Sridar Kuttan Pootheri
Characterizing and counting classes of unlabeled 2–connected graphs
(Under the direction of Robert W. Robinson)

Applying the Tutte decomposition of 2–connected graphs into 3–block trees we provide unique structural characterizations of several classes of 2–connected graphs, including minimally 2–connected graphs, minimally 2–edge–connected graphs, critically 2–connected graphs, critically 2–edge–connected graphs, 3–edge–connected graphs, 2–connected cubic graphs and 3–connected cubic graphs. We also give a characterization of minimally 3-connected graphs.

Cycle index sum equations for counting unlabeled minimally 2–connected graphs, unlabeled 2–connected minimally 2–edge–connected graphs and unlabeled 2–connected 3–edge–connected graphs are derived from the structural relations. Polynomial space counting algorithms are then developed using cycle index sum inversion techniques. The appendices contain tables of the numbers of these three classes of 2–connected graphs, by number of nodes and number of edges, obtained by computer implementation of the algorithms. These are listed for node orders up to 32, 25 and 34 respectively.

Index words: Inversion of cycle index sum relations,
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Minimally 2–edge–connected graphs, 3–edge–connected graphs,
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Critically 2–edge–connected graphs, 2–connected cubic graphs,
3–connected cubic graphs, Minimally 3–connected graphs.

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by

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Dedication

To the Divine Self who nourishes every living creature.

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I owe my achievements to my beloved parents whose constant encouragements and prayers made my period of study successful. I also want to extend my thanks to all our friends. Thank you for making the bad times bearable and the good times all the more special. I will always cherish your friendship.

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CHAPTER 1

Introduction

§1.1 Overview

The exact enumeration of graphs is one of the classical problems in combinatorics. In this dissertation exact counting algorithms for three classes of unlabeled 2–connected graphs are derived : minimally 2–connected graphs, 3–edge–connected blocks and minimally 2–edge–connected blocks.

The enumerations are based on relations satisfied by cycle index sums. These in turn are derived from the following characterization theorems, which are based on the minimal Tutte decomposition $T(G)$ of a 2–connected graph G into components which are cycles, bonds, or 3–connected graphs.

A 2–connected graph G is :

1. *minimally 2–connected if and only if each free edge in $T(G)$ belongs to a cyclic component;*
2. *minimally 2–edge–connected if and only if each free edge in $T(G)$ belongs to a cyclic component that contains at least one other free edge;*
3. *3–edge–connected if and only if each cyclic component has at most one free edge.*

These facts are developed and proved in Chapter 2. The cycle index sum relations are transformed algebraically to space efficient forms in Chapters 4, 5 and 6. The resulting recurrence relations were implemented in C++ to calculate the numbers, which are tabulated in the appendices.

As a function of N, the maximum of number of nodes for which the calculations are to be performed, the space requirements are transformed from exponential to polynomial order by means of inverting to power series. The number of operations required is still exponential in N, but the maximum for which computations are feasible is increased substantially since the workspace now fits in the RAM available on a workstation.

An important motivation for studying graphs is that they model interconnection networks for communication. These range in scale from the connection of processors in a multi–processor computer on up to a WAN connecting a global network. At any scale completeness of connectivity and efficiency are fundamental design considerations. The failure of a connection or a node can change the nature of the connectivity of the network. Minimality with respect to edge connectivity and node connectivity give ways of quantifying the fault tolerance of a network. Knowing the maximum number of configurations of networks satisfying a particular minimality condition helps in predicting the feasibility of synthesis or design approaches which require checking all possible network configurations with the specified properties and given parameters.

Of theoretical interest are the harmonic mean numbers of automorphisms of the graphs in our classes. These are obtained by comparing the numbers of unlabeled graphs with the corresponding numbers of labeled graphs which were calculated in [42]. The harmonic mean numbers of automorphisms are reported in the appendices of [42].

§1.2 Background on counting

When we seek to count configurations too numerous to be feasibly listed, we employ a formal power series of the form $\sum_{n=0}^{\infty} a_n x^n$. These series are not required to be convergent. They are formal expressions. By performing appropriate manipulations on these formal expressions, one arrives at the *counting series* of the desired configurations. As explained by Herbert Wilf in his book "Generatingfunctionology" [59], a counting series of a configuration is like a clothesline of numbers where each number a_n denotes the number of configurations of size n.

A *labeled graph* $\boldsymbol{G}$, is a pair $(V(\boldsymbol{G}), E(\boldsymbol{G}))$, where $V(\boldsymbol{G}) = \{1, 2, 3, \cdots, n\}$ for $n \geq 1$ and where $E(\boldsymbol{G})$ is a set of 2–element subsets of $V(\boldsymbol{G})$. This loops and multiple edges are not allowed in a graph. Two labeled graphs $\boldsymbol{G}$ and $\boldsymbol{H}$, both of order n, are *isomorphic* if there exists a permutation $\sigma \in S_n$, such that

$$\sigma(E(\boldsymbol{H})) = \Big\{(\sigma(i), \sigma(j)) : (i, j) \in E(\boldsymbol{H})\Big\} = E(\boldsymbol{G}).$$

We say that σ is an isomorphism from $\boldsymbol{H}$ to $\boldsymbol{G}$. In words, two labeled graphs are isomorphic if and only if there is a $1-1$ map between their node sets which preserves adjacency. An *unlabeled graph of order n* is an isomorphism class of labeled graphs of order n.

Let $\mathcal{A}$ denote the set of all the isomorphisms of a graph $\boldsymbol{G}$ onto itself. These form a group and there is a $1-1$ map between the left cosets of $\mathcal{A}$ in S_n and labeled graphs isomorphic to $\boldsymbol{G}$. Thus there are $n!/|\mathcal{A}|$ different ways of labeling $\boldsymbol{G}$. $\mathcal{A}$ is called the *automorphism group* of the graph $\boldsymbol{G}$.

An obvious method of counting unlabeled graphs is by enumerating them one by one, keeping count as they are produced. But to count this way, one has to eliminate duplicates. This involves isomorphism testing. For graphs, this is a rather hard problem. Even when there are orderly ways to enumerate graphs without isomorphism testing, the number of graphs grows so rapidly that it soon becomes impossible to manage. To overcome these limitations, one has to use generating functions or cycle index sums to count graphs. A cycle index sum encodes the cycle structure of the automorphisms of the graphs. For example, the cycle index sum of all complete graphs $\boldsymbol{K}$ starts,

$$\boldsymbol{K} = 1X^0 + 1a_1X^1 + \left(\frac{1}{2}a_1^2b_1 + \frac{1}{2}a_2c_1\right)X^2 + \left(\frac{1}{6}a_1^3b_1^3 + \frac{1}{2}a_1a_2b_2c_1 + \frac{1}{3}a_3b_3\right)X^3 + \cdots$$

where the coefficient of X^n is the cycle index of a complete graph on n nodes. In general, we will denote the cycle index of $\mathcal{A}$, $\boldsymbol{Z}(\mathcal{A})$, simply as $\boldsymbol{Z}(\boldsymbol{G})$, or just as $\boldsymbol{G}$ when no confusion will arise. For a class $\mathfrak{G}$ of graphs the cycle index sum will be denoted as $\boldsymbol{Z}(\mathfrak{G})$, or just as $\mathfrak{G}$ when no confusion will arise.

The earliest known example of enumerating a general class of unlabeled graphs (as opposed to a very special class, such as trees) is due to Redfield [48]. As noted by Harary [19], all graphical enumeration methods in current use were anticipated in this unique paper by Redfield, published in 1927 but unfortunately overlooked at that time. He relied on Burnside's Lemma, which is discussed in detail in the Chapter 3. There have been many elaborations in the basic method of counting graphs since then, but they all have Burnside's Lemma as their foundation. There have in fact been a number of independent re–discoveries of these basic methods, the most influential being Pólya's Theorem [41] and its applications to various types of graphs by Harary [17]. The latter includes connected graphs.

As noted at in the beginning of Chapter 8 of [19], a famous theoretical physicist, George E. Uhlenbeck, in his Gibbs Lecture entitled "Unsolved problems in statistical mechanics," given at a meeting of the American Mathematical Society in 1950, cited the enumeration of blocks as one of these problems. Subsequently Riddell, and Ford and Uhlenbeck counted labeled blocks, but it was Robinson [49] who succeeded in solving the unlabeled problem. The basic idea for cycle index sum enumeration appears already in Redfield [48], and was independently rediscovered by deBruijn [1], but it had not actually been applied to a graph counting problem prior to the enumeration of blocks. Blocks are discussed in section §**3.4**.

When counting unlabeled graphs, oftentimes one encounters a functional equation satisfied by cycle index sums in which a special operation called composition, occurs. Perhaps the simplest example is an equation of the form

$$\boldsymbol{Z}(W)[\boldsymbol{Z}(T)] = \boldsymbol{Z}(C),$$

where [ ] denotes a composition of cycle index sums (defined below). Here, W stands for a set of graphs which is to be counted in terms of a generating function $W(x)$. Further, W is related to an easily enumerable set of graphs C by the relation that the graphs in the set C are obtained by planting node–rooted graphs from a set T on the nodes of the graphs in W. (Node–rooted graphs are discussed in detail in Chapter 3.) This is exactly the case if we let W be the set of all connected graphs without endnodes excepting the single node, let C be the set of all connected graphs which are not trees, and let T be the set of all rooted trees. Clearly, each graph in C can be obtained from a unique graph γ in W by placing rooted trees on the nodes of γ. Then, Robinson's composition theorem, which is discussed in detail in Chapter 3, gives

$$\boldsymbol{Z}(C) = \boldsymbol{Z}(W)[\boldsymbol{Z}(T)], \tag{1.1}$$

where the composition product on the right is defined by

$$\boldsymbol{Z}(W)[\boldsymbol{Z}(T)] = \boldsymbol{Z}(W)[a_i \leftarrow \boldsymbol{Z}(T)[a_k \leftarrow a_{ik}]]$$

Equation (1.1) is typical of equations that come up in such an enumeration in that the cycle index of the set to be counted appears on the left side of the composition product. One can calculate $\boldsymbol{Z}(W)$ from (1.1) by a simple comparison of coefficients, but computationally such a procedure is very demanding of time and space. For one thing, $\boldsymbol{Z}(W)$ must be stored during the computation. For another, the eventual outcome is $\boldsymbol{Z}(W)$. For enumeration purposes, the ordinary generating function $W(x)$ of the set W is sufficient; $\boldsymbol{Z}(W)$ includes much more information than is needed.

To present another, even more basic, scenario to show upper bound on computational complexity of cycle index sum enumeration, we consider determining the number g_n of all unlabeled graphs of order n. This requires the calculation of $p(n)$ terms, where $p(n)$ is the number of partitions of n. It is well known from analytic number theory that $p(n) \sim k_1 \exp(k_2\sqrt{n})/n$ where $k_1 = 1/4\sqrt{3}$ and $k_2 = \pi\sqrt{2/3}$. To put this in perspective, note that $p(20) = 627$, $p(40) = 37,338$, $p(60) = 966,467$, $p(80) = 5,769,476$, and $p(100) = 190,569,292$. Each term requires a slowly growing number of arithmetic operations and contributes to a running total. Thus the memory requirements are minimal, and the factor which determines the feasibility of computation is the growth of $p(n)$ and the CPU speed.

Once cycle index sums of the automorphism groups of the objects being counted must be stored and manipulated, storage requirements become a major problem, as they too are proportional to $p(n)$. When primary memory no longer suffices and secondary storage must be utilized, huge increases in access times are incurred. Also, a simple multiplication of two such cycle index sums requires roughly $p(2n)$ arithmetic operations, so the feasible problem size is cut half insofar as CPU time is concerned, even if access times could be held constant.

