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ENUMERATION OF LADDER GRAPHS*

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Abstract. Consider N points labelled cyclically forming the vertex set of a planar convex polygon, pairs of which may be connected by bonds. Define a ladder graph to be a graph with no crossing bonds. New simple proofs are given for two results of Kirkman relating to the total number of ladder graphs with k bonds. Relations are derived for calculating the number of connected ladder graphs with k bonds, and asymptotic expressions are obtained.

1.

In the investigation of certain lattice walks (see, for example, [5]) we encounter the following geometric problem: "What is the number of connected line graphs which can be constructed from the vertex set of a planar convex polygon, with no two lines crossing?" This problem leads immediately to the more general one: "What is the total number of line graphs which can be constructed on the vertex set of a planar convex polygon, with no two lines crossing?" This second problem has been solved previously. It was proposed in 1857 by the Rev. T.P. Kirkman [7], who in his paper gave the solution without proof. This solution consists of an expression for the number of k-line graphs on an N-point vertex set, and a recurrence relation for this number. The former was proved by Cayley [2] some thirty years later. Cayley had also attempted a proof of the recurrence relation but this was only satisfactorily given by Watson [9] over seventy years later. (We are grateful to Professor H.N.V. Temperley for bringing this paper to our attention.)

Upon reading Cayley's paper, one is immediately struck by the amount and complexity of the algebra required to prove such simple results. Watson's paper, though simpler, is nonetheless still involved, and

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in fact the overriding characteristic of this seemingly simple problem appears to be the exceedingly intricate mathematics required in its solution.

The problem is similar in character to other enumerations in graph theory which have been solved more directly (see, for example, [8]). We have based our approach on the fact that the symmetry exhibited by this problem is that of the permutation group rather than that of the symmetric group. This symmetry has also determined the labelling scheme used.

As mentioned at the beginning, this problem is of importance in the study of lattice walks. In addition, it has application to perturbation expansions for a polymer chain [3, 5] and Ising model expansions [1].

2.

The points of the vertex set of a planar convex polygon are labelled cyclically 1, 2, ..., N. Any two points may be connected by a bond. Following Chikahisa we define a ladder graph to be a graph with no crossing bonds. We distinguish between basic bonds which connect adjacent points of the vertex set, and bridges which connect non-adjacent points. Let the total number of k-bond ladder graphs be a(N, k), the number of k-bond ladder graphs constructed solely from bridges be b(N, k) and the number of connected k-bond ladder graphs be c(N, k).

Define the generating functions

(1)
$$a_N(w) = \sum_{k=0}^{2N-3} a(N,k) w^k, \qquad A(x,w) = \sum_{N=1}^{\infty} a_N(w) x^N,$$

(2)
$$b_N(w) = \sum_{k=0}^{N-3} b(N,k) w^k$$
, $B(x,w) = \sum_{N=3}^{\infty} b_N(w) x^N$,

(3)
$$c_N(w) = \sum_{k=0}^{2N-3} c(N,k) w^k$$
, $C(x,w) = \sum_{N=1}^{\infty} c_N(w) x^N$.

(The upper limit to the k-summation is well-known (see, for example, [10, pp. 14, 118–120]).) By direct enumeration, we have

(4)
$$a_1(w) = 1,$$

 $a_2(w) = 1 + w,$
 $a_3(w) = 1 + 3w + 3w^2 + w^3,$
 $a_4(w) = 1 + 6w + 14w^2 + 16w^3 + 9w^4 + 2w^5,$

(5)
$$b_3(w) = 1,$$

 $b_4(w) = 1 + 2w,$

(6)
$$c_1(w) = 1,$$

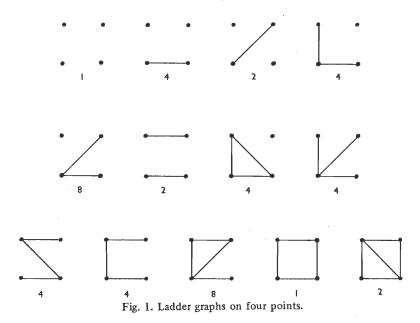
 $c_2(w) = w,$
 $c_3(w) = 3w^2 + w^3,$
 $c_4(w) = 12w^3 + 9w^4 + 2w^5.$

The graphs for N = 4 are presented in Fig. 1.

