebraic notions (e.g., the score of a tournament and the monoid of its endomorphisms); (2) theorems concerned with strictly algebraic aspects of tournaments (e.g., characterizing the lattice of congruences of a tournament). Our main result is that the group of automorphisms and the lattice of congruences of a tournament are in general independent. In the last part of the paper we give some examples and applications to other fields.

0. Introduction

The program of systematic study of algebraic properties of graphs and relations in general was carried out by K. Čulík, G. Sabidussi, Z. Hedrlín and A. Pultr. While this approach led undoubtedly to success in applications of graph theory to various branches of mathematics (see [5]), within graph theory itself the role of this approach is still debatable and arguments can be given to support both sides. Certainly, there are parts of graph theory where the study of properties of graphs by means of homomorphisms between them is generally known (e.g., chromatic numbers and polynomials). But this being more the exception than the rule, it is not very surprising that there are graphs – namely tournaments – which are basically the same as algebras of a certain kind, but which have not yet been studied from this point of view. As far as we know, [1] is the only paper dealing with the subject, apart from the work done on automorphism groups of tournaments, see [8]. In 1965,

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Z. Hedrlín observed that a tournament $\mathcal{T} = (T, t)$ can be made into an algebra $A_{\mathcal{T}} = \langle T, \cdot \rangle$ by defining $x \cdot y = y \cdot x = x \Leftrightarrow (x, y) \in t$. This correspondence between the class of all tournaments and the class of all commutative groupoids $\langle X, \cdot \rangle$ satisfying $x \cdot y \in \{x, y\}$ for all $x, y \in X$, is clearly a bijection. A moment of reflexion is enough to check that the tournament-homomorphisms and the algebraic homomorphisms are also in $1-1$ correspondence and actually coincide. It is the aim of this paper to go on in this direction and to study tournaments in an algebraic way (i.e., by means of homomorphisms between them).

The paper is divided into five parts. In Section 1 we state the basic definitions and theorems on the structure of homomorphisms and congruences of tournaments. We relate them to some of the known results. In Section 2 we prove a theorem which shows the independence of the degree sequence (score vector) and the endomorphisms of tournaments; indeed, we prove that given a strong tournament there is a tournament with the same degree sequence every endomorphism of which is either a constant or an automorphism: these tournaments are called simple. We also show that the absence of non-trivial automorphisms may be forced by the degree sequence, and we discuss this in detail in Section 5, where the corresponding characterization theorems are proved. In Section 3 we characterize all finite lattices which are isomorphic to the lattice of all congruences of a tournament; such lattices we call admissible. Investigating the simple tournaments in Section 4 we are able to prove:

Main Theorem. *Let L be an admissible lattice, G an odd group. Then there exists a tournament $\mathcal{T}$ such that*

(1) *the group of all automorphisms of $\mathcal{T}$ is isomorphic to G;*
(2) *the lattice of all congruences of $\mathcal{T}$ is lattice-isomorphic to L.*

A few applications, remarks and open problems conclude the paper. Those not interested in tournaments can read only this last paragraph.

What are the advantages of this approach? Tournament-homomorphisms are strongly related to the inner structure of the underlying tournaments. Since every surjective homomorphism is a retraction, the use of a homomorphism permits the reduction of a tournament to another, hopefully simpler one. At any rate, we decrease the number of vertices. This is often done intuitively in practive (blocks of a league, small groups in sociology). On the other hand, a homomorphism constitutes in itself a type of lexicographic decomposition of a tournament.

It is possible to say that the cyclic decomposition of non-strong tournaments and the simple decomposition of strong tournaments are the two tools for reduction of tournaments. The formulas in Section 5 can then be regarded as giving the number of essentially different non-isomorphic tournaments with n vertices.

Our Main Theorem gives a sharpening of a result of Moon [8] on the automorphism group of a tournament. On the other hand, it is an example of a type of theorem considered in universal algebra. Only recently, Lampe [7] proved that for universal algebras the congruence lattice and the group of automorphisms are independent. This is very difficult even in the class of all universal algebras, and Lampe proved it using mainly unary operations. For a restricted class of tournaments and the corresponding algebras our theorem gives the best result. All the tournaments considered here are finite, although many of the results can easily be generalized to the infinite case.

Remark: Recently, simple tournaments were studied in a different context by Erdös, Fried, Hajnal, Milner and Moon [2]. One of their main theorems states that (with one exception) every tournament $\mathcal{T}$ can be extended to a simple tournament $\mathcal{S}$ by adding one vertex only.

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1. Basic definitions and properties

A tournament $\mathcal{T} = (T, t)$ is a finite set T with a relation $t \subset T \times T$ which is reflexive and satisfies $(x, y) \in t \Leftrightarrow (y, x) \notin t$ for any two distinct vertices x, y of T.

Let $\mathcal{T} = (T, t)$ and $\mathcal{S} = (S, s)$ be tournaments. A mapping $f : T \to S$ is called a *homomorphism* if $(f(x), f(y)) \in s$ whenever $(x, y) \in t$. In an obvious sense we use the terms *endomorphism, automorphism, isomorphism.* Denote the set of all homomorphisms from $\mathcal{T}$ into $\mathcal{S}$ by $H(\mathcal{T}, \mathcal{S})$. We put $H(\mathcal{T}) = H(\mathcal{T}, \mathcal{T})$. This is obviously a monoid under composition of mappings. We denote by $A(\mathcal{T})$ the group-part of $H(\mathcal{T})$ which consists exactly of all 1–1 homomorphisms (as T is finite).

If a tournament $\mathcal{S}$ is a subtournament of $\mathcal{T}$, we write $\mathcal{S} \leqslant \mathcal{T}$. Since every constant mapping is a homomorphism we have $H(\mathcal{T}, \mathcal{S}) \neq \emptyset$ for any two tournaments. There are tournaments for which the constants are the only endomorphisms (see Section 2 and [1]). On the other hand, if $f \in H(\mathcal{T}, \mathcal{S})$ and f is onto, then $\mathcal{S}$ can be regarded as a subtournament of $\mathcal{T}$ (it is enough to take any set M of representatives of the family $\{f^{-1}(x): x \in S\}$, and to show that the subtournament of $\mathcal{T}$ generated by M is isomorphic to S). Consequently, we have that every epimorphism is a retraction (see [6]). We shall use the notation $\mathcal{T}|_M$ for the subtournament generated by M, i.e., $\mathcal{T}|_M = (M, t \cap M \times M)$.

The algebra $A_{\mathcal{T}}$ of a tournament $\mathcal{T}$ is the set T together with the binary operation on T defined by $x \cdot y = y \cdot x = x$ iff $(x, y) \in t$. We state explicitly:

Proposition 1. *$f \in H(\mathcal{T}, S)$ iff f is an algebraic homomorphism from $A_{\mathcal{T}}$ into $A_{\mathcal{S}}$.*

Hence from the homomorphical point of view we can regard a tournament either as a relation or as an algebra; we shall frequently make use of this possibility. In particular, an equivalence η on the set T is a *congruence* on a tournament $\mathcal{T}$ iff $(x, y) \in t \Rightarrow (x', y) \in t$ whenever $x \,\eta\, x'$. Denote by $\mathcal{T}/\eta$ the *factor tournament* under the congruence η. Every set $\eta[x] = \{y: x \,\eta\, y\}$ is called a *congruence class*. Denote by $(E(\mathcal{T}), \wedge, \vee)$ the lattice of all congruences of $\mathcal{T}$. The following is, of course, true for the congruence lattice of any algebra:

Proposition 2. *Let η_1, η_2 be congruences on $\mathcal{T}$. Then $\eta_1 \wedge \eta_2$ is the intersection of η_1 and η_2; $\eta_1 \vee \eta_2$ is the smallest equivalence containing η_1 and η_2.*

We list some other simple properties.

Proposition 3. *Let $\mathcal{T}$ be a tournament, M and N congruence classes of $\mathcal{T}$. Then the following holds:*

(1) *$(x, y) \in t$ iff $(x', y') \in t$ for any $x, x' \in M$, $y, y' \in N$;*

(2) *$M \cap N \neq \emptyset \Rightarrow M \cup N$ is a congruence class.*

Tournaments with the simplest congruence lattice are of a special importance: A tournament $\mathcal{T}$ is called *simple* if $E(\mathcal{T}) = \{0, 1\}$. Clearly, for a simple tournament the only congruences are T^2 and Δ_T (the diagonal on T).

Proposition 4. *Let $\mathcal{T}$ be a tournament. Then the following statements are equivalent:*

(1) *$\mathcal{T}$ is simple.*

(2) *$|H(\mathcal{T}, \mathcal{S})| = |S|$ for every $\mathcal{S} = (S, s) \ngeq \mathcal{T}$.*

(3) *For any proper subset M of T there exists $z \in T - M$ such that neither $\{z\} \times M \subseteq t$ nor $M \times \{z\} \subseteq t$.*

A sequence $(x_1, ..., x_n)$ of vertices of $\mathcal{T}$ is called a *cycle* in $\mathcal{T}$ if $(x_i, x_{i+1}) \in t$, $i = 1, ..., n-1$, and $(x_n, x_1) \in t$. If $T = \{x_1, ..., x_n\}$ and $(x_1, ..., x_n)$ is a cycle in $\mathcal{T}$, then $\mathcal{T}$ is called a *strong* tournament provided that $\mathcal{T}$ has at least two vertices.