In fact, Walsh in his paper [56] derived all the equations necessary to count 3–connected graphs and homeomorphically irreducible 2–connected graphs. But since he was performing all computations with cycle index sums, he was only able to carry out the computations up to 9 nodes.

To combat these inefficiencies, Robinson devised a cycle index sum inversion Theorem [12]. Using this, (1.1) is inverted to an equation of the form

$$\boldsymbol{Z}(C)[\boldsymbol{Z}(M)] = \boldsymbol{Z}(W) \tag{1.2}$$

and then simplified to

$$\boldsymbol{Z}(C)[M(x)] = W(x) \tag{1.3}$$

by replacing a_i with x^i. When $W(x)$ is computed using (1.3), no cycle indices need be stored and no more information than the ordinary generating function is produced. Computationally, (1.3) is very efficient. A combinatorial interpretation of the inversion theorem under some circumstances is provided in [11].

Using this method of extracting counting series from systems of equations involving cycle index sums, Hanlon and Robinson were able to count unlabeled 2–edge–connected graphs in [12] and Robinson and Walsh were able to count unlabeled 2– and 3–connected graphs and homeomorphically irreducible 2–connected graphs in [51]. Once the equations had been inverted, extracting the numbers by computer involved only counting series operations and summing $O(n)$ times over the coefficients of a cycle index sum. The inverted equations could then be solved recursively without storing any cycle index sum. Essentially the time– and space–complexity of the solution was reduced to that required by the direct use of Pólya's Theorem to count all unlabeled graphs. In [51] this enabled the authors to compute the above–mentioned classes of graphs with up to 18 nodes when the number of edges is included as a parameter, and up to 25 or 26 nodes otherwise.

§1.3 Outline of later chapters

Our method for counting graphs with prescribed properties relies on decomposing a graph into a core and components. In Chapter 2, we present decomposition characterizations of the classes of graphs counted in the later chapters, along with the characterizations of several other classes of 2–connected graphs. All these characterizations are unique, as they all build on a version of the Tutte decomposition which is unique. It should be noted that to perform unlabeled counting for a class of graphs, one not only needs a unique characterization for that class, but also one which lends itself to an algebraic interpretation.

Tutte's theorem decomposes blocks into 3–connected graphs, bonds with at least 3 edges and cycles with at least 3 edges. This can be translated to an equation relating the cycle index sums of the cores and the components. The classes that we have counted are minimally 2–connected graphs, 3–edge–connected blocks and minimally 2–edge–connected blocks. For each of these classes, our decomposition theorem given in the Chapter 2 is translated into cycle index sum equations. Each of these systems of equations and the one for blocks is appropriately inverted. The resulting equations have a set of terms that are common. Solving for the desired counting series, we obtain equations that involve only five unknown power series $\mu(x, y)$, $\nu(x, y)$, $\eta(x, y)$, $\delta(x, y)$ and $\epsilon(x, y)$. Note that in all power series used in this dissertation, x and y denote nodes and edges respectively, so that each coefficient of x in all our power series is a polynomial in y.

Five equations for these unknown five power series are derived (three of the five in Chapter 3 and the remaining two in the respective Chapters) so that they can be expressed in terms of explicitly known cycle index sums $\boldsymbol{K}$, $\boldsymbol{K}_a$, $\boldsymbol{K}_b$ and $\boldsymbol{K}_c$, and their terms can be extracted step by step as we follow a recursive procedure. More explicitly, if these series are known through order $(n-1)$ in x (order n in the case of $\eta(x, y)$) :

- the order n terms of $\mu(x, y)$ and $\nu(x, y)$ can be explicitly computed from the lower order terms of $\mu(x, y)$, $\nu(x, y)$, $\delta(x, y)$ and $\epsilon(x, y)$;
- the order $(n+1)$ term of $\eta(x, y)$ can be computed by substituting the lower order terms of $\eta(x, y)$, $\mu(x, y)$ and $\nu(x, y)$ in the cycle index sum of all complete graphs $\boldsymbol{K}$ and in the cycle index sum of all node–rooted complete graphs $\boldsymbol{K}_a$;
- the order n term of $\delta(x, y)$ can be computed by substituting the lower order terms of $\eta(x, y)$, $\mu(x, y)$ and $\nu(x, y)$ in $\boldsymbol{K}$ and in the cycle index sum of all positively edge–rooted complete graphs $\boldsymbol{K}_b$;

- the order n term of $\epsilon(x, y)$ can be computed by substituting the lower order terms of $\eta(x, y)$, $\mu(x, y)$ and $\nu(x, y)$ in $\boldsymbol{K}$ and in the cycle index sum of all negatively edge–rooted complete graphs $\boldsymbol{K}_c$.

Such an order by order computation is common to all the three classes. In Chapter 3, assuming that $\mu(x, y)$ and $\nu(x, y)$ are given, the relations meant for the above recursive computation for $\delta(x, y)$, $\epsilon(x, y)$ and $\eta(x, y)$ are are derived in equations (3.78), (3.79) and (3.80) respectively. Power series $p(x, y)$ and $s(x, y)$ depend on blocks. So, they are also derived in Chapter 3 in equations (3.83) and (3.84) respectively. Chapters corresponding to each of the three classes provide their own $q(x, y)$ and $t(x, y)$. These, along with $p(x, y)$ and $s(x, y)$ are used to compute the corresponding $\mu(x, y)$ and $\nu(x, y)$.

For example, unlabeled minimally 2–connected graphs are counted by solving equations (3.78), (3.79), (3.80), (4.8), and (4.12) simultaneously for $\delta(x, y)$, $\epsilon(x, y)$, $\eta(x, y)$, $\mu(x, y)$ and $\nu(x, y)$. This will involve using equations from Chapter 3 for $p(x, y)$, $s(x, y)$ and equations from Chapter 4 for $q(x, y)$ and $t(x, y)$ as intermediate steps in the computation. Computations for the other two classes are also similar in nature, but substitute equations from Chapter 5 or Chapter 6 for those of Chapter 4.

In the above recursive procedure to extract the nth order term of a power series, we compose with the corresponding cycle index sum only with the terms that correspond to graphs of at most n nodes. This works well, because those terms of the cycle index sums require the power series to be exponentiated up to degree n. Using the on–line power series exponentiation routine, due to Euler, listed in Knuth's book [31, pp. 525], it is possible to raise a power series up to the nth power when its first n terms are known. We use the *Next partition of Integer n (NEXTPAR)* algorithm listed in "Combinatorial Algorithms" [58, pp. 69-70] to generate each of the cycle index sums up to degree n from the explicit formulas given in equations (3.29) through (3.32).

CHAPTER 2

Decomposition characterizations

§2.1 Introduction

Every graph in this dissertation will be a loopless multigraph. A multigraph is said to be *simple* if between each pair of adjacent nodes there is exactly one edge. The *trivial graph* (a graph with just one node) and an isolated edge (a graph with just two nodes and one edge) are both considered to be 1–connected and 1–edge–connected but not 2–connected or 2–edge–connected. For a graph G on $p \geq 3$ nodes and $n \geq 1$, we say that G is n–connected if and only if $p \geq n+1$ and G cannot be disconnected by removing fewer than n nodes and their incident edges. A simple non–trivial graph is defined to be *n–edge–connected* if it is connected and cannot be disconnected by removing fewer than n edges. A *block* is a non–trivial connected graph with no cut node. In other words, a graph is a block if and only if it is a 2–connected graph or else an isolated edge. Blocks are also called *non–separable graphs.* A graph is said to be *minimal* (respectively, *critical*) with respect to a property P if it has P but loses property P when any one of its edges is removed (respectively, when any one of its nodes along with all the edges incident to it are removed). Bonds (graphs with two nodes joined by at least two edges) are the only kind of non–simple multigraphs considered in this dissertation. All graphs not otherwise specified are assumed to be finite, loopless, simple, undirected 2–connected graphs. We refer the reader to [18] for any graph theoretic terminology not defined in this dissertation.

Following Whitney's pioneering decomposition of connected graphs into non–separable graphs in [57], Mac Lane [33] was the first to introduce a decomposition of 2–connected graphs. He used this decomposition to extend Whitney's results on planar graphs. Later, in his seminal work [54], Tutte defined and established many properties of a decomposition of 2–connected graphs into graphs each of which is either a cycle, a bond or a 3–connected graph. Tutte's decomposition is equivalent to that of Mac Lane's when applied to planar graphs. Hopcroft and Tarjan independently discovered the same decomposition [27] and exploited the uniqueness of its minimal version for algorithmic purposes. Trakhtenbrot [53] also proved the existence and uniqueness of essentially the same decomposition, in the guise of a canonical decomposition of 2–pole networks. Cunningham and Edmonds [4] proved the existence and uniqueness of the minimal decomposition in a rather general framework. We refer the reader to [4, pp. 744–755] for a brief comparison of the various approaches to decomposing 2–connected graphs. We attribute the basic decomposition to Tutte, since he was the first to present it explicitly.

More recently, Di Battista and Tamassia in [7] have introduced a versatile data structure named SPQR–tree representing the decomposition of a biconnected graph with respect to its triconnected components and have used it for planarity testing.

In [32], D. R. Lick surveyed the properties of critically and minimally n–connected and n–edge–connected graphs. Since then there have been a number of studies, in particular on construction methods for these classes of graphs. However, all the proposed constructions have been non–unique, and so entail considerable duplication and isomorphism testing when used as a basis for constructing catalogs of graphs. Here we exploit the uniqueness of the Tutte decomposition of 2–connected graphs to provide unique structural characterization conditions for the construction of several widely studied subclasses of the 2–connected graphs.

§2.2 The Tutte decomposition

For the purposes of this dissertation, a *bond* is defined to be a connected multigraph with exactly two nodes and at least two edges. A *cycle* is a connected simple graph with all nodes of degree two containing at least three edges and three nodes. A recent introduction to the Tutte decomposition can be found in [9], where it is used to give a structural characterization of locally finite 2–connected graphs. Below we follow the approach of Hopcroft and Tarjan [27].

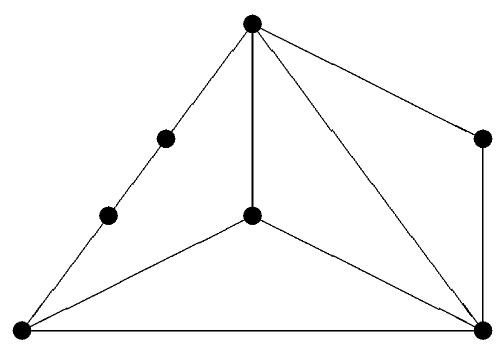

Figure 2.1: An example of a 2–connected graph G

For a 2–connected graph G, a decomposition into smaller 2–connected graphs is performed by the following process. Choose two nodes u and v in G, and let E_1, E_2,..., E_n be the equivalence classes in G such that two edges are in the same class if and only if they lie on a common path not containing u or v except as an endpoint. Let $F_1 = \bigcup_{i=1}^{k} E_i$ and $F_2 = \bigcup_{i=k+1}^{n} E_i$ be such that each F_i has at least 2 edges. Form $G_i = (V(F_i), F_i \cup (u,v))$, for $i = 1, 2$. The new edge (u,v) in each G_i is called a *virtual edge.* If (u,v) is already an edge in G then a bond with three edges is included as a component, the middle edge being the original (u,v) and the other two being virtual edges for G_1 and G_2. In this case, since $\Big\{(u,v)\Big\}$ is a singleton equivalence class, (u,v) cannot be used by itself as one of the G_i's. The reason for including the virtual edge in G_i is to maintain 2–connectivity throughout the decomposition.