For future reference we give the results obtained by Kirkman [7]: the expression is

(7)
$$b(N, k) = \frac{(N)_k (N - k - 2)_k}{k! (k + 1)!},$$

where $(N)_k = N(N+1) \dots (N+k-1)$. The recurrence relation is



(8)
$$b(N,k) = \frac{N}{2k} \sum_{h=3}^{N-1} \sum_{s=0}^{k-1} b(N,s)b(N-h+2,k-s-1).$$

3.

Since k basic bonds can be placed on the N points in $\binom{N}{k}$ ways, it is not hard to see that the generating function for basic bond graphs is $(1+w)^N$.

Furthermore, since graphs are constructed only of basic bonds and bridges, it follows that (for $N \ge 3$)

(9)
$$a_N(w) = (1 + w)^N b_N(w)$$
.

Multiplying both sides by x^N , and summing, yields the relation

(10)
$$A(x, w) = B[x(1+w), w] + a_1 x + a_2 x^2.$$

We can also obtain a functional relation between A and C. The $a_N(w)$ are decomposed into mutually exclusive groups made up of

- (i) a single connected set,
- (ii) two disjoint connected sets,
- (iii) three disjoint connected sets, and so on the standard manner (see, for example, [8]). We consider these in turn.
 - (i) A single term $c_N(w)$ has generating function C(x, w).
- (ii) The two sets can contain r and s points, where r and s are non zero integers (r + s = N). For any particular r and s $(r \neq s)$, there are N possible choices of the labelled points. When r = s, there are $\frac{1}{2}N$ choices. Hence the total contribution is

$$\frac{1}{2!} \sum_{r, s} N c_r c_s, \qquad r + s = N ,$$

which has generating function

$$\frac{x}{2!} \frac{\mathrm{d}}{\mathrm{d}x} C^2 .$$

(iii) If we now introduce a third connected set of t points, we find

that there are r + s ways in which these can be introduced adjacently to one another, and t - 1 ways non adjacently. When any of r, s and t are equal, there are appropriate symmetry reduction factors. We then find for this contribution

$$\frac{N(N-1)}{3!} \sum_{r,s,t} c_r c_s c_t, \qquad r+s+t=N,$$

which has generating function

(12)
$$\frac{x^2}{3!} \frac{d^2}{dx^2} C^3$$
.

Proceeding in a similar manner the mth term has generating function

(13)
$$\frac{x^{m-1}}{m!} \frac{d^{m-1}}{dx^{m-1}} C^m .$$

We can therefore write

(14)
$$A(x,w) = C(x,w) + \frac{x}{2!} \frac{d}{dx} C^2 + \frac{x^2}{3!} \frac{d^2}{dx^2} C^3 + \dots$$

$$+\frac{x^{m-1}}{m!}\frac{\mathrm{d}^{m-1}}{\mathrm{d}x^{m-1}}C^m+\ldots$$

Comparing this with the Lagrange expansion (see, for example, [6])

(15)
$$z = x + x \phi(z)$$

$$= x + x \phi(x) + \frac{x^2}{2!} \frac{d}{dx} \phi^2 + \dots + \frac{x^m}{m!} \frac{d^{m-1}}{dx^{m-1}} \phi^m + \dots,$$

we see that if we identify (z-x)/x with A(x, w) and ϕ with C(x, w), we obtain the functional relation

(16)
$$A(x, w) = C[x(1+A), w].$$

This relation can be used to obtain the $c_N(w)$, given the $a_N(w)$; however, it is not very convenient.

4.

Preliminary to proving the Kirkman results (7) and (8), we establish equivalent differential equations with initial conditions for the generating funtions. From (8) we have

$$b(N,k) = \frac{N}{2k} \sum b(N,s) \ b(N-h+2, k-s-1), \qquad k \neq 0,$$

= 1, \qquad k = 0.