Proposition 5. *Let $\mathcal{T}$ be a tournament. Then there exists a simple tournament $\mathcal{S}$ with at least two vertices which is a homomorphic image of $\mathcal{T}$. $\mathcal{S}$ is unique up to isomorphism. If $\mathcal{T}$ is a strong tournament, then there exists a unique congruence η on $\mathcal{T}$ such that $\mathcal{T}/\eta = \mathcal{S}$.*

Proof. If $\mathcal{T}$ is not a strong tournament, then $\mathcal{S} = (\{0, 1\}, \leqslant)$. Let $\mathcal{T}$ be a strong tournament. Let

$$\eta = \bigvee \{\eta_i : \eta_i \in E(\mathcal{T}), \eta_i \neq T^2\}.$$

We claim $\eta \neq T^2$. Suppose the contrary, then for any two distinct points x, y there are congruence classes $M_1, ..., M_n$ such that $x \in M_1$, $y \in M_n$ and $M_i \cap M_{i+1} \neq \emptyset$, $i = 1, ..., n-1$. It is easy to see that this yields a contradiction. $\mathcal{T}/\eta$ is a simple tournament as every congruence ϵ on $\mathcal{T}/\eta$ induces a congruence on $\mathcal{T}$. Clearly, if $\mathcal{T}/\eta \cong \mathcal{T}/\eta'$, then $\mathcal{T}/\eta \vee \eta' \cong \mathcal{T}/\eta$, and hence $\eta = \eta'$ (again we have $\eta \vee \eta' \neq T^2$).

The maximal non-trivial congruence on a strong tournament will be called the *simple decomposition* of a tournament.

Proposition 6. *Let η be a congruence on $\mathcal{T}$ such that there is no congruence ϵ with $\Delta_T < \epsilon < \eta$ (i.e., η is an atom of $E(\mathcal{T})$). Then η has only one congruence class containing at least two points, and this congruence class determines a simple subtournament of $\mathcal{T}$.*

Let us add one remark concerning our definition of homomorphism. In [4] a homomorphism is defined as a finite composition of "elementary" homomorphisms, where the latter are those homomorphisms (according to our definition) which identify exactly two points. Of course, in the case of undirected graphs which are finite (and only for those) both definitions coincide. But for certain relations these two definitions do not coincide even in the finite case: every simple tournament with at least three points provides such an example.

2. Homomorphisms and cycles in tournaments

Homomorphisms are related to the cycles in tournaments. It is easy to see that a tournament $\mathcal{T}$ is not strong iff there is a homomorphism from $\mathcal{T}$ onto $\mathcal{T}_2$ (denote by $\mathcal{T}_n = (T_n, t_n)$ the natural ordering of the natural numbers $1, \ldots, n$). More generally the following holds:

Proposition 7. *$\mathcal{T}$ is a strong tournament iff $f(\mathcal{T})$ is a strong tournament for some homomorphism f, where $f(\mathcal{T}) = (f(T), f^2(t))$ is the homomorphic image of $\mathcal{T}$.*

Proof. If $f(\mathcal{T}) = \mathcal{S} = (S, s)$ is not strong, then $\mathcal{T}$ fails to be a strong tournament also.

Let g be a homomorphism from $\mathcal{T}$ onto $\mathcal{T}_2$, and suppose that $f(\mathcal{T})$ is a strong tournament. Put $M = g^{-1}(1)$ and $N = g^{-1}(2)$. Then $f(M) \cap f(N) \neq \emptyset$ for otherwise g can be factorized through f, and $f(\mathcal{T})$ is not strong. Let $f(x) = f(y) \in f(M) \cap f(N)$. Let $(f(x), f(z)) \in s$, $f(z) \neq f(x)$. Then $z \in N$, for otherwise $f(x) = f(z)$. Similarly, $y' \in M$ for every $(f(y'), f(x)) \in s$, $f(y') \neq f(x)$. As $f(\mathcal{T})$ is strong this finishes the proof.

Let $d^-(x, \mathcal{T})$ be the cardinality of the set

$$V^-(x, \mathcal{T}) = \{y: (x, y) \in t,\ x \neq y\}.$$

We put $d^+(x, \mathcal{T}) = |V^+(x, \mathcal{T})|$, where

$$V^+(x, \mathcal{T}) = T - (\{x\} \cup V^-(x, \mathcal{T})).$$

Tournaments $\mathcal{T}$ and $\mathcal{S}$ are said to be *degree-equivalent* if the sequences $(d^-(x, \mathcal{T}): x \in T)$ and $(d^-(x, \mathcal{S}): x \in S)$ coincide (in a convenient enumeration of the vertices).

By the reversal theorem, see [8], two tournaments $\mathcal{T}$ and $\mathcal{S}$ are degree-equivalent iff $\mathcal{T}$ can be transformed into $\mathcal{S}$ by "chasing triangles". Consequently, if $\mathcal{T}$ and $\mathcal{S}$ are degree-equivalent, then either both $\mathcal{T}$ and $\mathcal{S}$ are strong, or both $\mathcal{T}$ and $\mathcal{S}$ fail to be strong. Thus together with Proposition 7 we have a restriction on the degrees of homomorphic images of a tournament. In particular,

$$\sum \{d^-(x_i, \mathcal{T}): i = 1, ..., k\} = \binom{k}{2}, \qquad k < |T| ,$$

iff there exist $m < |f(T)|$, vertices $y_1, ..., y_m$ of $f(\mathcal{T})$ such that

$$\sum_{i=1}^{m} d^-(y_i, f(\mathcal{T})) = \binom{m}{2} .$$

Now we show that this restriction on the degrees of a homomorphic image is best possible:

Theorem 1. *Let $\mathcal{T} = (T, t)$ be a strong tournament, $|T| \neq 4$. Then there exists a simple tournament $\mathcal{S} = (T, s)$ such that $\mathcal{T}$ and $\mathcal{S}$ are degree-equivalent.*

In the proof we employ the following notation: Let $\mathcal{T} = (T, t)$ be a tournament, let $(a_1, ..., a_k)$ be a cycle in $\mathcal{T}$. Denote by $(a_1, ..., a_k)\mathcal{T}$ the tournament which arises from $\mathcal{T}$ by reversing all arrows in the cycle $(a_1, ..., a_k)$, i.e., $(a_1, ..., a_k)\mathcal{T} = (T, \bar{t})$, where

$$\bar{t} = (t \setminus (\{(a_i, a_{i+1}): i = 1, ..., k-1\} \cup \{(a_k, a_1)\}))$$
$$\cup \{(a_{i+1}, a_i): i = 1, ..., k-1\} \cup \{(a_1, a_k)\} .$$

Proof. The case $|T| = 3$ can be handled easily.

Let $|T| = 5$ and let η be the simple decomposition of $\mathcal{T}$. If $\mathcal{T}$ fails to be simple, then necessarily $|\mathcal{T}/\eta| = 3$, and we may suppose that one of the following possibilities occurs:

(i) $\eta = \{\{1, 2, 3\}, \{4\}, \{5\}\}$,

(ii) $\eta = \{\{1, 2\}, \{3, 4\}, \{5\}\}$.

In case (i), it may be further assumed that

$$\{(5, 1), (1, 4), (4, 5)\} \subset t \quad \text{and} \quad \{(1, 2), (2, 3)\} \subset t .$$

But then $(2, 4, 5)\mathcal{T}$ is simple. In case (ii), we may assume that

$$\{(5, 1), (1, 3), (3, 5), (1, 2), (3, 4)\} \subset t,$$

and then $(2, 4, 5)\mathcal{T}$ is simple.

For $|T| \geqslant 5$ the statement will be proved by induction on $|T|$.

Let $|T| = k + 1$. As $\mathcal{T}$ is strong, there exists $a \in T$ such that $\mathcal{T}|_{T-\{a\}}$ is again a strong tournament. By the induction hypothesis there exists a tournament $\mathcal{S} = (T - \{a\}, s)$ which is degree-equivalent to $\mathcal{T}|_{T-\{a\}}$. Put

$$t' = s \cup (\{a\} \times T \cap t) \cup (T \times \{a\} \cap t),$$

then the tournament $\mathcal{T}' = (T, t')$ is obviously degree-equivalent to $\mathcal{T}$. If $\mathcal{T}'$ is simple, the statement follows. If $\mathcal{T}'$ is not simple, let η be the simple decomposition. But then necessarily $\eta = \Delta_T \cup \{a, b\}$, where we can assume $(a, b) \in t'$. As $\mathcal{S}$ is a strong tournament there are vertices $c, d \in T - \{a, b\}$ such that

$$\{(d, a), (d, b), (a, c), (b, c), (c, d)\} \subset t'.$$

Put $\mathcal{T}_a = (a, c, d)\mathcal{T}'$ and $\mathcal{T}_b = (b, c, d)\mathcal{T}'$. Then the following statements hold.

(1) There is no non-trivial $\eta \in E(\mathcal{T}_a)$ ($\eta \in E(\mathcal{T}_b)$, respectively) such that $\eta \cap \{a, c, d\}^2 \nsubseteq \Delta_T$ ($\eta \cap \{b, c, d\}^2 \nsubseteq \Delta_T$, respectively).

(2) If K is a congruence class of $\mathcal{T}_a$ ($\mathcal{T}_b$, respectively), containing a (b, respectively), then $|K| \leqslant 2$.

(3) If $K_1 \neq K_2$ are non-trivial congruence classes of $\mathcal{T}_a$ ($\mathcal{T}_b$, respectively), then there does not hold:

(i) $c \in K_1$, $d \in K_2$;

(ii) $a \in K_1$, $c \in K_2$ ($b \in K_1$, $d \in K_2$, respectively).

(4) If K_1 is a non-trivial congruence class of $\mathcal{T}_a$ ($\mathcal{T}_b$, respectively), and either $d \in K_1$ or $c \in K_1$, then also $b \in K_1$ ($a \in K_1$).

If either $\mathcal{T}_a$ or $\mathcal{T}_b$ is simple, the statement follows. If both $\mathcal{T}_a$ and $\mathcal{T}_b$ are not simple, let η_a (η_b, respectively) be the simple decomposition of $\mathcal{T}_a$ ($\mathcal{T}_b$, respectively).