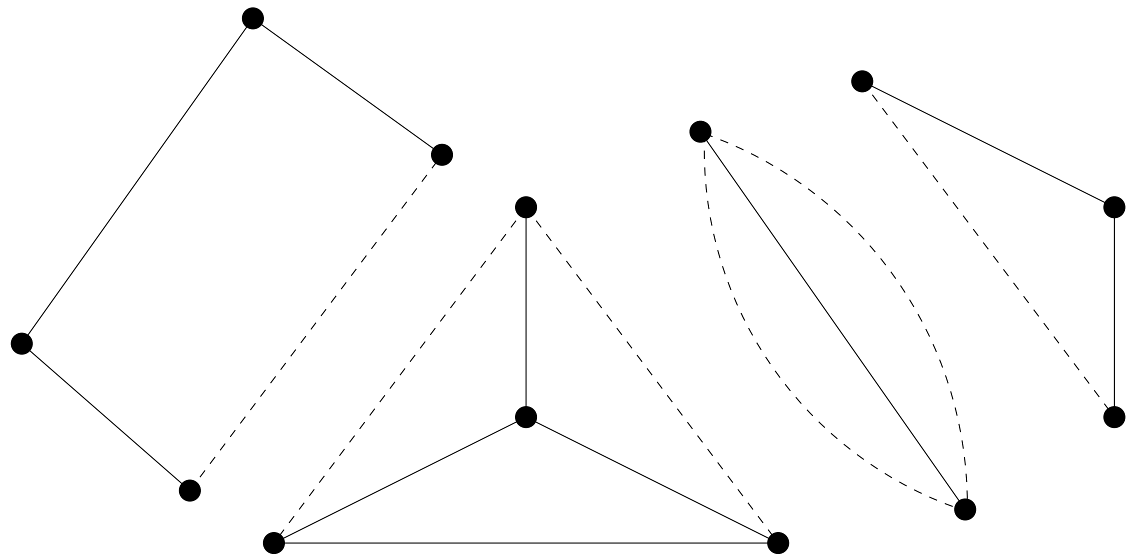

Figure 2.2: Tutte Decomposition $T(G)$ of the graph G

G_1 and G_2 are then further decomposed until every graph in the decomposition is indecomposable. We say that G is *indecomposable* if for every pair of nodes in G, (i) there is only one class, or (ii) there are three classes and each class consists of a single edge, or (iii) there are exactly two classes and one class consists of a single edge. It turns out that the indecomposable elements are 3–connected graphs, bonds with exactly 3 edges and cycles with exactly 3 edges. In the essentially equivalent decomposition defined by Tutte, any cycle or bond with at least 3 edges is considered indecomposable.

Theorem 1 (Tutte, 1966). *For any* 2*–connected graph* G*, the indecomposable elements of the Tutte decomposition are* 3*–connected graphs, bonds with at least* 3 *edges and cycles with at least* 3 *edges. The underlying graph is a tree.*

This is Theorem **11.63** in [54]. The indecomposable components were termed 3*–blocks* by Tutte, but here we refer to them as *basic components* or just *components*. The reason for our choice of indecomposable components is the uniqueness criterion described below. Given a graph G, by $T(G)$ we mean the Tutte decomposition of G,

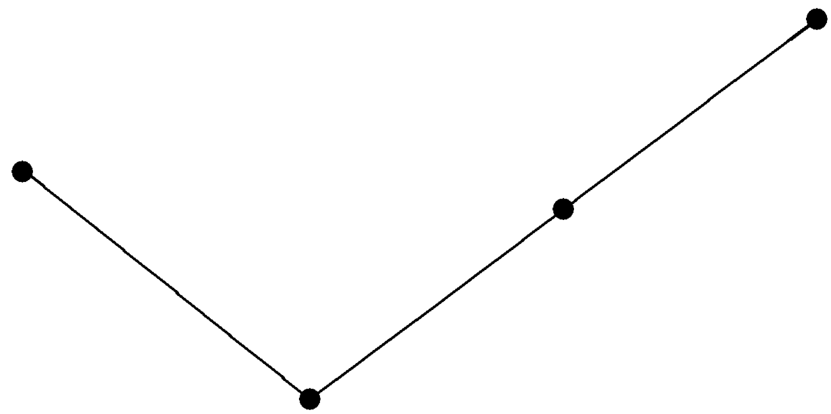

Figure 2.3: Underlying graph $\Gamma(G)$ of $T(G)$

which consists of the indecomposable elements of G and an associated underlying graph denoted by $\Gamma(G)$. If a 2–connected graph G is decomposed into G_1 and G_2 with a virtual edge (u, v), an edge is included in the underlying graph between the subgraphs corresponding to G_1 and G_2. If G_1 or G_2 is indecomposable, then it is represented by a node g_1 or g_2 in the underlying graph. If G_1 or G_2 can be further decomposed, then it corresponds to a subtree representing its decomposition. Thus, in $\Gamma(G)$ each node is a basic component and each edge corresponds to a pair of basic components (say, A and B) that share a single virtual edge (say, (u, v)). In this case, we say the nodes u and v are used as an *attachment pair* by the components A and B. When the two components are attached (and the virtual edge joining them is removed) for every pair adjacent in $\Gamma(G)$, the original graph G is recovered. Note that the order of attachment makes no difference to the recovery of G from $T(G)$.

For any 2–connected graph G, if no two bonds and no two cycles are adjacent in $T(G)$, then the corresponding decomposition is unique. This very useful uniqueness condition has been proved only in [3] and [28], as observed in [56, pp. 14] and in [4, pp. 745]. It may be observed that cycles (bonds) with more than 3 edges can be decomposed into cycles (bonds) of smaller size in a non–unique way. In [27], after decomposing a 2–connected graph using the procedure outlined above, it is suggested that adjacent cycles (bonds) should be merged into larger cycles (bonds) so as not to have any cycle (bond) adjacent to another cycle (bond). The resulting components are taken to be the triconnected components of the given graph. This decomposition is minimal in the sense that the number of components cannot be reduced. It is the unique decomposition of Cunningham and Edmond, and of Hopcroft and Tarjan. This uniqueness was not explicitly noted by Tutte. From now on we restrict our attention to the minimal decomposition and denote it by $T(G)$.

We remark that since the minimal decomposition for any 2–connected graph is unique and there is a $1-1$ correspondence between the set of all 2–connected graphs and the set of all minimal decompositions, any characterization based on minimal decompositions leads to a unique way of constructing the graphs under consideration, thereby avoiding duplicate representations and reducing considerably the need for isomorphism testing.

It is well understood that graphs with given connected components, and connected graphs with given blocks, have unique decompositions and constructions, given their connected components or their blocks. For this reason we restrict our attention to 2–connected graphs. Following Chaty and Chien [2], we note that it is straightforward to construct all separable minimally 2–edge–connected graphs by identifying two or more non–separable minimally 2–edge–connected graphs at nodes which become cut nodes after the identifications. Similarly, it is straightforward to construct all 3–edge–connected graphs by identifying two or more non–separable 3–edge–connected graphs at nodes which become cut nodes after the identifications.

Given the separation property of the basic components of a graph G, it follows that any two basic components of G can have at most two nodes in common. In $T(G)$, any edge which is not virtual is called *free*. For a component H, we denote by $E'(H)$ the set of free edges of H. There is a $1-1$ correspondence between the free edges in $T(G)$ and the edges of G. In our discussion, we often make no notational distinction between an edge in G and its corresponding free edge in $T(G)$. Although an edge e of G belongs to a unique component of $T(G)$, a node of G may belong to more than one component. We denote by H_e the component containing e as a free edge. For a node v, let H_v denote some component of $T(G)$ which contains v. By $\pi(H)$, we denote the node in $\Gamma(G)$ corresponding to the component H. A *terminal component* H is a component such that $\pi(H)$ is a terminal node in $\Gamma(G)$. An *internal component* H is a component such that $\pi(H)$ is an internal node in $\Gamma(G)$. It should be noted that removing an edge from a component does not affect the edge or node connectivity of any other component, while removing a node affects the edge or node connectivity of no component other than the ones to which it belongs. Also, removing an edge e of G from some component other than H leaves every free edge in H unaffected.

Lemma 1. *Let H be a basic component in the Tutte decomposition of G. For every virtual edge (u,v) in H, there is a corresponding $u-v$ path in $G-E'(H)$.*

Proof. Let K denote a basic component that shares u and v as attachment pair with H. Let e denote the edge in $\Gamma(G)$ that joins $\pi(H)$ and $\pi(K)$. Since $\Gamma(G)$ is a tree, $\Gamma(G)-e$ is disconnected and has exactly two connected components. Using the basic components in each of these connected components, construct graphs G_H and G_K by identifying the virtual edges of the adjacent components and then removing the virtual edges. By the uniqueness of the decomposition, we can recover G by identifying the nodes u and v in G_H and G_K and removing the virtual edge (u,v). Also, $E'(H) \subseteq E(G_H)$.

By the Tutte decomposition Theorem, both G_H and G_K are 2–connected. Therefore there is a $u-v$ path in $G_K - \{(u,v)\}$. This path contains no virtual edge, so is contained in $G-E'(H)$. □

It is easy to see that any cycle with at least 3 edges is minimally 2–connected, minimally 2–edge–connected, critically 2–connected and critically 2–edge–connected. Whenever an edge or a node is removed from a 3–connected graph, the resulting graph is 2–connected and 2–edge–connected. Thus 3–connected graphs are never minimally or critically 2–connected or 2–edge–connected. In a bond with at least 3 edges, the removal of an edge does not create a cut node or a bridge. But the removal of a node creates a trivial graph. So, bonds are critically but not minimally 2–connected and critically but not minimally 2–edge–connected. Thus we have the following lemma:

Lemma 2. *Let H be a basic component in the Tutte decomposition of a graph, e an edge in H and v a node in H. If $H - e$ or $H - v$ contains a cut node or a bridge, then H is a cycle.*

Lemma 3. *Let S be a basic component in $T(G)$. If $s_1, s_2 \in V(S) \cup E'(S)$, and $S - s_1 - s_2$ is disconnected then $G - s_1 - s_2$ is also disconnected.*

Proof. By Lemma 2, $S - s_1 - s_2$ disconnected implies that S is a cycle. If $G - s_1 - s_2$ is connected then there is a path P which is internally disjoint from S joining different connected components of $S - s_1 - s_2$. All edges of P are equivalent with respect to any two nodes on S. So, by Lemma 1, there must be an attachment pair $\{u, v\}$ separating $E(P)$ from $E(S) - (u, v)$, and (u, v) must be a virtual edge of S. Since u and v are separated in $S - s_1 - s_2$, one of s_1, s_2 must be (u, v). But neither of s_1, s_2 is a virtual edge, a contradiction. Thus $G - s_1 - s_2$ cannot be connected. □

Lemma 4. *Let $e = (u, v)$ be an arbitrary edge in a 2–connected graph G.*

(a) If $G - e$ contains a cut node w then w is a cut node in $H_e - e$.

(b) If $G - e$ contains a bridge f then f is a bridge in $H_e - e$.

Proof. (a) Removing e from G cannot create a cut node w in a component that does not contain e. If w is not a cut node of $H_e - e$ then the reconstruction of $G - e - w$ from basic components of G modified by removing e and w gives a connected graph. For, each component is still connected and any two components joined by a virtual edge still have at least one node in common. Thus $G - e - w$ is connected, a contradiction. Thus w must be a cut node of $H_e - e$.

(b) This is similar to (a), except that now any two components joined by a virtual edge in $T(G)$ have two nodes in common so $G - e - f$ would be connected, contradiction. □

§2.3 Decomposition characterizations

§2.3.1 Minimally 2–connected graphs

Non-unique constructive characterizations of minimally 2–connected graphs are given in [8, 43, 25, 26, 5, 47, 29]. They are studied in [22, 23, 24].

Theorem 2. *A 2–connected graph G is minimally 2–connected if and only if each free edge in $T(G)$ belongs to a cyclic component.*

Proof. If G is minimally 2–connected, then for any edge e, $G - e$ is connected but not 2–connected and has at least 3 nodes, so $G - e$ must contain a cut node. Then by Lemma 4(a), $H_e - e$ has a cut node and by Lemma 2, H_e is a cycle.