Multiply both sides by w^k , differentiate w.r.t. w, and sum over k to get

$$\frac{\mathrm{d}}{\mathrm{d}w} b_N(w) \equiv D_w b_N(w) = \frac{N}{2} \sum_{n=3}^{N-1} b_n b_{N+2-n}.$$

Multiply by t^{N-1} and integrate from 0 to x w.r.t. t:

$$D_w \int_0^x b_N t^{N-1} dt = \frac{1}{2} \sum_{n_1, n_2} b_{n_1} b_{n_2} x^N \qquad (n_1 + n_2 = N + 2).$$

Define $h_N(x, w) = b_N(w)/x$; $H = \sum h_N x^N = B/x$. We now have

$$D_w \int_0^x h_N t^N dt = \frac{1}{2} \sum_{n_1} h_{n_2} x^{N+2} \qquad (n_1 + n_2 = N+2)$$

which when summed over $N \ge 3$ gives

$$D_w \int_0^x H(t, w) dt = \frac{1}{2} H^2(x, w)$$
.

Differentiating both sides w.r.t. x yields

$$D_w H = H D_x H$$

or

(17)
$$D_w B - B D_x (B/x) = 0$$
.

It is necessary to establish an initial condition in x and one in w. The first is clearly

(18)
$$B(0, w) = 0$$
.

The w-condition is obtained from $b_N(0) = 1$ which when substituted into (2) yields

(19)
$$B(x, 0) = x^3/(1-x).$$

The differential equation and initial conditions equivalent to (8) is (17) with (18) and (19).

To find an equivalent equation for (7) we first write it as a recurrence relation

(20)
$$b(N+1,k+1) = \frac{(N+k)(N+k+1)(N-2)}{N(k+1)(k+2)} b(N,k).$$

Proceeding as before, one obtains the differential equation

(21)
$$[\mathfrak{D}^3 + O_1 \mathfrak{D}^2 + O_2 \mathfrak{D} - O_3 - O_4 \mathfrak{D}(1/x)] B = 0,$$

where

$$\begin{array}{ll} \mathcal{D} &= x \, D_x \; , \\ O_1 &= 2w \, D_w - 1 \; , \\ O_3 &= 2w \, D_w \, (w \, D_w + 1) \; , \end{array} \quad \begin{array}{ll} O_2 &= (w \, D_w)^2 - 3w \, D_w - 2 \; , \\ O_4 &= D_w^2 \, w \; . \end{array}$$

This equation requires three initial conditions in x and two in w. The x conditions are

(22)
$$B(0, w) = B_x(0, w) = B_{xx}(0, w) = 0$$
.

The first w condition is (19). To obtain the second we note that

(23)
$$b(N, 1) = N(N-3)/2!$$
.

Hence

(24)
$$D_{w}B(x,0) = \sum_{N=4}^{\infty} b(N,1) x^{N} = \frac{x^{2}}{(1-x)^{3}} + x(1+x).$$

The differential equation equivalent to (7) is (21) with initial conditions (22), (19) and (24).

We have established differential equations whose solutions satisfy Kirkman's results. If we can show that the generating functions for the ladder graphs are solutions of these equations, then we shall have proved that (7) and (8) are results valid for ladder graphs.

5.

For convenience both $a_N(w)$ and $c_N(w)$ will be represented in this section by $d_N(w)$ and both A and C by D.

We subdivide the $d_N(w)$ into mutually exclusive classes $d_N^{(r)}$ which enumerate graphs of the appropriate type having r bridges which terminate at the point labelled 1. Define $e_N(w)$ and E(x,w), generating functions for graphs of the appropriate type, with no bridges terminating at 1, and which contain a given basic bond adjacent to 1. Similarly, define $f_N(w)$ and F(x,w) to enumerate all such graphs which contain two basic bonds with common point 1.