By (1)–(4), the list of all possibilities for η_a and η_b is then as follows (where e is an element distinct from a, b, c, d):

Ia:	$\eta_a = \Delta_T \cup \{b, d\}^2$,	Ib:	$\eta_b = \Delta_T \cup \{a, c\}^2$,
IIa:	$\eta_a = \Delta_T \cup \{b, d\}^2 \cup \{a, e\}^2$,	IIb:	$\eta_b = \Delta_T \cup \{a, c\}^2 \cup \{b, e\}^2$,
IIIa:	$\eta_a = \Delta_T \cup K \times K$, $\lvert K\rvert \geqslant 3$, $K \subset T - \{a, c\}$,	IIIb:	$\eta_b = \Delta_T \cup L \times L$, $\lvert L\rvert \geqslant 3$, $L \subset T - \{b, d\}$,
IVa:	$\eta_a = \Delta_T \cup K \times K \cup \{a, e\}^2$, $\lvert K\rvert \geqslant 3$, $K \subset T - \{a, c, e\}$,	IVb:	$\eta_b = \Delta_T = L \times L \cup \{b, e\}^2$, $\lvert L\rvert \geqslant 3$, $L \subset T - \{b, d, e\}$,
Va:	$\eta_a = \Delta_T \cup \{a, e\}^2$,	Vb:	$\eta_b = \Delta_T \cup \{b, e\}^2$.

We prove that one of the cases Va, Vb must occur, and that either of them implies that $\mathcal{T}'$ can be reversed in a simple tournament. For example, consider the case Va: either (c, e, a, d) or (c, a, e, d) is a cycle in $\mathcal{T}_a$. Put $(c, a, e, d)\mathcal{T}_a = \mathcal{T}'_a$. Suppose by way of contradiction that

$\mathcal{T}'_a$ is not simple. Let K, $|K| \geqslant 2$, be a congruence class of $\mathcal{T}'_a$. Then

(i) $\mathcal{T}'_a|\{a, b, c, d, e\}$ is simple;

(ii) $|K \cap \{a, c, d, e\}| \leqslant 1$, $b \notin K$;

(iii) $a \notin K$.

The proof of (i) is a matter of routine. To establish the other two claims it is enough to prove the impossibility of $K \cap \{c, d, e\} \neq \emptyset$. Choose $f \in K - \{a, b, c, d, e\}$ (recall that $|T| > 5$). Then either $\{(f, a), (f, b)\} \subset t'_a$ or $\{(a, f), (b, f)\} \subseteq t'_a$, and also either $\{(f, a), (f, e)\} \subseteq t_a$ or $\{(a, f), (e, f)\} \subseteq t'_a$ which is a contradiction with $K \cap \{e, c, d\} \neq \emptyset$. Clearly also

(iv) $|K| = 2$.

Since (f, b, c) is a cycle, it is enough to prove that $(f, b, c)\mathcal{T}'_a$ is simple. But this is now routine as every congruence class would contain f as well as some of the points a, b, c, d, e, but between these points we have constructed enough 3-cycles. The case Vb can be handled symmetrically.

Now we prove that the only possible cases are Va and Vb. As the situation is symmetrical in a and b it suffices to prove the impossibility of the joint cases

(1) Ia and Ib,	(2) Ia and IIb,	(3) Ia and IIIb,
(4) Ia and IVb,	(5) IIa and IIb,	(6) IIa and IIIb,
(7) IIa and IVb,	(8) IIIa and IIIb,	(9) IIIa and IVb,
(10) IVa and IVb.		

The cases (1), (2), ..., (7) are almost self-evident.

Case (8): If follows easily that $K - \{b, d\} = L - \{a, c\}$, and consequently $K = \{e\}$ by the simplicity of $\mathcal{S}$. Let $f \in T - \{a, b, c, d, e\}$. Then $(f, b) \in t_a$ implies $(f, a) \in t_a$, and consequently $(f, x) \in t_a$ for every $x \in \{a, b, c, d, e\}$. This is a contradiction.

Case (9): Again $K - \{b, d\} = L - \{a, c\} = \{f\}$. Then $(f, e) \in t_b$, hence $(f, d) \in t_a$, and this is a contradiction. (10) may be proved in the same way. This finally proves Theorem 1.

According to Theorem 1 there are many simple tournaments and their number on a set of cardinality k is increasing with k. But in this respect Theorem 1 is not of much use, for it can be shown much more easily that $\lim_{n\to\infty} S(n)/T(n) = 1$, where $S(n)$ is the number of non-isomorphic simple tournaments and $T(n)$ the total number of non-isomorphic tournaments on a set with n points (see Section 5).

Theorem 1 can be used for the construction of rigid tournaments. A tournament is said to be *rigid* if the constants and the identity mapping are its only endomorphisms. It is proved in [1] that on every set M, $|M| > 4$, there exists a rigid tournament. Since there exists a degree sequence $d_1, \ldots, d_n$ of a strong tournament such that every tournament $\mathcal{T}$ with the same degree sequence possesses no non-identity automorphism (one such sequence is 2, 2, 3, 4, ..., $n-2$, where $n > 4$), we get as a corollary the result of [1]. See Section 5 for further results in this direction. There we characterize those degree sequences which simultaneously force simplicity and the identity as the only automorphism.

If a tournament fails to be strong, then the simple decomposition is not unique. There is another natural "reduction" – into cyclic parts. This may be formulated as follows (see [8]):

Proposition 8. *Let $\mathcal{T}$ be a non-strong tournament. Let f be a homomorphism from $\mathcal{T}$ onto $\mathcal{T}_n$ such that there is no homomorphism from $\mathcal{T}$ onto $\mathcal{T}_m$ for any $m > n$. Then f is determined uniquely, and $\mathcal{T}|_{f^{-1}(x)}$ is a strong tournament for every $x \in \mathcal{T}_n$.*

The partition $\{f^{-1}(x)\colon 1 \leqslant x \leqslant n\}$ is called the *cyclic* decomposition of $\mathcal{T}$. Of course, the cyclic decomposition of a tournament $\mathcal{T}$ is a congruence, but it is not a maximal one (with the exception of cyclic decompositions with exactly two elements).

3. Characterization of the lattice of all congruences of a tournament

Let $(L, \wedge, \vee)$ be a finite lattice. Denote by $I(L)$ the set of all join-irreducible elements of L (i.e., the set of all the elements $x \in L$ for which $x \neq \bigvee\{y\colon y < x\}$; as usual, we take the least element of L as the join of the empty set). Consider the set $I(L)$ endowed with the partial ordering induced by L. Denote $I(E(\mathcal{T}))$ by $I(\mathcal{T})$ for a tournament $\mathcal{T}$. We shall split the proof of the main theorem of this paragraph into two parts:

A: The characterization of $I(\mathcal{T})$;

B: The characterization of $I(\mathcal{T})$ in $E(\mathcal{T})$.

A: $I(\mathcal{T})$

First, we list a few lemmas which will be needed in the sequel.

Lemma 1. *Let $\mathcal{T} = (T, t)$ be a tournament, $M \subseteq T$. Then $M \cap N$ is a congruence class in $\mathcal{T}|_M$ for any congruence class N of $\mathcal{T}$.*

Lemma 2. *Let $\mathcal{T} = (T, t)$ be a tournament, let $U \subseteq T$ be a congruence class in $\mathcal{T}$. Then any congruence class V of $\mathcal{T}|_U$ is also a congruence class in $\mathcal{T}$.*

Lemma 3. *Let $\mathcal{T} = (T, t)$ be a tournament, let M, N be congruence classes of $\mathcal{T}$ such that $M - N \neq \emptyset$, $N - M \neq \emptyset$, and $M \cap N \neq \emptyset$. Then the tournament $\mathcal{T}|_M$ is not strong.*

Proof. Choose $x \in M - N$, $y \in M \cap N$, $z \in N - M$, and suppose, for example, $(x, y) \in t$. Then $(x, z) \in t$ and $(x', z) \in t$ for any $x' \in M - N$, and consequently $(x', y') \in t$ for any $x' \in M - N$ and $y' \in M \cap N$.

Lemma 4. *Let $\mathcal{T} = (T, t)$ be a tournament whose cycle decomposition contains exactly two strong tournaments C_1, C_2. Let M be a congruence class such that $M \cap C_i \neq \emptyset$, $i = 1, 2$. Then $M = T$.*

Proof. We proof $M \supseteq C_1$. Suppose that, on the contrary, $C_1 - M \neq \emptyset$. As C_1 is a congruence class, it follows by Lemma 3 that C_1 is not strong. Since $T = C_1 \cup C_2$, we have $M = T$.

Proposition 9. *Let $\mathcal{T} = (T, t)$ be a tournament. Then the following two statements are equivalent:*

(1) $\rho \in I(\mathcal{T})$;

(2) $\rho = K \times K \cup \Delta_T$ *for a set* $K \subseteq T$, $|K| \geqslant 2$, *and the cycle decomposition of* $\mathcal{T}|_K$ *has at most two elements.*

Proof. (1) $\Rightarrow$ (2).

(a) ρ has only one non-trivial congruence class, for otherwise

$$\rho = \{K_i \times K_i \cup \Delta_T : i = 1, \ldots, k\},$$

where K_i, $i = 1, \ldots, k$, $k \geqslant 2$, are congruence classes of ρ

(b) If ρ has only one non-trivial congruence class K, and $\mathcal{T}|_K$ has the cycle decomposition $C_1, \ldots, C_k$ for a $k \geqslant 3$, then we get again a contradiction with the irreducibility

$$\rho = [(C_1 \cup C_2)^2 \cup \Delta_T] \vee [(C_2 \cup C_3 \cup \ldots \cup C_k)^2 \cup \Delta_T] .$$

(2) ⇒ (1).

(a) First, suppose that $\mathcal{T}|_K$ is a strong tournament. Then

$$\bigvee\{\sigma \in E(\mathcal{T}): \sigma \subsetneq K \times K \cup \Delta_T\} = \eta \cup \Delta_T \neq K \times K \cup \Delta_T,$$

where η is the maximal congruence on the tournament $\mathcal{T}|_K$.

(b) Secondly, suppose the cyclic decomposition of $\mathcal{T}|_K$ contains exactly two strong tournaments C_1 and C_2. But then

$$\bigvee\{\sigma \in E(\mathcal{T}): \sigma \subsetneq K \times K \cup \Delta_T\} = C_1 \times C_1 \cup C_2 \times C_2 \cup \Delta_T \neq \rho$$

(see Lemma 4).