Conversely, suppose G is 2–connected but not minimally 2–connected. Then for some edge e, $G - e$ is 2–connected. If H_e were a cycle, we would have a node w in H_e such that $H_e - e - w$ is disconnected. By Lemma 3, $G - e - w$ would be disconnected too, contradicting the hypothesis that $G - e$ is 2–connected. Thus H_e cannot be a cycle. □

§2.3.2 Minimally 2–edge–connected graphs

Non-unique constructive characterizations of minimally 2–edge–connected graphs are given in [2, 37, 13, 63, 65, 62].

Theorem 3. *A 2–connected graph G is minimally 2–edge–connected if and only if each free edge in T(G) belongs to a cyclic component that contains at least one other free edge.*

Proof. If G is 2–connected and minimally 2–edge–connected, then for any edge e, $G - e$ must contain a bridge, say f. Then f is a bridge in $H_e - e$, by Lemma 4 (b), so H_e is a cycle, by Lemma 2. Note that e and f are free edges belonging to H_e.

Conversely, suppose G is 2–connected but not minimally 2–edge–connected, so that for some edge e, $G - e$ does not contain a bridge. Suppose H_e is a cycle; then $H_e - e$ contains no free edge. For, if f were such a free edge, then $H_e - e - f$ would be disconnected, hence, by Lemma 3, $G - e - f$ would be disconnected. This is contrary to the hypotheses that $G - e$ is connected and bridgeless. □

§2.3.3 Critically 2–connected graphs

Critically 2–connected graphs are non–uniquely characterized in [64] by B. W. Zhu.

Theorem 4. *A 2–connected graph G with at least 4 nodes is critically 2–connected if and only if each node that belongs to a 3–connected component belongs to at least one other component and every component that is a 3–cycle, has at most one free edge.*

Proof. Suppose G is critically 2–connected and suppose a node v of G belongs to a 3–connected component H_v and to no other component. Then $H_v - v$ is 2–connected, as there cannot be a cut node in H_v after removing a single node v. Since v belongs only to H_v, the node connectivity of all other components of $T(G)$ is unaffected by the removal of v. This means $G - v$ cannot have a cut node, contradicting the hypothesis that v is critical in G. Also, suppose v is a node belonging to a 3–cycle H_v and is incident to two free edges of H_v then $H_v - v$ is a virtual edge since there are at least 4 nodes in G. This means that $G - v$ has no cut nodes.

We treat the converse part in four cases, (1) Suppose that an arbitrary node v belonging to a 3–connected component H_v also belongs to at least one other basic component, say A. Then v along with another node, say u, is an attachment pair for the components H_v and A in $T(G)$. It can be seen that u is a cut node in $G - v$ since $G - v$ is connected but $G - v - u$ is disconnected.

(2) Suppose that an arbitrary node v belongs to a bond, say H_v. Since multiple edges are not allowed in G and bonds have at least 3 edges, H_v must have been associated with at least two other basic components, say A and B. Then v along with the other node, say u, of H_v is an attachment pair for the components A and B with H_v. As before, u is a cut node in $G - v$.

(3) Suppose that an arbitrary node v belongs to a cyclic component H_v having more than three edges. Then $H_v - v$ has a cut node, say u, so $H_v - v - u$ is disconnected. By Lemma 3, this implies $G - v - u$ is disconnected. Thus u is a cut node in $G - v$.

(4) Suppose that v is in a 3–cycle H_v with at most one free edge. At least one of the edges incident to v must be a virtual edge, say (v, u). Then, as before, u is a cut node in $G - v$. □

§2.3.4 Critically 2–edge–connected graphs

Critically 2–edge–connected graphs are non–uniquely characterized in [10] by X. F. Guo and B. W. Zhu.

Theorem 5. *A 2–connected graph G is critically 2–edge–connected if and only if each node of G belongs to a cyclic component which has a free edge not incident to the node.*

Proof. If G is 2–connected and critically 2–edge–connected then for any node v, $G - v$ contains a bridge. Let f be a bridge in $G - v$. Then obviously in $T(G)$, f is a free edge not incident to v. Then v must be a cut node in $G - f$, so by Lemma 4(a), v is a cut note in $H_f - f$, hence by Lemma 2 H_f is a cycle and clearly H_f contains v.

Conversely, let an arbitrary node v belong to a cyclic component H_v and let e be a free edge in it not incident to v. Then e is an edge in $G - v$, since it is free and not incident to v. Then $H_v - v - e$ is disconnected. By Lemma 3, this implies $G - v - e$ is disconnected. Since this holds for each node of G, G is critically 2–edge–connected.

□

§2.3.5 3–edge–connected graphs

A constructive but non–unique characterization of k-edge–connected (multi) graphs is given in [36].

Theorem 6. *A 2–connected graph G is 3–edge–connected if and only if each cyclic component has at most one free edge.*

Proof. Suppose that a cyclic component H in $T(G)$ contains two free edges e_1 and e_2. Then removing both from H would disconnect H. By Lemma 3, $G - \{e_1, e_2\}$ is disconnected, so G is not 3–edge–connected.

Conversely, suppose a graph G is 2–connected but not 3–edge–connected, *i.e.*, suppose there are two edges e_1 and e_2 in G such that $G - \{e_1, e_2\}$ is disconnected. Then by Lemma 4(b), e_2 is a bridge in $H_{e_1} - e_1$, so H_{e_1} is a cycle by Lemma 2. □

Note that since we are considering only simple graphs here, components that are bonds also cannot contain more than one free edge.

§2.3.6 Minimally 3–connected graphs

Constructive but non–unique characterizations of minimally 3–connected graphs are given in [14, 15, 16, 34, 35, 5, 6].

In this subsection, an edge e of a 3–connected graph G is said to be *essential* (respectively, *non–essential*) if $G - e$ is not 3–connected (respectively, is still 3–connected). In a minimally 3–connected graph all edges are essential. Consequently, for any edge e in a minimally 3–connected graph G, $G - e$ is 2–connected but not 3–connected and hence has a non–trivial Tutte decomposition. We give a characterization of minimal 3–connectivity in terms of conditions which must be satisfied for every edge. An existential characterization would be more useful for counting or constructing minimally 3–connected graphs, but we have been unable to find one.

Theorem 7. *A 2–connected graph G is minimally 3–connected if and only if for every edge $(u_1, u_2) = e$ in $E(G)$ the following conditions hold :*

1. *$G - e$ is 2–connected.*

2. *$\Gamma(G - e)$ is a path.*

3. *u_1 and u_2 belong to different terminal components of $T(G - e)$.*

4. *u_1 and u_2 are disjoint from the cut–pairs for those components.*

Proof. Let G be a 2–connected graph satisfying the conditions of the Theorem, for every edge in it. Let e be an edge in G. Since adding an edge e to $G - e$ cannot increase the number of cut–pairs, we can say that if a cut–pair exists in G then it must have been in $G - e$. A cut–pair in $G - e$ must either belong to a component in $T(G - e)$ or must be an attachment pair in $T(G - e)$. If a cut–pair belongs to a component then by lemma 2, the component has to be a cycle. By condition (2), no cyclic component in the decomposition of $G - e$ can have more than two virtual edges. Adding an edge e to $G - e$ makes any cut–pair from a cyclic component of $T(G - e)$ no longer a cut–pair. On the other hand, all cut–pairs arising from attachment pairs of $G - e$, are no longer a cut–pair by conditions (2), (3) and (4) when e is added back to make G. Hence G is a 3–connected graph.

Now, since condition (1) also holds for every edge in G, G is minimally 3–connected.

Conversely, for (1) let G be a minimally 3–connected graph. Then by the definition of connectivity, $G - e$ is 2–connected for any edge e in G.

(2) If for some edge $e = (u_1, u_2)$ in G, $\Gamma(G - e)$ is not a path then in $\Gamma(G - e)$ there is a node z with degree more than 2. Then there is a component H in $T(G-e)$ which is not on the path containing u_1 and u_2 and has an attachment pair (v_1, v_2) attaching H with the component corresponding to z. This attachment pair (v_1, v_2) is a cut–pair in G. This contradicts our assumption that G is 3–connected.

(3) If for some edge $e = (u_1, u_2)$ in G, $\Gamma(G - e)$ is a path, but one of $\left\{u_1, u_2\right\}$, say u_1, is not on a terminal component of $\Gamma(G - e)$, then there must be a terminal component not containing either u_1 or u_2. But then its attachment pair is still a cut–pair in G. This again contradicts the 3–connectivity of G.

(4) If for some edge $e = (u_1, u_2)$ in G, if one of $\left\{u_1, u_2\right\}$, say u_1, and some node v forms an attachment pair for a terminal component in $T(G - e)$ then u_1 and v form a cut–pair for G. This again contradicts the 3–connectivity of G. □

§2.3.7 2–connected cubic graphs

A graph is said to be *cubic* (or *3–regular*) if every node has degree three. Counting methods for cubic graphs are reported in [44, 45, 50, 60]. In [60], N. C. Wormald discusses enumeration of labeled cubic graphs with given connectivity. The connectivity possible for cubic graphs are only 1, 2 and 3. Since we deal here with

Tutte decomposition which handles graphs only with connectivity at least two, our characterizations will not address 1–connected graphs.

In the following Theorem we provide a unique constructive characterization of 2–connected cubic graphs. One of the conditions is that the 3–connected components used in such a construction must also be cubic. In Theorem 9 we provide a recursive constructive characterization of 3–connected cubic graphs. Thus using Theorems 8 and 9 together it should be possible to construct, and perhaps to count, all unlabeled 2–connected cubic graphs.

Theorem 8. *A 2–connected graph G is cubic if and only if in $T(G)$ all the following conditions hold :*

1. *Each 3–connected component is cubic and is not adjacent to another 3–connected component.*
2. *Each bond component has exactly three edges.*
3. *No cyclic or bond component is a terminal component.*
4. *Bond components are never adjacent to 3–connected components.*
5. *Each cyclic component has an even number of edges with alternating free and virtual edges.*

Proof. By conditions (1) and (2), 3–connected components and bond components are cubic. By conditions (3) and (5), if an arbitrary node in G belongs to a single component, it belongs to a 3–connected cubic component. So, it has degree three in G. Suppose that an arbitrary node v in G belongs to more than one component. By conditions (1) and (4), between two cubic components there is a cyclic component. Thus by condition (5), v belongs to at most one non–cyclic component. The nodes belonging to a cubic component that are attachment pairs used to attach a cyclic component with the cubic component have the same degree after the attachment. Moreover, by condition (5), each node in a cyclic component is in only one attachment pair of the cyclic component. Thus, G is a cubic graph.

Conversely, for (1) every node in a 3–connected component has degree at least three in it. If a node from a 3–connected component is of degree more than three then G cannot be cubic, as the attachments never reduce the degree of any node.

(2) If a bond component has more than three edges then it means that each of its nodes have degree more than three. As before, since the attachments never reduce the degree of any node, G cannot be a cubic graph.

(3) If a bond component is a terminal component then G is a multigraph. If a cyclic component is a terminal component then since only one edge of the component is virtual, G contains a node of degree two. So, G cannot be a cubic graph.

(4) If a bond component is adjacent to a 3–connected component then the nodes of the bond component are of degree more than three, as this type of attachment increases the degree of the nodes.

(5) If a node of a cyclic component is incident to two virtual edges then it has degree at least four in G. □

§2.3.8 3–connected cubic graphs

Labeled cubic graphs are counted by connectivity by N. C. Wormald in [60]. Here we give a characterization in terms of the existence of an edge with certain properties. This might be of use in counting unlabeled 3–connected cubic graphs, though it is not immediate on the basis of existing counting techniques.