Consider $d_N^{(1)}$ which counts all configurations with one bridge terminating at 1 (see-Fig. 2). This bridge divides the N-gon into an m-gon and an n-gon (m+n=N+2). There are e_m graphs on the m-gon containing this bridge and e_n on the n-gon. Thus

$$d_N^{(1)} = \frac{1}{w} \sum_{m,n} e_m e_n, \qquad m+n=N+2.$$

The factor 1/w arises because the bridge is counted twice, once in each sub-polygon. Multiplying by x^N and summing over $N \ge 4$, we find

(25)
$$D^{(1)} = (1/wx^2) E^2$$
.

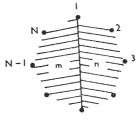


Fig. 2. Graphs with one bridge terminating at 1.

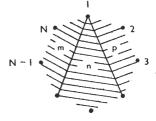


Fig. 3. Graphs with two bridges terminating at 1.

Looking at $d_N^{(2)}$ (see Fig. 3) we have two bridges, which divide the polygon into an m-gon, an n-gon and a p-gon (m+n+p=N+4). There are e_m graphs on the m-gon containing one of the given bridges, e_p graphs on the p-gon containing the other given bridge, and f_n graphs on the n-gon containing both bridges. Thus

$$d_N^{(2)} = \frac{1}{w^2} \sum_{m,n,p} e_m f_n e_p, \qquad m+n+p = N+4,$$

from which we obtain

(26)
$$D^{(2)} = (1/wx^2)^2 E^2 F$$
.

The generalization to the case of graphs with r bridges terminating at 1 is straightforward: the result is

$$d_N^{(r)} = \frac{1}{w^r} \sum_{n_1 \dots n_{r+1}} e_{n_1} f_{n_2} \dots f_{n_r} e_{n_{r+1}},$$

$$n_1 + n_2 + \dots + n_{r+1} = N + 2r,$$

(27)
$$D^{(r)} = (1/wx^2)^r E^2 F^{r-1}.$$

Now, writing

(28)
$$D = D^{(0)} + D^{(1)} + D^{(2)} + \dots$$

we have

$$D = D^{(0)} + (E^2/wx^2)[1 + (F/wx^2) + (F/wx^2)^2 + \dots]$$

or

(29)
$$D(x,w) = D^{(0)}(x,w) + E^2(x,w)/(wx^2 - F(x,w)).$$

For this to be a useful expression, we must for the cases D = A and D = C ascertain E and F, and also $A^{(0)}$ and $C^{(0)}$.

To obtain an expression for $a_N^{(0)}(w)$, we isolate the point 1 and allow all possible graphs on the remaining N-1 points (see Fig. 4). In addition, we can have 0, 1 or 2 basic bonds which terminate at 1. Thus we have immediately

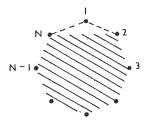


Fig. 4. Graphs with no bridges terminating at 1.

(30)
$$a_N^{(0)} = (1+w)^2 a_{N-1}(w)$$
.

Multiplying by x^N and summing over $N \ge 3$, we have on the r.h.s.

$$x(1+w)^2 (A-x)$$
.

To include the cases N = 1, 2, we add to this, from (4),

$$a_1 x + a_2 x^2 = x + (1 + w) x^2$$

to get

(31)
$$A^{(0)} = x + (1+w) x^2 + (1+w)^2 x (A-x).$$

Using similar procedures it is not difficult to show that

(32)
$$E_A = x w (1 + w) (A - x), F_A = x w^2 (A - x).$$

Substitution of these results into (29) yields, after some manipulation,

(33)
$$A^2w + A[(1+w)x^2 - (1+2w)x] + (1+w)x^2 = 0.$$

This can be solved formally to give

(34)
$$A(x,w) =$$

$$= x[(1+2w) - x(1+w) - \{[x(1+w) - 1]^2 - 4xw(1+w)\}^{1/2}/2w.$$

The sign of the radical being chosen from the condition $(dA/dx)_{x=0} = 1$. (34) can give an expression for B(x, w), since from (10),

$$B[x(1+w), w] = A(x, w) - x - x^{2}(1+w)$$

$$= [x(1+w) - x^{2}(1+w)^{2}(1+2w)$$

$$-x(1+w)\{[x(1+w) - 1]^{2} - 4xw(1+w)\}^{1/2}]/2w(1+w)$$

which suggests

(35)
$$B(x,w) = x[1-x(1+2w) - \{(x-1)^2 - 4xw\}^{1/2}]/2w(1+w).$$

This quantity satisfies the differential equations (17) and (21) together with the associated boundary conditions, thus proving both of Kirkman's results.