Definition. Let $O = (X, \leqslant)$ be a partial ordering. Define $n(O) = (X, n(\leqslant))$ by $(x, y) \in n(\leqslant)$ iff $x < y$, and there exists no $z \in X$ such that $x < z < y$ (i.e., iff y covers X). We put $(x, y) \in n(\mathcal{T})$ if $x, y \in I(\mathcal{T})$ and $(x, y) \in n(O)$ for the partial ordering $O = (I(\mathcal{T}), \supseteq)$.

Lemma 5. *Given a tournament $\mathcal{T} = (T, t)$, let $M, N, P, \in I(\mathcal{T})$ be mutually distinct (this is to be understood in the sense that M, N, P are strong congruence classes corresponding to three irreducible congruences of $\mathcal{T}$, see Proposition 9) and suppose that $(M, P) \in n(\mathcal{T})$ and $(N, P) \in n(\mathcal{T})$. Then $\mathcal{T}|_{M \cup N}$ has the cycle decomposition C_1, C_2, C_3, where $C_2 = P$, $C_1 \cup C_2 = M$ and $C_2 \cup C_3 = N$.*

Proof. It follows from the definition that $M \cap N \supseteq P \neq \emptyset$ and $M \not\supseteq N$, $N \not\supseteq M$. It follows from Lemma 3 that $\mathcal{T}|_M$ is not a strong tournament, and since M is an irreducible congruence class, the cyclic decomposition of $\mathcal{T}|_M$ consists of exactly two parts C_1', C_2'. Similarly, one can prove that the cyclic decomposition of $\mathcal{T}|_N$ has precisely two parts C_3' and C_4'. If follows by Lemma 4 that we may assume $P \cap C_1' = \emptyset$, and hence $P \subseteq C_2'$; similarly, we may take $P \subseteq C_3'$. Hence C_2' is an irreducible congruence class, $M \supseteq C_2' \supseteq P$ and $(M, P) \in n(\mathcal{T})$, so that $C_2' = P$. It follows in the same way that $C_3' = P$. The rest of the statement is clear.

Quite analogously one can prove:

Lemma 6. *Let $M, N \in I(\mathcal{T})$, $M - N \neq \emptyset \neq N - M$, $M \cap N \neq \emptyset$. Then $\mathcal{T}|_{M \times N}$ has the cycle decomposition C_1, C_2, C_3, and $M = C_1 \cup C_2$, $N = C_2 \cup C_3$.*

Let $O = (X, \geqslant)$ be a partially ordered set. Put

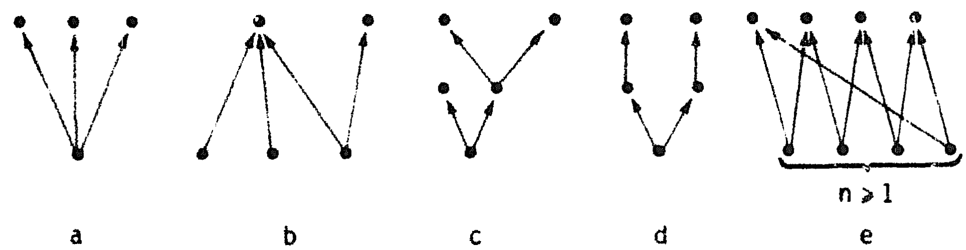
$$\max(O) = \{y \in X: y \leqslant x \Rightarrow x = y\},$$

$$\max^{-1}(O) = \{y \in X: y < x \Rightarrow x \in \max(O)\} - \max(O).$$

We can now prove:

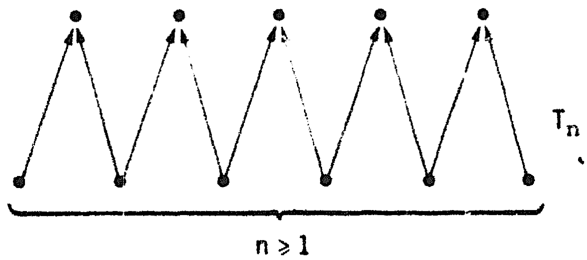
Theorem A. *Let $O = (X, \geqslant)$ be a partially ordered set. Then O is isomorphic to $(I(\mathcal{T}), \supseteq)$ for a tournament $\mathcal{T}$ iff*

(A1) *$n(O)$ does not contain the following subgraphs:*



for every $n > 1$ (arrows lead from smaller to bigger elements).

(A2) *The partial ordering induced on $\max(O) \cup \max^{-1}(O)$ by $\leqslant$ is isomorphic either to the graph T_n or to a graph which arises from it by deletion of some of the lower vertices.*



Proof. I. *Necessity*:

(1.a) Let $\mathcal{T} = (T, t)$ be a tournament such that there are $M, N, P, Q \in I(\mathcal{T})$ with $(M, Q) \in n(\mathcal{T})$, $(N, Q) \in n(\mathcal{T})$, $(P, Q) \in n(\mathcal{T})$. According to Lemma 5 there are strong subtournaments C, C_1, C_2, C_3 such that $Q = C$, $M = C \cup C_1$, $N = C \cup C_2$, $P = C \cup C_3$. Choose $x \in C$, $x_i \in C_i$, $i = 1, 2, 3$. But then $(x_1, x) \in t$ implies $(x, x_2) \in t$ and $(x, x_3) \in t$, which is a contradiction against Lemma 5 used for the triple N, P, Q.

(1.b) Let $\mathcal{T} = (T, t)$ be a tournament which violates (1.a): there are

$P, Q, R, M, N \in I(\mathcal{T})$ such that $\{(P, M), (Q, M), (R, M), (R, N)\} \subset n(\mathcal{T})$. By Lemma 5, there are strong subtournaments C_1, C_2, C_3 such that $M = C_1 \cup C_2$, $N = C_2 \cup C_3$ and $R = C_2$, and these unions are cyclic decompositions. We prove $P \subseteq C_1$; according to the definition of $n(\mathcal{T})$, it is impossible that $P \subseteq C_2$. But if $P \cap C_i \neq \emptyset$, $i = 1, 2$, then by Lemma 4, $P = M$. Hence necessarily $P \subseteq C_1$. As C_1 determines an irreducible congruence, we have $P = C_1$ by the definition of $n(\mathcal{T})$. Quite analogously it can be proved that $Q = C_1 = P$. This is a contradiction.

(1.c) Let $\mathcal{T} = (T, t)$ be again a tournament with the following properties: there are $M, N, O, P, Q \in I(\mathcal{T})$ such that $\{(M, P), (N, P), (O, Q), (P, Q)\} \subset n(\mathcal{T})$. Using Lemma 5 for the triple M, N, P, we get that $T|_P$ is a strong tournament, while by applying the same lemma to the triple O, P, Q we obtain that $T|_P$ fails to be a strong tournament, a contradiction.

(1.d) Let $\mathcal{T} = (T, t)$ be a tournament with irreducible congruences P, M_1, M_2, N_1, N_2 such that $\{(M_1, P), (N_1, P), (M_2, M_1), (N_2, N_1)\} \subset n(\mathcal{T})$. By Lemma 5, there are strong subtournaments C_1, C_2, C_3 such that $N_1 = C_1 \cup C_2$, $M_1 = C_2 \cup C_3$, $P = C_2$. We prove $N_2 \cap C_3 \neq \emptyset$. Assume $N_2 \cap C_3 = \emptyset$. Fix $x_3 \in C_3$, $x \in N_2$, let e.g. $(x, x_3) \in t$. By Lemma 2, C_3 is a congruence class in $\mathcal{T}$ and hence $(x', x_3) \in t$ for every $x' \in N_2$. Choose $y_1 \in C_1$, $y_2 \in C_2$, $y_3 \in N_2 - N_1$ arbitrarily. As M_1, N_1 are congruence classes, $(y_3, x_3) \in t$, we get $(y_3, y_1) \in t$ and $(y_3, y_2) \in t$. Consequently, $(y_1, y_2) \in t$. As y_1, y_2, y_3 were arbitrarily chosen, $\mathcal{T}|_{N_2}$ has at least three components in its cyclic decomposition, viz., $C_2, C_1, N_2 - N_1$. This is a contradiction (Proposition 9).
$(P, Q)\} \subset n(\ )$. Using Lemma 5 for the triple M, N, P, we ge that $T\ _P$ is a strong tournament, while by applying the same lemma to the triple O, P, Q we obtain that $T\ _P$ fails to be a strong tournament, a contradiction.

$$(N_2, K_1) \in n(\mathcal{T}),$$

$$(K_i, K_{i+1}) \in n(\mathcal{T}), \qquad i = 1, \ldots, k-1,$$

$$(K_k, M_1) \in n(\mathcal{T}).$$

Then necessarily $K_k = M_2$ (by (1.c)), hence $M_2 \subseteq N_2$. Using the same argument for M_2, C_1, we get $M_2 \subseteq N_2$, a contradiction.

(1.e) Let $\mathcal{T} = (T, t)$ be a tournament which violates (1.e): there are $M_1, \ldots, M_n$ and $N_1, \ldots, N_n \in I(T)$ such that $(M_i, N_i) \in n(\mathcal{T})$ and $(M_{i+1}, N_i) \in n(\mathcal{T})$ for $i = 1, \ldots, n$, $n \geqslant 2$ (the subscripts are taken mod n).

By a subsequent application of Lemma 5, we get that there are strong subtournaments $C_1, \ldots, C_n$ such that $N_i = C_i$ and $M_{i+1} = C_i \cup C_{i+1}$, $i = 1, \ldots, n$, and these unions are the cycle decompositions of M_i. Choose $x_i \in C_i$, $i = 1, \ldots, n$, and let $(x_1, x_2) \in t$. As $M_3, \ldots, M_n$ are congruence classes, we get $(x_1, x_i) \in t$, $i = 2, \ldots, n$. But by Lemma 5, we have $\{(x_2, x_3), (x_3, x_4), \ldots, (x_n, x_1)\} \subseteq t$, a contradiction.