Theorem 9. *A 2–connected graph G is 3–connected cubic if and only if there is an edge $(u_1, u_2) = e$ in $E(G)$ for which all the following conditions hold for $T(G - e)$ and $\Gamma(G - e)$:*

1. *$G - e$ is 2–connected, $\Gamma(G - e)$ is a path in which the poles u_1 and u_2 belong to different terminal components of $T(G - e)$ and are disjoint from the cut–pairs for those components.*

2. *All bond components occur only as internal components and have exactly three edges.*

3. *The terminal components are 3–cycles.*

4. *Any cyclic internal component has exactly four edges with alternating free and virtual edges.*

5. *No bond component is adjacent to a 3–connected component.*

6. *Each 3–connected component is cubic and is not adjacent to another 3–connected component.*

Proof. Let G be a 2–connected graph having an edge $(u_1, u_2) = e$ satisfying all the conditions (1) – (6). Note that condition (1) is similar to the conditions of Theorem 7. Therefore, by the proof of that Theorem, G is a 3–connected graph. Now, the conditions (2) – (6) are similar to the conditions of Theorem 8, except that in this Theorem terminal components are 3–cycles and there cycles cannot be terminal components. Adding e back to $G - e$ brings u_1 and u_2 up to three. Since adding e to $G - e$ affects only the degrees of u_1 and u_2 we can conclude that all nodes in G are of degree three by the proof of Theorem 8,

Conversely, choose an edge e in G. If condition (1) fails then G is not a 3–connected graph, as in the proof of Theorem 7. Noting again that removing e from G alters the degree only for the nodes u_1 and u_2, proceeding just as in the proof of Theorem 8 we can prove that if every node in G is of degree three then conditions (2) through (6) hold. □

Corollary 1. *A 2–connected graph G is 3–connected cubic if and only if* $1-6$ *hold for every edge.*

Proof. The proof of Theorem 9 actually showed the necessity of $1-6$ for every edge of a 3–connected cubic graph. □

§2.4 Related results and problems

There has been plenty of interest in problems related to dissection of polygons. It was only in [21] the solution was given for the problem of enumerating dissections of n–gon into k–gons ($k \geq 3$) subject to equivalence under rotations and reflections. The stratagem used in [21] to enable the symmetries of the configurations to be easily perceived was to convert the problem into a type of cell–growth problem. This turns the dissection of polygons problem inside out; instead of starting with a polygon and dividing it up, one starts with a number of small polygons (cells) and sticks them together to make the larger polygon (or a homeomorph of it). Such an assemblage of regular cells will be called a "cluster." Since no vertex of the dissected n–gon can belong to more than two cells, it follows that the structure of these clusters is essentially tree–like, as observed in [21]. It is this which makes their enumeration feasible, in contrast to the general cell–growth problem, which appears to be quite intractable.

Based on the theory developed in [21] one of the three authors, R. C. Read in [46], using Pólya's Theorem, has counted the number of dissections of an arbitrary polygon by the number of cells and by the number of sides of the polygon being dissected but by allowing the cells to be with unequal number of sides.

We remark here that the cell–growth problem can be characterized, and hence perhaps be counted, using Tutte decomposition. If bonds with only three edges are used between every pair of cyclic components with at least three edges, we get a cluster.

§2.5 Conclusions

Since the characterizing conditions for minimally 2–connected graphs are based on edges and the characterizing conditions for critically 2–connected graphs are based on nodes, the conjunction of the two sets of conditions gives a unique characterization for critical minimally 2–connected graphs. Such graphs are non–uniquely characterized in [30] by K. Huang.

Similarly, the conjunction of the characterizing conditions of Theorem 3 with those of Theorem 5 give a unique characterization for non–separable critical minimally 2–edge–connected graphs. The connected graphs having these as blocks are characterized non–uniquely in [61].

The unique characterizations of minimally 2–connected graphs, 2–connected minimally 2–edge–connected graphs and 2–connected 3–edge–connected graphs given in Theorems 2, 3 and 6 are the structural basis for unlabeled counting algorithms developed in Chapter 3 for these classes of graphs. Each of these algorithms is the first of its kind.

Theorems 4, 5, 7, 8 and 9 do not seem to lend themselves to efficient use as bases of unlabeled counting methods. Theorem 9 shows more promise than the others. It might provide a basis for counting unlabeled 3–connected cubic graphs. However, the fact that it deals with edge–rooted graphs suggests that this would require the development of new methods for counting unlabeled graphs.

CHAPTER 3

Cycle index sums and reversions

§3.1 Preliminaries

Let $\mathcal{A}$ be a permutation group with object set $X = \Big\{1, 2, \cdots, n\Big\}$. Each permutation σ in $\mathcal{A}$ can be written uniquely as a product of disjoint cycles. Let $j_k(\sigma)$ denote the number of cycles of length k in the disjoint cycle decomposition of σ. Then the *cycle index* of $\mathcal{A}$, denoted by $\boldsymbol{Z}(\mathcal{A})$, is a polynomial over the rational numbers $\mathbb{Q}$ in indeterminates $a_1, a_2, \cdots, a_n$ defined by

$$\boldsymbol{Z}(\mathcal{A}) = \frac{1}{|\mathcal{A}|} \sum_{\sigma \in \mathcal{A}} \prod_{k=1}^{n} a_k^{j_k(\sigma)}. \tag{3.1}$$

Each permutation σ of n objects can be associated with the partition of n which has exactly $j_k(\sigma)$ parts of each size k. We shall denote a partition of n by the vector $\boldsymbol{j} = (j_1, j_2, \cdots, j_n)$ where j_k is the number of parts of size k. Thus,

$$n = \sum_{k=1}^{n} k j_k.$$

In the following, let $\boldsymbol{j} \vdash n$ denote that $\boldsymbol{j}$ is a partition of n.

For example, the automorphism group of the complete graph $\boldsymbol{K}_n$ on n nodes is S_n, the symmetric group on n objects. Let $h(\boldsymbol{j})$ be the number of permutations σ in S_n whose cycle decomposition corresponds to the partition $\boldsymbol{j}$, so that $j_k(\sigma) = j_k$ for all k. Then

$$h(\boldsymbol{j}) = n! / \prod_{k=1}^{n} k^{j_k} j_k!$$

and the cycle index sum of S_n is given by

$$\boldsymbol{Z}(S_n) = \frac{1}{n!}\sum_{\boldsymbol{j}\vdash n} h(\boldsymbol{j}) \prod_{k=1}^{n} a_k^{j_k}.$$

Let $\mathcal{A}$ and $\mathcal{B}$ be two groups with disjoint object sets X and Y, respectively. The *product* of $\mathcal{A}$ and $\mathcal{B}$, denoted by $\mathcal{A}\mathcal{B}$, is a permutation group with object set $X \cup Y$. Each pair of permutations α in $\mathcal{A}$ and β in $\mathcal{B}$ determines a permutation $\alpha\beta$ in $\mathcal{A}\mathcal{B}$ such that for each z in $X \cup Y$,

$$(\alpha\beta)z = \begin{cases} \alpha z, & z \in X \\ \beta z, & z \in Y. \end{cases}$$

Pólya observed the elementary but useful fact that the cycle index of a product is the product of the cycle indices of the constituent groups [41, pp. 28].

Theorem 10. *The cycle index of the product* $\mathcal{A}\mathcal{B}$ *is given by*

$$\boldsymbol{Z}(\mathcal{A}\mathcal{B}) = \boldsymbol{Z}(\mathcal{A})\boldsymbol{Z}(\mathcal{B}).$$

Now, again let $\mathcal{A}$ be a permutation group with object set $X = \Big\{1, 2, \cdots, n\Big\}$. Then $x, y \in X$ are called $\mathcal{A} - $*equivalent* if there is a permutation σ in $\mathcal{A}$ such that $\sigma x = y$. It is a classical and immediate result that this is an equivalence relation. The $\mathcal{A}$–equivalence classes are called *orbits* of $\mathcal{A}$. For each x in X, let

$$Stab(x) = \Big\{\sigma \in \mathcal{A} \mid \sigma x = x\Big\}.$$

Then $Stab(x)$ is called the *stabilizer* of x. In some situations the stabilizer is more naturally thought of as $Aut(x)$, the group of automorphisms of x.

Burnside's Lemma. [38] *The number* $N(\mathcal{A})$ *of orbits of* $\mathcal{A}$ *is given by*

$$N(\mathcal{A}) = \frac{1}{|\mathcal{A}|}\sum_{\sigma\in\mathcal{A}} j_1(\sigma).$$

Proof. Let $m = N(\mathcal{A})$, let $X_1, X_2, \cdots, X_m$ be the orbits of $\mathcal{A}$, and let $x_i \in X_i$ for all i. Now, let $\sigma_1, \sigma_2 \in \mathcal{A}$. Since

$$\sigma_1 x_i = \sigma_2 x_i \Leftrightarrow \sigma_1^{-1}\sigma_2 x_i = x_i \Leftrightarrow \sigma_1^{-1}\sigma_2 \in Stab(x_i) \Leftrightarrow \sigma_2 Stab(x_i) = \sigma_1 Stab(x_i),$$

it follows that the map given by $\sigma Stab(x_i) \mapsto \sigma x_i$ is a well–defined bijection of the set of left cosets of $Stab(x_i)$ in $\mathcal{A}$ onto the orbit $X_i = \left\{\sigma x_i \mid \sigma \in \mathcal{A}\right\}$. Hence the number of elements in the orbit of any x_i is the index of the stabilizer of x_i in $\mathcal{A}$. Thus we have,

$$m|\mathcal{A}| = \sum_{i=1}^{m} |Stab(x_i)| \cdot |X_i|.$$

If x and x_i are in the same orbit, $Stab(x)$ and $Stab(x_i)$ are conjugate subgroups of $\mathcal{A}$, so $|Stab(x)| = |Stab(x_i)|$. Hence,

$$m|\mathcal{A}| = \sum_{x \in X} |Stab(x)|,$$

which can be also be expressed as

$$m|\mathcal{A}| = \sum_{x \in X} \sum_{\sigma \in Stab(x)} 1.$$

On interchanging the order of summation and modifying the summation indices accordingly, we have

$$m|\mathcal{A}| = \sum_{\sigma \in \mathcal{A}} \sum_{x = \sigma x} 1.$$

But the second sum is just $j_1(\sigma)$. Thus the proof is completed on division by $|\mathcal{A}|$ on both sides and recalling that $m = N(\mathcal{A})$. □

Burnside's Lemma forms the basis for numerous solutions to counting problems for unlabeled graphs. Basically, it says that number of orbits of a permutation group is the average number of fixed points of the permutations in the group. We will need a slight generalization of this famous lemma called the *weighted form* of Burnside's Lemma.

Let $\Re$ be any commutative ring containing the rationals and let w be a function, called the *weight function*, from the object set X of $\mathcal{A}$ into the ring $\Re$ which respects $\mathcal{A}$–equivalence. That is, if $x \sim_{\mathcal{A}} y$ then $w(x) = w(y)$, so that w can be considered to be defined on orbits of $\mathcal{A}$ by $w(X_i) = w(x_i)$. Sometimes such a weight function is called a *class function*. Then

Lemma 5. *The sum of the weights of the orbits of $\mathcal{A}$ is given by*

$$\sum_{i=1}^{m} w(X_i) = \frac{1}{|\mathcal{A}|} \sum_{\sigma \in \mathcal{A}} \sum_{\sigma x = x} w(x).$$

The proof is similar to that of Burnside's Lemma.

We will now develop the terminology to express Pólya's main enumeration Theorem, which is based on the weighted form of Burnside's Lemma.