6.

Stirling's approximation is readily applied to the expression (7) for b(N, k) to obtain asymptotic results. Writing (7) in terms of factorials we have

(36)
$$b(N,k) = \frac{(N+k-1)!}{(N-1)(N-2)(N-k-3)! \ k! \ (k+1)!}.$$

For the case N and k both large we apply Stirling's formula to (N+k-1)!, k! and (k+1)! and find

(37)
$$b(N,k) \sim \frac{(N+k-1)^{N+k-1/2} e^{k+2-N}}{\sqrt{2\pi} (N-1) (N-2) (N-k-3)! k^{k+1/2} (k+1)^{k+3/2}}.$$

For the case $k \leq N$, we find

(38)
$$b(N,k) \sim \frac{(N+k-1)^{N+k-1/2} e^{-2(k+1)}}{(N-1)(N-2)(N-k-3)^{N-k-5/2} k! (k+1)!}$$

In order to obtain further asymptotic results we shall make use of a theorem by Darboux [4] for a function f(z) which has a singularity at z = a on the circle of convergence and satisfies

(39)
$$f(z) = \theta(z) + (z - a)^k \phi(z),$$

where θ and ϕ are functions regular at z = a. Then, according to Darboux, the asymptotic form of the coefficients in the expansion of f is the same as that of the coefficients of $\phi(a)$ $(z - a)^k$. Higher approximations can also be obtained.

We now apply this to obtain an asymptotic expression for $b_N(w)$ for the special case w = 1. $b_N(1)$ is the total number of ladder graphs which can be constructed solely from bridges on N points. We note from (9) that

(40)
$$\sigma_N(1) = 2^N b_N(1)$$
.

From (35),

(41)
$$B(x, 1) = \frac{1}{4}x - \frac{3}{4}x - \frac{1}{4}x\{(x - 1)^2 - 4x\}^{1/2}$$
$$= \frac{1}{4}x - \frac{3}{4}x - \frac{1}{4}x(1 - \alpha x)^{1/2} (1 - \beta x)^{1/2},$$

where $\alpha = 3 + 2^{3/2}$, $\beta = 3 - 2^{3/2}$.

Applying Darboux's theorem,

$$B(x, 1) \sim -\frac{1}{4}x(1 - \alpha x)^{1/2} (1 - \beta/\alpha)^{1/2} = \Lambda_0 x(1 - \alpha x)^{1/2}$$
,

where

$$\Lambda_0 = -\frac{1}{4}(1-\beta/\alpha)^{1/2} .$$

Expanding

Table 1

	b_N (1)	<i>c</i> _N (1)		
N	Asymptotic	Exact	Asymptotic	Exact	
10	20 856	20 793	49 692 × 10 ²	56 162 × 10 ²	
20	30.055×10^7	30 016 x 10 ⁷	$25\ 289 \times 10^{12}$	20.871×10^{12}	
30	$71\ 601 \times 10^{14}$	71.542×10^{14}	20.098×10^{22}	20 925 X 10 ²³	
40	20700×10^{22}	20.687×10^{22}	19 121 x 10 ³²	19 708 × 10 ³	
50	66362×10^{29}	66330×10^{29}	20.067×10^{42}	20.557×10^4	
60	22692×10^{37}	22 683 × 10 ³⁷	22403×10^{52}	22 858 x 10 ⁵	
70	$81~094 \times 10^{44}$	81 067 × 10 ⁴⁴	26 099 x 10 ⁶²	26552×10^6	
80	29.925×10^{52}	29 916 x 10 ⁵²	$31\ 368 \times 10^{72}$	31.844×10^7	
90	11 315 × 10 ⁶⁰	$11\ 312 \times 10^{60}$	38606×10^{62}	39 126 × 10 ⁸	

$$(1 - \alpha x)^{1/2} = \sum (-1)^N {1/2 \choose N} \alpha^N x^N$$

from which it follows that

(42)
$$b_N(1) \sim \Lambda_0(-1)^{N-1} {1/2 \choose N-1} \alpha^{N-1}$$
.