(2) Let $|\max(I(\mathcal{T}), \supseteq)| \geqslant 2$. Then $T^2 \notin I(\mathcal{T})$, and hence $\mathcal{T}$ is not a strong tournament. Let $C_1, \ldots, C_n$ be the cycle decomposition of $\mathcal{T}$. Then

$$\{C_i \cup C_{i+1} : i = 1, \ldots, n-1\} = \max(I(\mathcal{T}), \supseteq),$$

$$\{C_i : i = 1, \ldots, n, |C_i| \geqslant 2\} = \max^{-1}(I(\mathcal{T}), \supseteq).$$

Hence we obtain condition (A2) (here we delete a vertex in $\max^{-1}(I(\mathcal{T}), \supseteq)$ whenever C_i is the trivial tournament).

II. *Sufficiency*:

Let $O = (X, \geqslant)$ be a partial ordering satisfying (A1), (A2).

$$G_1 = \max(O), \qquad G_2 = \max^{-1}(O),$$

$$G_i = \left\{ x \in X - \bigcup_{j=1}^{i-1} G_j : \text{there exists } y \in G_{i-1} \text{ and } (y, x) \in n(O) \right\},$$

$$k(O) = \max\{i : G_i \neq \emptyset\}.$$

We prove the statement by induction on $k(O)$.

If $k(O) = 1$, then $O \simeq (I(\mathcal{T}), \supseteq)$ for the tournament $\mathcal{T} = (T, t)$, where $|T| = |G_1| + 1$, and t is a chain on T. If $|G_1| = 1$, then also very simple tournament represents O.

Supposing that the statement holds for every O with $k(O) \leqslant k - 1$, we prove it for k.

(a) Let $|G_1| = 1$. Define the relation ϵ on G_2 by: $(x, y) \in \epsilon \Leftrightarrow$ there exists $z \in G_3$, $(x, z) \in n(O)$ and $(y, z) \in n(O)$. Let $\bar{\epsilon}$ be the smallest equivalence on G_2 containing ϵ, and $[x]$ the $\bar{\epsilon}$-equivalence class containing x. Put

$$[\bar{x}] = \{y \in X : z \geqslant y \text{ for some } z \in [x]\}.$$

Then every partial ordering $([\bar{x}], \geqslant)$ satisfies conditions (A1), (A2), and as $k([\bar{x}], \geqslant) \leqslant k - 1$, there exists a tournament $\mathcal{T}_{[x]}$ such that $I(\mathcal{T}_{[x]}, \supseteq) \simeq ([\bar{x}], \geqslant)$. Let $|G_2/\bar{\epsilon}| = n$. Let $\mathcal{S} = (S, s)$ be a simple tournament with more than n vertices. Let $\phi : G_2/\bar{\varepsilon} \to S$ be one-one. Put

$$T = \bigcup\{T_{[x]} : x \in G_2\} \cup S - \phi(G_2/\bar{\varepsilon})\,.$$

Define the relation t on T: $(x, y) \in t$ iff either $(x, y) \in t_{[z]}$ for some $z \in G_2$,

or

$$x \in T_{[z]}, \quad y \in T_{[u]}, \quad [z] \neq [u] \quad \text{and} \quad (\phi[z], \phi[u]) \in s,$$

or

$$x \in T_{[z]}, \quad y \in S - \phi(G_2/\bar{\varepsilon}) \quad \text{and} \quad (\phi[z], y) \in s,$$

or

$$x, y \in S - \phi(G_2/\bar{\varepsilon}) \quad \text{and} \quad (x, y) \in s.$$

It is easy to prove that $(I(\mathcal{T}), \supseteq) \simeq O$ and $\mathcal{T} = (T, t)$ is a strong tournament.

(b) Let $|G_1| > 1$. Take $x, y \in G_2$, $x \neq y$. Then $\{x\} \cap \{y\} = \emptyset$. By induction hypothesis there are strong tournaments $\mathcal{T}_x$, $x \in G_2$, such that $I(\mathcal{T}_x, \geqslant) \simeq (\{\bar{x}\}, \geqslant)$. Let A be a set disjoint from G_2, $|A| = |G_1| + 1 - |G_2|$. For $a \in A$ let $\mathcal{T}_a$ be the trivial one-point tournament. Choose a total order $<$ of $G_2 \cup A$, and define the tournament $\mathcal{T} = (T, t)$ with $T = \bigcup\{T_x : x \in G_2 \cup A\}$ by setting $(x, y) \in t$ iff either $x, y \in T_z$ and $(x, y) \in t_z$ or $x \in T_a$, $y \in T_b$ and $a < b$. It follows again easily that there exists an ordering $<$ such that $(I(\mathcal{T}), \geqslant) \simeq O$ (see the proof of the necessity of (A2)).

As a consequence of the above proof we obtain:

Corollary. *$\mathcal{T}$ is not a strong tournament if* $|\max(O)| \geqslant 2$. *$\mathcal{T}$ is a strong tournament if* $|\max(O)| = 1$ *and* $|\max^{-1}(O)| > 2$. *If* $|\max(O)| = 1$ *and* $|\max^{-1}(O)| \leqslant 2$, *then $\mathcal{T}$ can be chosen both strong or not strong.*

B: $E(\mathcal{T})$.

Since all the lattices considered here are finite, the following lemma is well known:

Lemma 7. $x = \bigvee\{y \in I(\mathcal{T}) : y \leqslant x\}$ *for every* $x \in E(\mathcal{T})$.

Lemma 8. *Let $\mathcal{T} = (T, t)$ be a tournament, $\rho, \rho_1, \rho_2, \ldots, \rho_k \in I(\mathcal{T})$. If $\bigvee\{\rho_i : i = 1, \ldots, k\} \geqslant \rho$, there exists an i such that $\rho_i \geqslant \rho$.*

Proof. By way of contradiction let us assume that $\rho_i \not\geq \rho$, $i = 1, \ldots, k$, and that $\sigma = \bigvee\{\rho_i : i = 1, \ldots, k\} \geq \rho$. Let ρ be determined by the congruence class K (Proposition 9). We distinguish two cases.

(a) $\mathcal{T}|_K$ is a strong tournament. Let η be the simple decomposition of $\mathcal{T}|_K$. Choose an i, $1 \leq i \leq k$. Then $\rho_i \cap K \times K \subseteq \eta$, and also $\rho_i \subseteq K \times K \cup (T-K) \times (T-K)$, for if there exists a congruence class L of $\mathcal{T}$ with $L \cap K \neq \emptyset$ and $L - K \neq \emptyset \neq K - L$, then $\mathcal{T}|_K$ is not a strong tournament by Lemma 3. Hence

$$\sigma \subseteq \eta \cup (T-K) \times (T-K),$$

and consequently,

$$\sigma \subseteq K \times K \cup \Delta_T,$$

a contradiction.

(b) $\mathcal{T}|_K$ is not strong and has the cyclic decomposition C_1, C_2 with $(x_1, x_2) \in t$ for all $x_1 \in C_1$, $x_2 \in C_2$. As $\sigma \geq \rho$, there are irreducible congruence classes (Proposition 9) K_j, $j = 1, \ldots, n$, such that $K_1 \cap C_1 \neq \emptyset$, $K_j \cap K_{j+1} \neq \emptyset$, $j = 1, \ldots, n-1$, and $K_n \cap C_2 \neq \emptyset$. We can further assume that n is the smallest natural number with this property. By Lemma 6, there are disjoint strong subtournaments C_i, $i = 3, \ldots, n+1$, such that $K_1 = C_1 \cup C_3$, $K_{i-1} = C_i \cup C_{i+1}$, $i = 3, \ldots, n$, $K_n = C_{n+1} \cup C_2$. As C_i, $i = 1, \ldots, n+1$, are irreducible congruence classes we have a contradiction against Theorem A, (1.e).

Lemma 9. *Let $\mathcal{T} = (T, t)$ be a tournament, $\rho \in E(\mathcal{T})$. Then the set M of irreducible congruences satisfying*

(1) $\bigvee\{\epsilon : \epsilon \in M\} = \rho$,

(2) $\epsilon' \leq \epsilon, \epsilon \in M \Rightarrow \epsilon' \in M$

is uniquely determined, and $M = \{\epsilon \in I(\mathcal{T}) : \epsilon \leq \rho\}$.

This follows directly from Lemma 8.

It is well known that the properties given in Lemma 8 and Lemma 9 are equivalent to the distributivity of the lattice $E(\mathcal{T})$. Thus we finally have:

Theorem 3. *Let L be a finite lattice. Then the following two statements are equivalent:*

(1) *$L \simeq E(\mathcal{T})$ for a tournament $\mathcal{T}$.*

(2) *L is distributive and $I(\mathcal{T})$ satisfies conditions* (A1) *and* (A2) (*Theorem A*).

The finite distributive lattices satisfying (A1) and (A2) will be called *admissible*.

Remark. Condition (A2) shows that one cannot describe the class of all congruence lattices of tournaments by giving a list of forbidden sublattices.

4. Tournaments with given group of automorphisms and given lattice of congruences

In the proof of our main theorem the principal role is played by the following:

Proposition 10. *For any group G of odd order there exists a simple tournament $\mathcal{T}$ such that $A(\mathcal{T}) \simeq G$.*

We shall prove this using the classical Cayley technique, see e.g. [8]. Let $G = \{g_1, \ldots, g_n\}$ be the given odd group, and let $H = \{g_1, \ldots, g_k\}$ be a fixed minimal system of generators of G. Furthermore, fix any total order $\leqslant$ of G.

Define a relational system $\mathcal{G} = (G; \{R_i: i = 0, 1, \ldots, k\})$ by

$$R_i = \{(g, g_i g): g \in G\} \cup \{(g, g): g \in G\}, \qquad i = 1, \ldots, k\ ,$$

$$R_0 = \{(g, h): g^{-1}h > h^{-1}g\} \cup \{(g, g): g \in E\}.$$

Then

$$R_i \cap R_i^{-1} = \Delta_G\ , \qquad R_i \cap R_j = \Delta_G \quad \text{for } i \neq j\ ,$$

$$\bigcup\{R_i: i = 1, \ldots, k\} = G \times G\ ,$$

as can easily be shown using that G is odd and the definition of R_i.