Let $\mathcal{A}$ be a permutation group with object set $X = \left\{1, 2, \cdots, n\right\}$ and let $\mathcal{E}$ be the identity group on a countable object set Y of at least two elements. Then the *power group*, denoted by $\mathcal{E}^A$, has the set Y^X of functions from X to Y as its object set. The notation of $\mathcal{E}^A$ and terminology power group were established in a context in which $\mathcal{E}$ could be replaced by a different group $\mathcal{B}$. The permutations of $\mathcal{E}^A$ consist of all ordered pairs, written $(\alpha; \iota)$, of permutations α in $\mathcal{A}$ and ι in $\mathcal{E}$. The image of any function ϕ in Y^X under $(\alpha; \iota)$ is given by

$$((\alpha; \iota)\phi)(x) = \iota\phi(\alpha x) = \phi(\alpha x), \text{ for each } x \text{ in } X. \tag{3.2}$$

It is easily seen that $\alpha \mapsto (\alpha^{-1}; \iota)$ is a homomorphism of $\mathcal{A}$ onto $\mathcal{E}^A$. From now on we consider $\mathcal{E}^A$ to be the natural representation of $\mathcal{A}$ as a permutation group on Y^X, and rely on context to determine whether $\mathcal{A}$ is acting on X or on Y^X. We also note that this representation need not be faithful.

Let $w : Y \to \left\{0, 1, 2, \cdots\right\}$ be a function whose range is the set of nonnegative integers, and for which $|w^{-1}(k)| < \infty$ for all k. Then w is a *weight function* for Y, and the elements of Y are called *figures*. For each $k = 0, 1, 2, \cdots$ let

$$F_k = |w^{-1}(k)|,$$

the number of figures with weight k.

The series in the indeterminate x

$$F(x) = \sum_{k \geq 0} F_k x^k$$

which enumerates the elements of Y by weight is called the "figure counting series".

The *weight of a function* ϕ in Y^X is defined by

$$w(\phi) = \sum_{x \in X} w(\phi(x)).$$

It is easily seen that functions in the same orbit of the power group $\mathcal{E}^A$ have the same weight. We can therefore define the weight $w(\Theta)$ of an orbit Θ of $\mathcal{E}^A$ to be $w(\phi)$ for any ϕ in Θ. Since $|w^{-1}(k)| < \infty$ for each $k = 0, 1, 2, \cdots$, there are only a finite number of orbits of each weight. We let f_k denote the number of orbits of functions of weight k. Then the series in the indeterminate x

$$f(x) = \sum_{k \geq 0} f_k x^k$$

is called the *function counting series*, or the *configuration counting series* following Pólya. Now we can finally state

Pólya's Enumeration Theorem (Pólya [41]). *The function counting series $f(x)$ is determined by substituting $F(x^k)$ for each indeterminate a_k in $\boldsymbol{Z}(\mathcal{A})$. Symbolically,*

$$f(x) = \boldsymbol{Z}(\mathcal{A})[a_k \leftarrow F(x^k)] \qquad \text{or} \qquad f(x) = \boldsymbol{Z}(\mathcal{A})[F(x)]. \tag{3.3}$$

Proof. Let ι be the identity permutation on Y. For each α in $\mathcal{A}$, and each $k = 0, 1, 2, \cdots$, let $\mathfrak{n}(\alpha, k)$ denote the number of functions of weight k fixed by $(\alpha; \iota)$.

Now for each k, on restricting the power group $\mathcal{E}^{A}$ to the functions of weight k and applying Burnside's Lemma, we have

$$f_k = \frac{1}{|\mathcal{A}|}\sum_{\alpha\in\mathcal{A}} \mathfrak{n}(\alpha, k). \tag{3.4}$$

$$\text{Therefore, } f(x) = \sum_{k\geq 0}\frac{1}{|\mathcal{A}|}\sum_{\alpha\in\mathcal{A}} \mathfrak{n}(\alpha, k)x^k, \tag{3.5}$$

and on interchanging the order of summation we have

$$f(x) = \frac{1}{|\mathcal{A}|}\sum_{\alpha\in\mathcal{A}}\sum_{k\geq 0} \mathfrak{n}(\alpha, k)x^k. \tag{3.6}$$

Now, $\sum_{k\geq 0}\mathfrak{n}(\alpha, k)x^k$ is the counting series for all functions fixed by $(\alpha;\iota)$ and we seek an alternative form for this series.

Suppose ϕ in Y^X is fixed by $(\alpha;\iota)$. Then $(\alpha;\iota)\phi(x) = \phi(x)$, for all x in X, but from equation (3.2) we have $(\alpha;\iota)\phi(x) = \iota\phi(\alpha x)$. Thus we must have $\phi(\alpha x) = \phi(x)$ for all x, and hence ϕ must be constant on the disjoint cycles of α. Conversely, all functions constant on the cycles of α are fixed by $(\alpha;\iota)$.

Let $\mathfrak{z}_r$ be a cycle of length r in α. If ϕ maps the elements of $\mathfrak{z}_r$ to one of the F_k elements of Y of weight k, then the contribution to the weight of ϕ is rk. Thus it can be seen that the series

$$F(x^r) = \sum_{k\geq 0} F_k x^{rk}$$

has as its coefficient of x^{rk}, the number of ways ϕ can be defined on the elements of $\mathfrak{z}_r$ so that ϕ is fixed by $(\alpha;\iota)$ and the contribution to $w(\phi)$ is rk. It follows that $F(x^r)^{j_r(\alpha)}$ enumerates by weight the ways of defining functions which are fixed on all the cycles of length r in α.

On considering cycles of α of all lengths we can then express the series for fixed functions as the product

$$\sum_{k\geq 0}\mathfrak{n}(\alpha, k)x^k = \prod_{k=1}^{n} F(x^k)^{j_k(\alpha)}.$$

Equation (3.3) follows from this, equation (3.6) and the definition of $\boldsymbol{Z}(\mathcal{A})$. □

From now on we assume that the members of $Y = \Big\{Y_1, Y_2, \cdots\Big\}$ are graphs Y_i with order as weight and automorphism group $\mathcal{B}_i$. We define $\mathcal{B}$ to be the multi-set union of the automorphism groups $\mathcal{B}_i$ of graphs in Y. Define $\boldsymbol{Z}(\mathcal{B}) = \sum_{i=1}^{m} \boldsymbol{Z}(\mathcal{B}_i)$. Then $F(x)$, can be replaced in Pólya's Theorem by the cycle index sum $\boldsymbol{Z}(\mathcal{B})$. The replacement $a_i \leftarrow F(x^i)$ generalizes to $a_i \leftarrow \boldsymbol{Z}(\mathcal{B})[a_k \leftarrow a_{ik}]$, and the latter is abbreviated to $[\boldsymbol{Z}(\mathcal{B})]$. We can then state Robinson's [49] generalization of Pólya's Theorem,

Composition Theorem. *The cycle index sum of the $\mathcal{A}$–inequivalent functions in Y^X is $\boldsymbol{Z}(\mathcal{A})[\boldsymbol{Z}(\mathcal{B})]$.*

We refer the reader to [19] for a proof of this Theorem. To prove the Composition Theorem, Robinson generalized *Redfield's Lemma* [48]. We will need that here in order to prove Norman's Theorem. Let $\mathcal{A}$ be a permutation group with object set X and let $\Re$ be a commutative ring containing the rationals. Let χ be any *type* (generalizing the cycle type), $\chi : \mathcal{A} \times X \to \Re$, which satisfies the following condition for all α and β in $\mathcal{A}$ and all x in X :

$$\text{if } \alpha x = \beta y = x, \quad \text{then} \quad \chi(\alpha, x) = \chi(\beta^{-1}\alpha\beta, y). \tag{3.7}$$

The orbit of x determined by $\mathcal{A}$ will be denoted by $[x] = \Big\{\alpha x \mid \alpha \in \mathcal{A}\Big\}$. Recall that the stabilizer of x is $Stab(x) = \Big\{\alpha \in \mathcal{A} \mid \alpha x = x\Big\}$. Then

Lemma 6. *For any function χ satisfying equation* (3.7)*,*

$$\sum_{[x]} \frac{1}{|Stab(x)|} \sum_{\alpha \in Stab(x)} \chi(\alpha, x) = \frac{1}{|\mathcal{A}|} \sum_{\alpha \in \mathcal{A}} \sum_{x = \alpha x} \chi(\alpha, x). \tag{3.8}$$

Proof.

Let $\boldsymbol{Z}_\chi(Stab(x)) = \dfrac{1}{|Stab(x)|} \displaystyle\sum_{\alpha \in Stab(x)} \chi(\alpha, x)$. Then since for any $\alpha \in \mathcal{A}$ and for any $x \in X$, $Stab(x) = \alpha^{-1}(Stab(\alpha x))\alpha$, we have

$\boldsymbol{Z}_\chi(Stab(x)) = \boldsymbol{Z}_\chi(Stab(\alpha x))$. Thus for any $x \in X$ we can define

$$\begin{aligned}
\boldsymbol{Z}_\chi([x]) &= \boldsymbol{Z}_\chi(Stab(x)) \\
&= \frac{1}{|\mathcal{A}|}\sum_{x\in[x]} |Stab(x)|\boldsymbol{Z}_\chi(Stab(x)), \text{ since } |[x]| = \frac{|\mathcal{A}|}{|Stab(x)|} \\
&= \frac{1}{|\mathcal{A}|}\sum_{x\in[x]}\sum_{\alpha\in Stab(x)} \chi(\alpha, x).
\end{aligned}$$

Summing over all $[x]$ and reversing the summations yields the right side of (3.8). □

Now, let $\mathcal{A}$ be the automorphism group of a graph $\boldsymbol{G}$. We denote $\boldsymbol{Z}(\mathcal{A})$ by $\boldsymbol{Z}(\boldsymbol{G})$, or just by $\boldsymbol{G}$ if no confusion will result. Denote by $\boldsymbol{G}'$ the set of non–isomorphic node–rooted graphs whose underlying graph is $\boldsymbol{G}$. Node–rooting can be though of as giving a node v the special status of "root–node" and requiring that the root–node be preserved by isomorphisms and automorphisms.

$$\boldsymbol{Z}(\boldsymbol{G}') = \sum_{H\in\boldsymbol{G}'} \boldsymbol{Z}(H).$$

By convention the root–node is not included in the object set of the automorphism group of any $H \in \boldsymbol{G}'$. Then we have the following Theorem :

Theorem 11 (Norman [39]).

$$\boldsymbol{Z}(\boldsymbol{G}') = \frac{\partial}{\partial a_1}\boldsymbol{Z}(\boldsymbol{G}).$$

Proof. We first note that, if v is a node in $\boldsymbol{G}$, then $Stab(v)$ is the automorphism group of the graph obtained from $\boldsymbol{G}$ by rooting at v So

$$\boldsymbol{Z}(\boldsymbol{G}') = \frac{1}{a_1}\sum_{[v]} \boldsymbol{Z}(Stab([v])),$$

where $[v]$ denotes the equivalence class of v with respect to $Aut(\boldsymbol{G})$.