A comparison of the values this expression gives with exact results is given in Table 1.

7.

The functional relation (16) does not permit convenient evaluation of the numbers of connected ladder graphs. It is more useful to make a direct attack using the approach of Section 5.

To find $c_N^{(0)}(w)$ we isolate the point 1, allowing all connected graphs on the remaining N-1 points. At least one of the basic bonds adjacent to the point 1 must be present. If both are present there will be additional graphs corresponding to two separate connected subgraphs on the N points, joined only by the two basic bonds whose common point is 1 (see Fig. 5).

Collecting all contributions we have

(43)
$$c_N^{(0)}(w) = w(w+2) c_{N-1} + w^2 \sum_{m,n} c_m c_n \qquad (m+n=N-1)$$

from which we find

(44)
$$C^{(0)} = x + x^2 w + x w^2 C^2 + x w (w+2) (C-x).$$

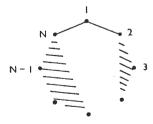


Fig. 5. Two connected subgraphs joined by the two basic bonds whose common point is 1.

The functions e_N and f_N for connected graphs can be similarly evaluated to give

$$e_N = w (1+w) c_{N-1} + w^2 \sum_{m,n} c_m c_n$$
 $(m+n=N-1),$

(45)
$$E_c = x w (1+w) (C-x) + x w^2 C^2,$$

and

$$f_N = w^2 c_{N-1} + w^2 \sum_{m,n} c_m c_n$$
 $(m+n=N-1)$,

(46)
$$F_c = x w^2 (C - x) + x w^2 C^2$$
.

Substitution of these results gives the cubic equation

(47)
$$w C^3 + w C^2 = x (1 + 2w) C + x^2 (1 + w) = 0$$

which can be formally solved to give

(48)
$$C(x,w) = \left[g - xh^{1/2}\right]^{1/3} + \left[x(1+2w)/3w + \frac{1}{9}\right] \left[g - xh^{1/2}\right]^{-1/3} - \frac{1}{3},$$

where

$$g(x, w) = -\frac{1}{27} - x(1+2w)/6w - x^2(1+w)/2w ,$$

$$h(x, w) = x^2(1+w)^2/4w^2 + x((1+2w)/3w^2)[(1+w)/2 - (1+2w)^2/9w] + (1+w)/27w - (1+2w)^2/108w^2 .$$

From the conditions that (48) have the desired expansion we determine the correct sign for $xh^{1/2}$, and we also ascertain that $g - xh^{1/2}$ lies on the first Riemann sheet.

From (47) we can obtain the recursion relation

(49)
$$c_N(w) = w \sum_{n=2}^{N-1} c_n c_{N+1-n} + w \sum_{n=1}^{N-1} \sum_{m=1}^{N-n} c_n c_m c_{N+1-m-n}$$

Table 2 Values of c(N, k)

k\N	1	2	3	4	5	6	7	8	9	10	
0	1									ι Λ .	7752
1		1	3							1 +1	2177
2			1	12					K	^ ~	901
4			-	9	55				NI	W 171	164
5				2	66	273			w.	, , ,	1
6					30	455	1 428			A	2409
7					5	315	3 060	7 752	42.262	* 1	0 71/1
8						105	2 856	20 349	43 263	246 675	H340°
9						14	1 428	23 940	134 596	888 030	
0							378	15 960	191 268 159 390	1 480 050	4108
1							42	6 300 1 386	83 490	1 480 050	1110
12								132	27 324	965 250	
13								132	5 148	418 275	
14									429	117 117	
15										19 305	
16										1 430	
17											

which is readily adaptable for use on a digital computer. We list in Table 2 values of C(N, k) to N = 10, as these are useful in physical applications.