Lemma 10. *Denote by $H(\mathcal{G})$ the monoid of all endomorphisms of the relational system $\mathcal{G}$ (i.e., the mappings $G \to G$ preserving simultaneously all the relations of $\mathcal{G}$). Then the group-part of $H(\mathcal{G})$ is isomorphic to G, and the only endomorphisms which are not automorphisms are constants.*

Proof. The statement about automorphisms is proved in [8].

Let F be an endomorphism of $\mathcal{G}$ for which $F(G) \neq G$. We show that if F is non-constant, then it is one-one (and hence belongs to the group-part of $H(\mathcal{G})$). Suppose $F(g) = F(g') \neq F(h)$. If $(F(g), F(h)) \in R_i$ for some $i > 0$, then $(g, h) \in R_i$, $(g', h) \in R_i$. Since each R_i, $i > 0$, is a one-one mapping $G \to G$, we have $g = g'$. Similarly, we can handle the case $(F(h), F(g)) \in R_i$, $i > 0$.

Since $g_1, ..., g_k$ are generators, for every $f \in G$, $F(f) \neq F(g)$, there is a sequence $g_{i(1)}, ..., g_{i(p)}$ of elements of H with

$$(g, g_{i(1)}) \in R_{i(1)} \cup R_{i(1)}^{-1}, ..., (g_{i(p-1)}, g_{i(p)}) \in R_{i(p-1)} \cup R_{i(p-1)}^{-1},$$

$$(g_{i(p)}, f) \in R_{i(p)} \cup R_{i(p)}^{-1}.$$

This proves the lemma.

Proof of Proposition 10. Let G be an odd group. Let $\mathcal{G} = (G; \{R_i: i = 0, 1, ..., k\})$ be the relational system from Lemma 10. Let $\mathcal{T}_i = (T_i, t_i)$, $i = 0, ..., k$, be fixed rigid tournaments such that $2|G| < |T_i|$, and $|T_i| < |T_j|$ whenever $i < j$. For each $i = 0, ..., k$, choose a vertex $a_i \in T_i$.

Define the tournament $\mathcal{S} = (S, s)$ as follows:

$$S = G \cup \{\overline{R}_i \times T_i: i = 0, ..., k\}, \quad \text{where } \overline{R}_i = R_i \setminus \Delta_G .$$

For the definition of the relation s we consider five cases:

(i) $s|_{G \times G} = \bigcup\{R_i: i = 0, 1, ..., k\}$.

(ii) $a \in \overline{R}_i \times T_i$, $b \in \overline{R}_j \times T_j$, $i \neq j$, then $(a, b) \in s$ iff $i < j$.

(iii) $a, b \in \overline{R}_i \times T_i$, $a = ((x, y), c)$, $b = ((x', y'), c')$. Then $a, b \in s$ iff either $x \neq x'$, $(x, x') \in s$ or $x = x'$, $y \neq y'$, $(y, y') \in s$ or $x = x'$, $y = y'$, $(c, c') \in t_i$.

(iv) $a \in G$, $b = ((y, z), c) \in R_i \times T_i$, $a \in \{y, z\}$. Then $(a, b) \in s$ iff either $y = a$, $c \neq a_i$ or $z = a$, $c = a_i$; $(b, a) \in s$ iff either $y = a$, $c = a_i$ or $z = a$, $c \neq a_i$.

(v) $a \in G$, $b = ((y, z), c) \in \overline{R}_i \times T_i$, $a \notin \{y, z\}$. Then $(a, b) \in s$ iff $(a, z) \in s$.

First, we prove that $\mathcal{S}$ *is a simple tournament*. In the following, let M, with $|M| > 1$, be a congruence class on $\mathcal{S}$, $M \neq S$.

(1) $|M \cap ((x, y) \times T_i)| \leqslant 1$ for every i and $(x, y) \in s$. For if $|M \cap ((x, y) \times T_i)| > 1$, then $M \supset \{(x, y) \times T_i\} \cup \{x, y\}$ as $\mathcal{T}_i$ is simple and because of (iv). Moreover, there exists an $a \in (x', y') \times T_i \cap (S \setminus M)$; otherwise $S = M$. Now if there exists $((x', y'), b) \in M$, then

$$\{((x', y'), b)\} = M \cap (x', y') \times T_i ,$$

and we get a straightforward contradiction with (iii). Considering (iii), (iv), it is now easy to find that $c, d, e \in M$ such that $\{(e, c), (d, e)\} \subset s$, a contradiction.

(2) $|M \cap G| \leqslant 1$. If x and y are two different elements of $M \cap G$, then by (iv) either $((x, y), a_i) \in M$ or $((y, x), a_i) \in M$, and consequently $(x, y) \times T_i \subset M$, which is impossible by (1).

Let $x \in M \cap \bar{R}_i \times T_i$, $y \in M \cap \bar{R}_j \times T_j$, $i \neq j$. Then it follows from (ii), (iii) and (1) that $M = S$. It is also easy to see that $|M \cap \bar{R}_i \times T_i| \leqslant 1$. The remaining case $|M \cap G| = 1$, $|M \cap \bar{R}_i \times T_i| = 1$ is also impossible. Hence is a simple tournament.

Secondly, we prove $A(\mathcal{S}) \simeq G$. Let $f \in A(\mathcal{S})$. By Lemma 10, it suffices to prove $f(G) = G$ and $f((x, y) \times T_i) = (f(x), f(y)) \times T_i$.

Claim. For every $x \in G$ and every $i = 0, 1, \ldots, k$, there exists $x \neq y \in G$ such that $(y, x) \in R_i$ ($(x, y) \in R_i$, respectively). For $i > 0$, this is clear. For $i = 0$, note that since G is an odd group, $x^{-1}y \neq y^{-1}x$ for any $y \in G$, $y \neq x$. Hence either $(x, y) \in R_0$ or $(y, x) \in R_0$. This establishes the claim.

Let $a \in \bar{R}_0 \times T_0$. Then

$$d^+(a, \mathcal{S}) > |R_0|\,|T_0| + |G|\,,$$

while

$$d^+(b, \mathcal{S}) > |\bar{R}_0|\,|T_0| + |T_1|$$

for every $b \in \bar{R}_i \times T_i$, $i > 0$. Taking a sufficiently large $\mathcal{T}_1$ (see Section 2), we have

$$f(\bar{R}_0 \times T_0) \subset G \cup (\bar{R}_0 \times T_0)\,.$$

Let $a \in \bar{R}_i \times T_i$, $i \geqslant 1$, and let $f(\bar{R}_j \times T_j) \subset G \cup (\bar{R}_j \times T_j)$ for all $j < i$. Then

$$d^+(a, \mathcal{S}) \leqslant |G| + \sum \{|\bar{R}_j \times T_j| : i \geqslant j\}\,,$$

while

$$d^+(b, \mathcal{S}) > \sum \{|\bar{R}_j|\,|T_j| : j < k\} + |T_k|$$

for every $b \in \bar{R}_k \times T_k$, $k > i$. Thus, if we choose the cardinalities of T_i appropriately, we may easily obtain

$$f(\bar{R}_i \times T_i) \subset G \cup (\bar{R}_i \times T_i)\,.$$

Moreover, $f(G) \subset G$ as $f(g) \in \bar{R}_i \times T_i$ implies $(f(g), f(a)) \in s$ ($(f(a), f(g)) \in s$, respectively) for at least $|T_j|$ vertices a from each $(x, y) \times T_j$, $j > i$ ($i < j$, respectively) (see (ii), (iv)). Further,

$f((x, y) \times T_i) = (x', y') \times T_i$ as the tournament $\mathcal{S}|_{S \setminus G}$ fails to be simple. From (iv) and (v) it is easy to prove that $f((x, y) \times T_i) = (f(x), f(y)) \times T_i$.

We now turn our attention to the lattice of all congruences of a tournament.

Proposition 11. *Let L be an admissible lattice. Then there exists a tournament $\mathcal{T}$ such that $E(\mathcal{T}) \simeq L$ and $A(\mathcal{T}) = \{1_T\}$.*

In fact this follows from the sufficiency part of the proof of Theorem A. There we proved by induction the existence of a tournament $\mathcal{T}$ with $E(\mathcal{T}) \simeq L$ for a given admissible lattice L. The induction step involves sufficiently large simple tournaments and orderings. There exist arbitrarily large rigid tournaments (Section 2), and every ordering has the identity as its only automorphism. From this Proposition 11 follows.

In the proof of our main theorem we make use of the following "doubling" construction. Let $\mathcal{T} = (T, t)$ be a tournament. Denote by $2\mathcal{T}$ the tournament with vertex-set $T \times \{0, 1\}$, and with the following set of arrows:

$$\{((x, i), (y, i)) \colon i = 0, 1, (x, y) \in t\} \cup \{((x, 1), (y, 0)) \colon x \neq y\} \\ \cup \{((x, 0), (x, 1)) \colon x \in T\}\ .$$

Proposition 12. *Let G be an odd group. Then there exists a tournament such that*

(1) $A(\mathcal{T}) \simeq A(2\mathcal{T}) \simeq G$;

(2) *both $\mathcal{T}$ and $2\mathcal{T}$ are simple tournaments.*

This follows from the construction of $2\mathcal{T}$ and the proof of Proposition 10. It suffices to take in the proof of Proposition 10 simple tournaments $\mathcal{T}_i$ with a large number of 3-cycles (i.e., with homogeneous degree sequence).

Theorem 4. *Let L be an admissible lattice, G an even group. Then there exists a tournament $\mathcal{T}$ with $A(\mathcal{T}) \simeq G$ and $E(\mathcal{T}) \simeq L$.*

Proof. The theorem will be proved by the following *triple-tournament* construction which allows us to use the results of Propositions 11 and 12 simultaneously.