Now, define a function $\xi : \mathcal{A} \times X \to \Re$ by

$$\xi(\alpha, x) = \prod_k a_k^{j_k(\alpha)},$$

where $j_k(\alpha)$ is the number of cycles of length k in the disjoint cycle decomposition of α. Since object x does not figure in the definition of ξ, we can define $\xi(\alpha) = \xi(\alpha, x)$. Then

$$\boldsymbol{Z}_\xi(\mathcal{A}) = \frac{1}{|\mathcal{A}|} \sum_{\alpha \in \mathcal{A}} \xi(\alpha)$$

is the cycle index of $\mathcal{A}$, as given in the equation (3.1). To establish that ξ satisfies the conjugacy condition (3.7), note that if $(x_1, x_2, \cdots, x_k)$ is a cycle of α, then $(\beta^{-1}(x_1), \beta^{-1}(x_2), \cdots, \beta^{-1}(x_k))$ is a cycle of $\beta^{-1}\alpha\beta$. Thus, $j_k(\alpha) = j_k(\beta^{-1}\alpha\beta)$, for all k. So we have

$$\begin{aligned}
\boldsymbol{Z}(\boldsymbol{G}') &= \frac{1}{a_1} \sum_{[v]} \boldsymbol{Z}(Stab([v])) \\
&= \frac{1}{a_1} \sum_{[v]} \boldsymbol{Z}_\xi(Stab([v])) \\
&= \frac{1}{a_1|\mathcal{A}|} \sum_{\alpha \in \mathcal{A}} \sum_{x = \alpha x} \xi(\alpha, x), \text{ in view of Lemma 6} \\
&= \frac{1}{a_1|\mathcal{A}|} \sum_{\alpha \in \mathcal{A}} j_1(\alpha)\xi(\alpha) \\
&= \frac{\partial}{\partial a_1} \boldsymbol{Z}(\boldsymbol{G}).
\end{aligned}$$

□

So far, the cycle index $\boldsymbol{Z}(\boldsymbol{G})$ has been the disjoint cycle decomposition of automorphisms of $\boldsymbol{G}$ considered as acting only on the nodes of $\boldsymbol{G}$. We now extend this by viewing the edges of $\boldsymbol{G}$ as objects, so that each automorphism gives rise to a disjoint cycle decomposition over the nodes and edges. In the corresponding cycle type, an orientation preserving (respectively, an orientation reversing) edge cycle of length k corresponds to an indeterminate b_k (respectively, corresponds to an indeterminate c_k).

More explicitly, suppose σ is an automorphism of the graph $\boldsymbol{G} = (V(\boldsymbol{G}), E(\boldsymbol{G}))$. Then $\sigma : V(\boldsymbol{G}) \to V(\boldsymbol{G})$ induces a permutation $\tilde{\sigma} : E(\boldsymbol{G}) \to E(\boldsymbol{G})$. Any cycle C

of length k of the permutation $\tilde{\sigma}$ is represented by a factor b_k (respectively, c_k), if for every edge $(u, v) \in C$, $\sigma^k(u) = u$ and $\sigma^k(v) = v$(respectively, $\sigma^k(u) = v$ and $\sigma^k(v) = u$). Since we would like to keep track of nodes as well as edges, in each term of the cycle index the corresponding a's, b's and c's are all multiplied together in a single monomial. In this way the cycle index sum of a class of graphs $\mathfrak{G}$ can be considered to be a member of the commutative ring $\mathbb{Q}[b_1, c_1, b_2, c_2, \cdots][[a_1, a_2, \cdots]]$. In fact, for a graph $\boldsymbol{G}$ with automorphism group $\mathcal{A}$, the cycle index in all three types of indeterminates a_k, b_k and c_k is

$$\boldsymbol{Z}(\mathcal{A}) = \frac{1}{|\mathcal{A}|} \sum_{\sigma \in \mathcal{A}} \prod_k a_k^{\alpha_k(\sigma)} \prod_k b_k^{\beta_k(\sigma)} \prod_k c_k^{\gamma_k(\sigma)},$$

where $\alpha_k(\sigma)$ denotes the number of cycles of length k in the disjoint cycle decomposition of σ and $\beta_k(\sigma)$ (respectively, $\gamma_k(\sigma)$) denotes the number of orientation preserving (respectively, orientation reversing) edge cycles of length k in the disjoint cycle decomposition of $\tilde{\sigma}$.

Corresponding to Theorem 11, there are analogous theorems for rooting with respect to an oriented edge and for rooting with respect to an unoriented edge. For a graph $\boldsymbol{G}$, let the set of all non–isomorphic graphs obtained by rooting (but not orienting) the edges of $\boldsymbol{G}$ be denoted by $\boldsymbol{G}^*$. Edge–rooting can be thought of as distinguishing and then deleting an edge of the graph. When the root edge is oriented before deletion, the nodes incident to the edge are distinguished and labeled with a positive or negative sign (or a 0 or ∞). These distinguished nodes in the graph must be preserved in isomorphisms and automorphisms. In the literature these nodes are called *poles*. Edge–rooted 2–connected graphs are termed *networks* by Walsh [56]. When the root edge was oriented he called them *2–pole–networks*. Let $N(\boldsymbol{G})$ denote the set of 2–pole–networks obtained from a graph $\boldsymbol{G}$.

For any network $g \in \boldsymbol{G}^*$, define $\mathcal{A}^+(g)$ to be the subgroup of *positive* automorphisms of g – those which preserve the ends of the root edge – and define $\mathcal{A}^-(g)$ to be the subset of *negative* automorphisms of g – those which reverse the ends of the root edge. Then $\mathcal{A}(g) = \mathcal{A}^+(g) \cup \mathcal{A}^-(g)$. Define the *positive cycle index* $\boldsymbol{Z}^+(g)$ of g to be $\boldsymbol{Z}(\mathcal{A}^+(g))$, and extend the notation to sets of networks by linearity. If g is *pole–symmetric* – that is, if $|\mathcal{A}^-(g)| > 0$ – then clearly $|\mathcal{A}^+(g)| = |\mathcal{A}^-(g)| = |\mathcal{A}(g)|/2$. In this case, define the *negative cycle index* $\boldsymbol{Z}^-(g)$ of g to be $\boldsymbol{Z}(\mathcal{A}^-(g))$; otherwise define $\boldsymbol{Z}^-(g)$ to be zero. Note that if g' is the (unique up to isomorphism) 2–pole–network corresponding to any pole–symmetric g, then $\mathcal{A}(g') = \mathcal{A}^+(g)$. Any pole non–pole–symmetric g corresponds to two 2–pole networks g_1 and g_2, where the labels on the poles of g_1 and g_2 are opposite. Since $\mathcal{A}^-(g) = 0$, $\mathcal{A}(g) = \mathcal{A}^+(g)$ and $\mathcal{A}(g) = \mathcal{A}(g_1) = \mathcal{A}(g_2)$. Thus by linearity $2\boldsymbol{Z}^+(\boldsymbol{G}^*) = \boldsymbol{Z}(N(\boldsymbol{G}))$, since this holds for each network whether pole-symmetric or not.

Let $\boldsymbol{G}_R$ denote the set of all non–isomorphic edge–rooted graphs obtained by rooting at the pole–symmetric edges of $\boldsymbol{G}$ and let $\boldsymbol{G}_I$ denote the set of all non–isomorphic edge–rooted graphs obtained by rooting at the non–pole–symmetric edges of $\boldsymbol{G}$. Then $\boldsymbol{G}^* = \boldsymbol{G}_R \cup \boldsymbol{G}_I$. As noted above, each member of $\boldsymbol{G}_R$ corresponds to one member of $N(\boldsymbol{G})$ while each member of $\boldsymbol{G}_I$ corresponds to two. Symbolically, $N(\boldsymbol{G}) = \boldsymbol{G}_R \cup 2\boldsymbol{G}_I$. Let $\boldsymbol{Z}^+(\boldsymbol{G})$ denote $\boldsymbol{Z}(N(\boldsymbol{G}))/b_1$ and let $\boldsymbol{Z}^-(\boldsymbol{G})$ denote $\boldsymbol{Z}^-(\boldsymbol{G}^*)$. In the literature $\boldsymbol{Z}^+(\boldsymbol{G})$ and $\boldsymbol{Z}^-(\boldsymbol{G})$ are called *positive* and *negative cycle index sums*, respectively.

Theorem 12.

$$(i) \quad \boldsymbol{Z}^+(\boldsymbol{G}) = \frac{2}{a_1^2}\frac{\partial}{\partial b_1}\boldsymbol{Z}(\boldsymbol{G}).$$

$$(ii) \quad \boldsymbol{Z}^-(\boldsymbol{G}) = \frac{2}{a_2}\frac{\partial}{\partial c_1}\boldsymbol{Z}(\boldsymbol{G}).$$

Proof. We can apply the method of the proof of Theorem 11 to $N(\boldsymbol{G})$. Since there are two ways of labeling the poles, this gives

$$\boldsymbol{Z}(N(\boldsymbol{G})) = \frac{2b_1}{a_1^2}\frac{\partial}{\partial b_1}\boldsymbol{Z}(\boldsymbol{G}), \tag{3.9}$$

in view of the convention that the poles themselves are deleted from the cycle index. Then by the definition of $\boldsymbol{Z}^+(\boldsymbol{G})$, we have (i). Now applying the method of proof of Theorem 11 to $\boldsymbol{G}^*$ gives

$$\boldsymbol{Z}(\boldsymbol{G}^*) = \frac{b_1}{a_1^2}\frac{\partial}{\partial b_1}\boldsymbol{Z}(\boldsymbol{G}) + \frac{c_1}{a_2}\frac{\partial}{\partial c_1}\boldsymbol{Z}(\boldsymbol{G}), \tag{3.10}$$

since the root edge may be left fixed (b_1 with ends a_1^2) or reversed (c_1 with ends a_2).

Now it is enough to show that

$$\boldsymbol{Z}^-(\boldsymbol{G}) = 2\boldsymbol{Z}(\boldsymbol{G}^*) - \boldsymbol{Z}(N(\boldsymbol{G})), \tag{3.11}$$

since (ii) will then follow from (3.9) and (3.10). In turn, (3.11) will follow by linearity once it is verified for an arbitrary network and corresponding 2–pole–network(s).

First, suppose g is a network in $\boldsymbol{G}_R$. Then

$$\begin{aligned}\boldsymbol{Z}(\mathcal{A}(g)) &= \frac{1}{|\mathcal{A}(g)|}\sum_{\sigma\in A(g)}\boldsymbol{Z}(\sigma)\\ &= \frac{1}{|\mathcal{A}(g)|}\Big\{\sum_{\sigma\in\mathcal{A}^+(g)}\boldsymbol{Z}(\sigma) + \sum_{\sigma\in\mathcal{A}^-(g)}\boldsymbol{Z}(\sigma)\Big\}.\end{aligned}$$

Since $|\mathcal{A}(g)| = 2|\mathcal{A}^+(g)| = 2|\mathcal{A}^-(g)|$,

$$\begin{aligned}\boldsymbol{Z}(\mathcal{A}(g)) &= \frac{1}{2|\mathcal{A}^+(g)|}\sum_{\sigma\in\mathcal{A}^+(g)}\boldsymbol{Z}(\sigma) + \frac{1}{2|\mathcal{A}^-(g)|}\sum_{\sigma\in\mathcal{A}^-(g)}\boldsymbol{Z}(\sigma)\\ &= \frac{1}{2}\boldsymbol{Z}(\mathcal{A}^+(g)) + \frac{1}{2}\boldsymbol{Z}(\mathcal{A}^-(g)).\end{aligned}$$

Now if g' is the unique 2–pole–network in $\boldsymbol{G}_R$ corresponding to g, then $\mathcal{A}(g') = \mathcal{A}^+(g)$ and, $\boldsymbol{Z}(\mathcal{A}(g')) = \boldsymbol{Z}(\mathcal{A}^+(g))$. Thus the contribution of $g \in \boldsymbol{G}_R$ to the right hand side of (3.11) is $2\Big\{\frac{1}{2}\boldsymbol{Z}(\mathcal{A}^+(g)) + \frac{1}{2}\boldsymbol{Z}(\mathcal{A}^-(g))\Big\} - \boldsymbol{Z}(\mathcal{A}^+(g)) = \boldsymbol{Z}(\mathcal{A}^-(g))$, in agreement with the left hand side of (3.11).