8.

Darboux's theorem may be used to obtain an asymptotic expression for $c_N(1)$, the total number of connected ladder graphs which may be drawn on N points. From (48),

(50)
$$C(x,1) = [g(x) - xh^{1/2}(x)]^{1/3} + (x + \frac{1}{9}) [g(x) - xh^{1/2}(x)]^{-1/3} - \frac{1}{3},$$
where
$$g(x) = -\frac{1}{27} - \frac{1}{2}x - x^2, \quad h(x) = x^2 - \frac{1}{108}.$$

In order to apply the technique of Darboux, it is necessary to know the singularities of C(x, 1). $g - xh^{1/2}$ has no zeroes, so the singularities occur only when

(51)
$$h = 0$$
 or $x_0 = \pm 108^{-1/2}$.

To put C(x, 1) into the Darboux form, we write

(52)
$$g - xh^{1/2} = g + u(x) (1 - x/x_0)^{1/2}$$

which defines u(x). Then

(53)
$$C = g^{1/3} \left[1 + (u/g) \left(1 - x/x_0 \right)^{1/2} \right]^{1/3}$$

$$+ \left(x + \frac{1}{9} \right) g^{-1/3} \left[1 + (u/g) \left(1 - x/x_0 \right)^{1/2} \right]^{-1/3}$$

$$= \sum \left[g^{1/3} {1/3 \choose 2N} + \left(x + \frac{1}{9} \right) g^{-1/3} {-1/3 \choose 2N} \right] (u/g)^{2N} (1 - x/x_0)^N - \frac{1}{3}$$

$$+ \left(1 - x/x_0 \right)^{1/2} \sum \left[g^{1/3} {1/3 \choose 2N+1} + \left(x + \frac{1}{9} \right) g^{1/3} {-1/3 \choose 2N+1} \right] (u/g)^{2N+1} (1 - x/x_0)^N$$

$$+ \left(x + \frac{1}{9} \right) g^{1/3} {-1/3 \choose 2N+1}$$

which is in the standard Darboux form (39). To find the asymptotic values of the expansion we replace C by

(54)
$$(1 - x/x_0)^{1/2} \left[\frac{1}{3} g_0^{1/3} - \frac{1}{3} (x_0 + \frac{1}{9}) g_0^{-1/3} \right] (u_0/g_0) \equiv$$

$$\equiv \phi_0 (1 - x/x_0)^{1/2} ,$$

where $u_0 = u(x_0)$, $g_0 = g(x_0)$. Expanding $(1 - x/x_0)^{1/2}$ gives

(55)
$$c_N(1) \sim \phi_0(-1)^N \binom{1/2}{N} (1/x_0^N)$$
.

This procedure must be followed for both singular points (51).

The values given by (55) are compared with exact values in Table 1.

9.

The connection between A and C (16) can be obtained very simply from equations (33) and (47). Multiply (33) by (1 + A) to get the cubic

equation

(56)
$$wA^3 + wA^2 - x(1+A)(1+2w)A + x^2(1+A)^2(1+w) = 0$$

whose solution is A. However, writing y = x(1+A) and comparing with (47) we see that C(y) is also a solution. In fact, as before, we have

$$A(x, w) = C[x(1 + A), w]$$
.

An important simplification in C occurs when w = -1 [1, 5]. Setting w = -1 in (47) gives the equation

(57)
$$C^3 + C^2 - x C = 0$$

which, if we disallow C = 0, becomes

(58)
$$C^2 + C - x = 0$$

which solves to give

(59)
$$C(x,-1) = \frac{1}{2}(-1 + [1+4x]^{1/2}).$$

The sign of the radical is chosen from C(0, -1) = 0. The expansion of (59) as a power series in x yields $c_N(-1)$.

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