Let $\mathcal{S}$, $\mathcal{T}$ be tournaments. Define the tournament $\mathcal{S} * 2\mathcal{T}$ on the set $S \cup (T \times \{0, 1\})$ by the following set of arrows:

$$s \cup t' \cup \{(y, (x, 0)) : y \in S, x \in T\} \cup \{((x, 1), y) : x \in T, y \in S\},$$

where t' denotes the set of arrows of $2\mathcal{T}$.

Now choose two tournaments $\mathcal{S}$ and $\mathcal{T}$ with the following properties: $E(\mathcal{S}) \simeq L$, $A(\mathcal{S}) = \{1_S\}$, $A(\mathcal{T}) \simeq A(2\mathcal{T}) \simeq G$, and $2\mathcal{T}$ are simple, and $|S| < |T|$. Then $E(\mathcal{S} * 2\mathcal{T}) \simeq E(\mathcal{S}) \simeq L$, since there is no non-trivial congruence class of $\mathcal{S} * 2\mathcal{T}$ which meets $T \times \{0, 1\}$.

Since

$$d^-(x, \mathcal{S} * 2\mathcal{T}) < d^-(y, \mathcal{S} * 2\mathcal{T})$$

for every $x \in T \times \{0\}$ and every $y \notin T \times \{0\}$, we have

$$A(\mathcal{S} * 2\mathcal{T}) \simeq A(2\mathcal{T}) \simeq G.$$

This completes the proof.

Remark. Using a simple induction argument one can prove:

Lemma 11. *Let $\mathcal{T}$ be a tournament. Then there is a simple tournament $\mathcal{S}$ which contains $\mathcal{T}$ as a subtournament.*

Hence with respect to the automorphisms and congruences there are globally no "forbidden parts".

Theorem 4'. *Given an admissible lattice L, an odd group G, and a tournament $\mathcal{T}$, there is a tournament $\mathcal{S}$ such that*

$$\mathcal{T} \leqslant \mathcal{S}, \quad E(\mathcal{S}) \simeq L \quad \textit{and} \quad A(\mathcal{S}) \simeq G.$$

5. Applications

5.1. Universal algebras

An n-ary operation ω which satisfies $\omega(x_1, \ldots, x_n) \in \{x_1, \ldots, x_n\}$ is called a quasitrivial algebra. Hence the tournaments are precisely the binary commutative quasitrivial algebras. Of course, they do not form a primitive class (variety). The question may be raised concerning the smallest primitive class T of algebras containing all finite tournaments. This is obviously the same as asking for all equations which tournaments satisfy. T clearly satisfies the equations $x \cdot x = x$, $x \cdot y = y \cdot x$, $x \cdot (x \cdot y) = x \cdot y$ and the equations derived from them. It can easily be proved that there are no other equations involving two variables only. However, we have found an infinite number of independent equations which T satisfies. Put

$$A_{1,k} = (\dots((x_1 x_2) x_3) \dots) x_k \,,$$

$$A_{n,k} = (\dots((A_{n-1,k}) x_1) \dots) x_k \,, \qquad n \geqslant 2 \,.$$

Then the equation $A_{n!+n,k} = A_{n,k}$ is satisfied by all tournaments with $\leqslant n$ vertices for every $k \geqslant 1$ (this follows from the periodicity of the sequence $A_{1,k}, A_{2,k}, \dots$). One can easily find an example of a tournament with more than n vertices which does not satisfy an equation $A_{n!+n,k} = A_{n,k}$ for a suitable k (e.g., $k = 2n! + 1$). Thus T is not generated by the tournaments with $\leqslant n$ vertices.

On the other hand, the equations $A_{n!+n,n} = A_{n,n}$, $n \geqslant 1$, are satisfied by every finite tournament. It can be proved easily that this set contains an infinite independent system of equations. The following question is unsolved: Can T be defined by a finite set of equations?

5.2. *Forcing of endomorphisms and automorphisms by degrees*

Let d be a degree sequence of a tournament. Denote by $[d]$ the set of all tournaments with the degree sequence d. We say that a property P is *forced by d* if $\mathcal{T}$ has P for every $\mathcal{T} \in [d]$. Theorem 1 (Section 2) may then be stated in the following way: there is no endomorphism except constants and automorphisms which may be forced by a strong degree sequence. We proceed to give a full discussion of the question of forcing of automorphisms and endomorphisms by degrees. The characterizations are quite simple.

First we show which non-identity automorphisms can be forced by a degree sequence (recall the trivial fact that a homogeneous tournament has a constant degree sequence).

Theorem 1′. *Let $\mathcal{T} = (T, t)$ be a tournament. Exactly one of the following cases must occur:*

(1) *There exists a tournament $\mathcal{S}$ such that $A(\mathcal{S}) = \{\mathrm{id}\}$, and $\mathcal{T}$ and $\mathcal{S}$ are degree-equivalent.*

(2) *$\mathcal{T}$ is the homogeneous tournament with 5 vertices.*

(3) *$\mathcal{T}$ is the homogeneous tournament with 3 vertices.*

(4) *$\mathcal{T}$ is not strong, and at least one of the components in the cycle decomposition of $\mathcal{T}$ satisfies either (2) or (3).*

Corollary. *The only groups which may be forced by a degree sequence are the identity and finite direct sums of cyclic groups of order 3 or 5.*

Outline of the proof. We prove (1) by induction of $|T| = n$. Clearly (2), (3), (4) give the complete solution of the case $|T| \leq 5$.

Let $n > 5$. If $\mathcal{T}$ is a homogeneous tournament, then (1) holds. If $\mathcal{T}$ is not strong and (4) does not hold, one can use the induction hypothesis for every cycle components of $\mathcal{T}$. Let $\mathcal{T}$ be strong. Let $T = \bigcup_{i=1}^{k} A_i$ be a disjoint union such that $d^-(x, \mathcal{T}) \neq d^-(y, \mathcal{T})$ for any $x \in A_i$, $y \in A_j$, $i \neq j$. We can then apply the induction hypothesis to $\mathcal{T}|_{A_i}$ (as $\mathcal{T}$ is strong, we can assume that $\mathcal{T}|_{A_i}$ is not a homogeneous tournament even when $|A| = 3, 5$).

Thus Theorems 1 and 1′ solve the question of forcing of a non-trivial automorphism and endomorphism. (The question of forcing of endomorphisms by a non-strong degree sequence is not interesting because it is a consequence of Theorem 1.)

We say that a degree sequence d is *forcibly identical* (FI) if $A(\mathcal{T}) = \{\mathrm{id}\}$ for every $\mathcal{T} \in [d]$. Similarly, d is called *forcibly simple* (FS) if $\mathcal{T}$ is a simple tournament for every $\mathcal{T} \in [d]$. We then have

FI-Theorem. *The following two statements are equivalent:*

(1) *d is an FI-degree sequence;*

(2) *no three elements of d are equal.*

FS-Theorem. *The following two statements are equivalent:*

(1) *d is an FS-degree sequence;*

(2) $d \in \{(0), (0, 1), (1, 1, 1), (2, 2, 2, 2, 2), (3, 3, 3, 3, 3, 3, 3)\}$.

In fact, the two theorems are related to each other, permitting us to give a proof from which both statements will follow at the same time.

Proof. Obviously, (2) ⇒ (1) in both theorems. The proof that (1) ⇒ (2) will be split into three parts. Throughout the proof let $d = (d_1, \ldots, d_n)$, $n > 2$, be a fixed degree sequence written in non-decreasing order.

(I) *There exists an* $i < n$ *such that*

$$d_{i+1} = d_i + 1 \quad or \quad d_{i-1} = d_i = d_{i+1}.$$

This is clear since

$$\sum_{i=0}^{k} 2 \cdot 2 \cdot i > \binom{2k+1}{2} \quad \text{for every } k.$$

(II) *Suppose that* $d_{i+1} = d_i + 1$ *for some* $i < n$. *Then there exists a tournament* $\mathcal{T} = (\{1, \ldots, n\}, t) \in [d]$ *such that* $d^-(i, \mathcal{T}) = d_i$ *and a congruence* $\eta \in E(\mathcal{T})$ *with* $\eta[i] = \{i, i+1\}$.

Proof. Let $\mathcal{S} = (\{1, \ldots, n\}, s)$ be a tournament with $d^-(i, \mathcal{S}) = d_i$, $i = 1 \ldots n$. We can write

$$\{1, \ldots, n\} - \{i, i+1\} = V \cup P \cup A \cup S,$$

where

$$V = \{x\colon \{(x, i), (x, i+1)\} \subset s\}, \qquad P = \{x\colon \{(i, x), (i+1, x)\} \subset s\},$$

$$A = \{x\colon \{(i, x), (x, i+1)\} \subset s\}, \qquad S = \{x\colon \{(x, i), (i+1, x)\} \subset s\}.$$

Clearly, $|A| = |S|$. Let $f\colon A \to S$ be one-one. Put $A = \{a_1, \ldots, a_k\}$.

Suppose first that $(i+1, i) \in s$. Define

$$g(a_j) = i \qquad \text{if } (a_j, f(a_j)) \in s,\ j \leqslant k,$$

$$g(a_j) = i + 1 \quad \text{if } (f(a_j), a_j) \in s,\ j \leqslant k.$$

Then $(a, f(a), g(a))$ is a 3-cycle in $\mathcal{S}$ for every $a \in A$, and, moreover, $a \neq a'$ implies that $(a, f(a), g(a))$ and $(a', f(a'), g(a'))$ are arrow-distinct 3-cycles. Now it is easy to prove that the tournament

$$\mathcal{T} = (a_k, f(a_k), g(a_k)) \ldots (a_2, f(a_2), g(a_2))(a_1, f(a_1), g(a_1))\mathcal{S}$$

(see the proof of Theorem 1) satisfies (II).

Secondly, if $(i, i+1) \in s$, there exists a cycle in $\mathcal{S}$ containing $(i, i+1)$, and hence we can suppose $(i+1, i) \in s$.

(III) *Let* d *be a strong degree sequence not belonging to*

$$\{(1,1,1), (2,2,2,2,2), (3,3,3,3,3,3,3), (2,2,2,3,3,3)\}.$$

Suppose that $d_{i-1} = d_i = d_{i+1}$ for some $i < n$. Then there exists a tournament $\mathcal{T} = (\{1, ..., n\}, t)$ such that $d^-(j, \mathcal{T}) = d_j$, $1 \leq j \leq n$, and a congruence $\eta \in E(\mathcal{T})$ with $\eta[i] = \{i-1, i, i+1\}$.

Proof. Let $\mathcal{S} = (\{1, ..., n\}, s)$ be a tournament with $d^-(j, \mathcal{S}) = d_j$. We can assume that $\{(i, i+1), (i+1, i-1), (i-1, i)\} \subset s$. We can then write

$$\{1, ..., n\} - \{i-1, i, i+1\} = A \cup B \cup P_0 \cup P_1 \cup P_2 \cup V_0 \cup V_1 \cup V_2,$$

where

$$A = \{x: \{(x, i), (x, i-1), (x, i+1)\} \subset s\},$$

$$B = \{x: \{(i, x), (i-1, x), (i+1, x)\} \subset s\},$$

$$P_0 = \{x: \{(x, i-1), (i, x), (i+1, x)\} \subset s\};$$

$$P_1 = \{x: \{(i-1, x), (x, i), (i+1, x)\} \subset s\};$$

$$P_2 = \{x: \{(i-1, x), (i, x), (x, i+1)\} \subset s\};$$

$$V_0 = \{x: \{(i-1, x), (x, i), (x, i+1)\} \subset s\};$$

$$V_1 = \{x: \{(x, i-1), (i, x), (x, i+1)\} \subset s\};$$

$$V_2 = \{x: \{(x, i-1), (x, i), (i+1, x)\} \subset s\}.$$

Clearly it suffices to prove that

$$m(\mathcal{S}) = \sum_{i=0}^{2} |P_i| + |V_i| = 0.$$

Let us suppose that $\mathcal{S}$ has the property that $m(\mathcal{S}) > 0$ is minimal among all tournaments with $d^-(j, \mathcal{S}') = d_j$ (this does not contradict the assumption that $(i-1, i, i+1)$ is a 3-cycle). From the minimality we have:

(1) $(x, y) \in s$ for every $x \in P_i$, $y \in V_i$, $i = 0, 1, 2$;

(2) there exists exactly one k such that $V_k \neq \emptyset$ and $P_k \neq \emptyset$.

Assume for example, $V_0 \neq \emptyset$, $P_0 \neq \emptyset$. Then obviously $P_0 = V_0$. Suppose $|V_0| \geq 2$, and let $x \in P_0$, $y \in V_0$. Then (i, x, y) is a 3-cycle, and the tournament $\mathcal{S}' = (i, x, y)\mathcal{S}$ does not satisfy (2), and consequently $m(\mathcal{S})$ is not minimal. Hence $|V_0| = |P_0| = 1$. Put $V_0 = \{v\}$, $P_0 = \{p\}$. Consider the tournament $\mathcal{S}|_{\{1,...,n\}-\{i-1,i,i+1\}} = \tilde{\mathcal{S}}$. $\tilde{\mathcal{S}}$ does not contain a cycle containing p and v, for otherwise chasing of a triangle in $\mathcal{S}$ would yield a tournament $\mathcal{S}'$ which violates (1), and hence $m(\mathcal{S}')$ would be not minimal. We distinguish two cases.

(a) There are vertices x, y of $\tilde{\mathcal{S}}$ such that $\{(v,x),(v,y),(x,y)\} \subset s$. But then we may assume $y \in A$ ($\mathcal{S}$ being a strong tournament), and we may consider the tournament $(y,i,p)(v,y,i-1)\mathcal{S}$ instead of $\mathcal{S}$ and we get a cycle in $\tilde{\mathcal{S}}$.

(b) There are vertices x, y of $\tilde{\mathcal{S}}$ such that $\{(y,x),(y,p),(x,p)\} \subset s$. This case can be handled similarly as (a).

Further it is clear that we must have either (a) or (b) whenever $n \geqslant 8$. Thus let $5 \leqslant n \leqslant 7$, and suppose that neither (a) nor (b) holds. Then the only possibilities for d are the sequences $(2,2,2,2,2), (2,2,2,3,3,3)$, $(3,3,3,3,3,3,3)$. This proves (III) and the FI-Theorem. The FS-Theorem follows by proving that $(3,3,3,3,3,3,3)$ is a forcibly simple degree sequence. This is easy to check.

Remark on matrices. Tournaments are precisely those 0, 1-matrices which are antisymmetric: $a_{ij} = 0 \Leftrightarrow a_{ji} = 1$, $a_{ii} = 1$. Every non-trivial homomorphism represents a decomposition of a matrix into blocks. The FS-Theorem asserts that with precisely four exceptions for any antisymmetric 0, 1-matrix $(a_{i,j})_{i,j=1}^{n}$ there exists an antisymmetric 0, 1-matrix $(b_{i,j})_{i,j=1}^{n}$ such that

(1) $\Sigma_j a_{i,j} = \Sigma_j b_{i,j}$;

(2) $(b_{i,j})$ can be decomposed into at least 2 blocks.

Remark on undirected graphs. It follows easily from the algorithm on degree sequences of (undirected) graphs (see [4]) that there are no non-trivial degree sequences of graphs which force the identity. This aspect of forcing being simpler for ordinary graphs than for tournaments it seems a very difficult question to characterize the degree sequences of asymmetric graphs. (In [4] these graphs are called identity graphs.)

5.3. Representation of monoids

Consider the monoid $H(\mathcal{T})$ of all endomorphisms of a tournament. The fact that a given abstract monoid M is isomorphic to $H(\mathcal{T})$ (i.e., that M has a representation) is very closely related to the realization of a certain permutation monoid. Put

$$C = \{a \in M\colon ab = a \text{ for every } b \in M\}.$$

Let $(M, \{L_a\colon a \in M\})$ be the regular representation of M by left translations. Obviously, $L_a(C) \subset C$ for every $a \in M$. Consider the permutation monoid $(C, \{L_a^C\colon a \in M\})$, where L_a^C is the restriction of L_a to C.

It is clear that $H(\mathcal{T}) \simeq M$ if and only if $H(\mathcal{T}) = (C, \{L_a^C : a \in M\})$. Thus it seems to be a hard question to characterize those monoids which can be represented as monoids of endomorphisms of a certain tournament. Nevertheless, one can say more about the structure of representable monoids. Let $H'(\mathcal{T}) = H(\mathcal{T}) - C$. Consider any maximal group G contained in $H'(\mathcal{T})$. Such a group is generated by an idempotent element f, $f \circ f = f$. On the other hand, G is isomorphic to the group of all automorphisms of the tournament on the set $f(T)$. Thus we have a necessary condition: every maximal group in $M - C$ is an even group.

5.4. The number of simple and rigid tournaments

Let $T(n)$ ($S(n)$, respectively) denote the number of all non-isomorphic tournaments (simple tournaments, respectively) with n vertices. The following is true:

Theorem 5.

$$\lim_{n\to\infty} \frac{S(n)}{T(n)} = 1 .$$

Proof. We have according to the simple decomposition:

$$T(n) - S(n) \leqslant \sum_{k=2}^{n-1} S(k)\, T(n-k+1) \leqslant \sum_{k=2}^{n-1} T(k)\, T(n-k+1).$$

We employ the following inequality which holds for every k, $2 \leqslant k \leqslant n-1$,

$$\frac{2^{\binom{k}{2}} 2^{\binom{n-k+1}{2}}}{k!\,(n-k+1)!} \leqslant \frac{2^{\binom{n-1}{2}}}{(n-1)!}$$

and the bounds

$$\frac{2^{\binom{n}{2}}}{n!} \leqslant T(n) \leqslant \frac{2^{\binom{n}{2}}\, 2^{n/2}}{n!}$$

which follows from [8]. Then

$$\frac{T(n) - S(n)}{T(n)} \leqslant \frac{n!}{2^{\binom{n}{2}}} \sum_{k=2}^{n-1} \frac{2^{\binom{k}{2}}\, 2^{\binom{n-k+1}{2}}\, 2^{(k/2)}\, 2^{(n-k+1)/2}}{k!\,(n-k+1)!}\, k$$

$$\leqslant \frac{n^3}{2^{(n-3)/2}} .$$

This proves the statement. Hence almost all tournaments are simple irreducible.

Let $A(n)$ be the number of all non-isomorphic tournaments $\mathcal{T}$ with n vertices and $A(\mathcal{T}) = \{1_T\}$ (i.e., asymmetric tournaments, see [3]). Let $R(n)$ be the number of all non-isomorphic rigid tournaments with n vertices.

Then we have again:

Theorem 6.

$$\lim_{n\to\infty} \frac{A(n)}{T(n)} = 1, \qquad \lim_{n\to\infty} \frac{R(n)}{T(n)} = 1 .$$

We have analogously as above

$$T(n) - A(n) \leqslant \sum_{\substack{k=3\\ k \text{ odd}}}^{n} 2^{(k-1)/2}(n-k)\,T(n-k) \leqslant 2^{n/2}\,T(n-3)\,(n-3).$$

Using the bounds for $T(n)$ from [8], we get $\lim_{n\to\infty}(T(n)-A(n))/T(n) = 0$. The second part of the statement follows from the fact that

$$T(n) - R(n) \leqslant T(n) + (A(n) + S(n)).$$

Hence almost all tournaments are rigid.

The number $S(n)$ grows rapidly. Table 1 gives the first few values of $S(n)$. The values $S(n)$ were computed using a method involving three recursive formulas. This will be discussed in a separate paper.

A3505 Table 1 A568 A3507

n	$S(n)$	$T(n)$	$R(n)$
1	1	1	1
2	1	1	1
3	1	2	0
4	0	4	0
5	3	12	2
6	15	56	13
7	203	456	199
8	3785	6880	3773

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