Second, suppose g is a network in $\boldsymbol{G}_I$. Then as noted before, g corresponds to two 2–pole networks g_1 and g_2, both with the same automorphism group as g. So g contributes $2\boldsymbol{Z}(\mathcal{A}(g))$ to $\boldsymbol{Z}(N(\boldsymbol{G}))$. Thus the contribution of $g \in \boldsymbol{G}_I$ to the right hand side of (3.11) is $2\boldsymbol{Z}(\mathcal{A}(g)) - 2\boldsymbol{Z}(\mathcal{A}(g)) = 0$. This agrees with the left hand side of (3.11) since $\boldsymbol{Z}(\mathcal{A}^-(g)) = 0$ in this case. □

In order to count the minimal Tutte decompositions rooted at specified components or at attachments between specified components we will need a generalization of the Composition Theorem in which $\boldsymbol{Z}(\mathcal{A})\,[a_i \leftarrow \boldsymbol{Z}(\mathcal{B})[a_k \leftarrow a_{ik}]]$ generalizes to

$$\boldsymbol{Z}(\mathcal{A})\Big[a_i, b_i \leftarrow \boldsymbol{Z}^+(\mathcal{D})[a_k \leftarrow a_{ik}, b_k \leftarrow b_{ik}, c_k \leftarrow c_{ik}],$$
$$c_i \leftarrow \boldsymbol{Z}^-(\mathcal{D})[a_k \leftarrow a_{ik}, b_k \leftarrow b_{ik}, c_k \leftarrow c_{ik}]\Big],$$

which is abbreviated to $\boldsymbol{Z}(\mathcal{A})[a_1, \boldsymbol{Z}^+(\mathcal{D}), \boldsymbol{Z}^-(\mathcal{D})]$.

We will prove a generalization to Pólya's Theorem and arrive at a theorem from which this generalization of the Composition Theorem will follow as a special case. But we need more definitions and facts before plunging into the statement and proof of the generalized Pólya's Theorem.

Let $\mathcal{A}$ be the automorphism group of a graph whose nodes are $X = \Big\{1, 2, \cdots, n\Big\}$ and whose edges are $Y = \Big\{1, 2, \cdots, \binom{n}{2}\Big\}$. Then the action of $\mathcal{A}$ on X induces an action on Y as described earlier in this section. Then the generalized cycle index of $\mathcal{A}$ is

$$\boldsymbol{Z}(\mathcal{A}) = \frac{1}{|\mathcal{A}|}\sum_{\sigma\in\mathcal{A}}\prod_k a_k^{\alpha_k(\sigma)}\prod_k b_k^{\beta_k(\sigma)}\prod_k c_k^{\gamma_k(\sigma)}.$$

Let $\mathcal{X} = X \cup Y$. We wish to count functions from $\mathcal{X}$ into $\mathcal{Z} = Z_1 \cup Z_2$ which map X into Z_1 and Y into Z_2. We can consider any such function as a pair in $Z_1^X \times Z_2^Y$. Further, elements of Z_2 will be 2–pole networks. A 2–pole network is a 2–connected graph with an edge say (u, v) removed from it and the labels with signs $+$ and $-$ are given to the poles u and v respectively. Let $\mathcal{F}$ denote the pole–reversal operation on 2–pole networks. That is, $\mathcal{F}$ maps G to the result of reversing G's poles. Let $\mathcal{R}$ be the set of pole reversible members of Z_2. Then $\mathcal{F}(G) = G$, for any $G \in \mathcal{R}$. Let $\mathcal{I} = Z_2 - \mathcal{R}$. Then the action of $\mathcal{F}$ on $\mathcal{I}$ is a disjoint set of 2–cycles. Then each σ in

$\mathcal{A}$ induces $\boldsymbol{\sigma}$ which acts on functions in $Z_1^X \times Z_2^Y$ as follows : for all x in X

$$(\boldsymbol{\sigma}\phi)(x) = \phi(\sigma^{-1}x), \text{ and for all } y = (u,v) \text{ in } Y \text{ with } u < v,$$

$$(\boldsymbol{\sigma}\phi)(y) = \begin{cases} \phi(\sigma^{-1}y), & \text{if } \sigma^{-1}u < \sigma^{-1}v, \\ \mathcal{F}\phi(\sigma^{-1}y), & \text{otherwise.} \end{cases} \tag{3.12}$$

Let $\boldsymbol{\mathcal{A}}$ denote the set of all $\boldsymbol{\sigma}$. It is easily seen that $\sigma \mapsto \boldsymbol{\sigma}$ is a homomorphism from $\mathcal{A}$ to $\boldsymbol{\mathcal{A}}$. From now on we consider $\boldsymbol{\mathcal{A}}$ to be a natural representation of $\mathcal{A}$ as a permutation group on $Z_1^X \times Z_2^Y$. We also note that this need not be faithful. In the following, the object set of σ should be clear from the context if not specified.

Let $\Omega = \{0, 1, \cdots\}$ and let an arbitrary element $(k_1, k_2, \cdots)$ of Ω be denoted by $\boldsymbol{k}$. A sequence of elements $(z_1, z_2, \cdots)$ in $\mathcal{Z}$ shall be denoted by $\boldsymbol{z}$ and $(z_1^{k_1}, z_2^{k_2}, \cdots)$ by $\boldsymbol{z}^{\boldsymbol{k}}$. Further, let Ω^Ω denote all sequences in Ω such that all but a finite number of terms are equal to 0. Let $w : \mathcal{Z} \to \Omega^\Omega$ be a function for which $|w^{-1}(\boldsymbol{k})| < \infty$, for all $\boldsymbol{k}$ in Ω^Ω. Then w is a *weight function* for $\mathcal{Z}$, and the elements of $\mathcal{Z}$ are called *figures*. Let $[z_1^{k_1}, \cdots, z_r^{k_r}]$ denote the coefficient of operator. Let $F(\boldsymbol{z})$ be a function such that the coefficient of $z_1^{k_1}, \cdots, z_r^{k_r}$ in $F(\boldsymbol{z})$

$$[z_1^{k_1}, \cdots, z_r^{k_r}]F(\boldsymbol{z}) = |w^{-1}(k_1, \cdots, k_r, 0, \cdots) \cap Z_1|, \text{ let } P(\boldsymbol{z}) \text{ be such that}$$

$$[z_1^{k_1}, \cdots, z_r^{k_r}]P(\boldsymbol{z}) = |w^{-1}(k_1, \cdots, k_r, 0, \cdots) \cap \mathcal{R}| + 2|w^{-1}(k_1, \cdots, k_r, 0, \cdots) \cap \mathcal{I}|$$

and let $R(\boldsymbol{z})$ be such that $[z_1^{k_1}, \cdots, z_r^{k_r}]R(\boldsymbol{z}) = |w^{-1}(k_1, \cdots, k_r, 0, \cdots) \cap \mathcal{R}|$.

The *weight of a function* ϕ in $Z_1^X \times Z_2^Y$ is defined by

$$w(\phi) = \sum_{x \in \mathcal{X}} w(\phi(x)),$$

where the summation is taken term–wise. It is easily seen that functions in the same orbit of the power group $\boldsymbol{\mathcal{A}}$ have the same weight. We can therefore define the weight $w(\Theta)$ of an orbit Θ of $\boldsymbol{\mathcal{A}}$ to be $w(\phi)$ for any ϕ in Θ. Since $|w^{-1}(\boldsymbol{k})| < \infty$ for each sequence $\boldsymbol{k}$ in Ω^Ω, there are only a finite number of orbits of each weight.

Let $f(\boldsymbol{z})$ be such that $[z_1^{k_1}, \cdots, z_r^{k_r}]f(\boldsymbol{z})$ is number of orbits of weight $(k_1, \cdots, k_r)$. Then $f(\boldsymbol{z})$ is called the *function counting series*, or the *configuration counting series* following Pólya. Then Pólya's Theorem generalizes to

Generalized Pólya's Enumeration Theorem. *The function counting series $f(\boldsymbol{z})$ is determined by substituting for each indeterminate a_k, b_k and c_k in $\boldsymbol{Z}(\mathcal{A})$, $F(\boldsymbol{z}^k)$, $P(\boldsymbol{z}^k)$ and $R(\boldsymbol{z}^k)$ respectively. Symbolically,*

$$f(\boldsymbol{z}) = \boldsymbol{Z}(\mathcal{A})[a_k \leftarrow F(\boldsymbol{z}^k), b_k \leftarrow P(\boldsymbol{z}^k), c_k \leftarrow R(\boldsymbol{z}^k)]$$

This is abbreviated as $f(\boldsymbol{z}) = \boldsymbol{Z}(\mathcal{A})[F(\boldsymbol{z}), P(\boldsymbol{z}), R(\boldsymbol{z})]$. (3.13)

Proof. For each σ in $\mathcal{A}$, and each sequence of non negative integers $\boldsymbol{k}$ in Ω^Ω, we let $\mathfrak{n}(\sigma, \boldsymbol{k})$ denote the number of functions of weight $\boldsymbol{k}$ fixed by $\boldsymbol{\sigma}$. Now for each $\boldsymbol{k}$, on restricting the power group $\boldsymbol{\mathcal{A}}$ to the functions of weight $\boldsymbol{k}$ and applying Burnside's Lemma, we have

$$\left[z_1^{k_1}, \cdots, z_r^{k_r}\right] f(\boldsymbol{z}) = \frac{1}{|\mathcal{A}|} \sum_{\sigma \in \mathcal{A}} \mathfrak{n}(\sigma, \boldsymbol{k}).$$

Therefore, $f(\boldsymbol{z}) = \sum_{k_1 \geq 0} \sum_{k_2 \geq 0} \frac{1}{|\mathcal{A}|} \sum_{\sigma \in \mathcal{A}} \mathfrak{n}(\sigma, \boldsymbol{k}) \boldsymbol{z}^{\boldsymbol{k}}$,

and on interchanging the order of summation we have

$$f(\boldsymbol{z}) = \frac{1}{|\mathcal{A}|} \sum_{\sigma \in \mathcal{A}} \sum_{\boldsymbol{k} \in \Omega} \mathfrak{n}(\sigma, \boldsymbol{k}) \boldsymbol{z}^{\boldsymbol{k}}. \tag{3.14}$$

Now, $\sum_{\boldsymbol{k} \in \Omega} \mathfrak{n}(\sigma, \boldsymbol{k}) \boldsymbol{z}^{\boldsymbol{k}}$ is the counting series for all functions fixed by $\boldsymbol{\sigma}$ and we seek an alternative form for this series.

Suppose ϕ in $Z_1^X \times Z_2^Y$ is fixed by $\boldsymbol{\sigma}$. Then $(\boldsymbol{\sigma})\phi(x) = \phi(x)$, for all x in $\mathfrak{X}$, but from equation (3.12) we have $(\boldsymbol{\sigma})\phi(x) = \phi(\sigma^{-1}x)$. Thus we must have $\phi(\sigma^{-1}x) = \phi(x)$ for all x, and hence ϕ must be constant on the disjoint cycles of σ. Conversely, all functions constant on the cycles of σ are fixed by $\boldsymbol{\sigma}$.

Let $\mathfrak{Z}_t$ be a node cycle of length t in σ. If ϕ sends the elements of $\mathfrak{Z}_t$ to one of the $[z_1^{k_1}, \cdots, z_r^{k_r}]F(\boldsymbol{z})$ elements of $\mathcal{Z}$ of weight $(k_1, \cdots, k_r)$, then the contribution to the weight of ϕ is $(tk_1, \cdots, tk_r)$, since ϕ must be constant on the nodes of $\mathfrak{Z}_t$. Thus it can be seen that the series $F(\boldsymbol{z}^t)$ has as its coefficient of $z_1^{tk_1}, \cdots, z_r^{tk_r}$ the number of ways ϕ can be defined on the elements of $\mathfrak{Z}_t$ so that ϕ is fixed by $\boldsymbol{\sigma}$ and the contribution to $w(\phi)$ is $(tk_1, \cdots, tk_r)$. It follows that $F(\boldsymbol{z}^t)^{\alpha_t(\sigma)}$ enumerates by weight the ways of defining functions which are fixed on all node cycles of length t in σ.

Similarly, let $\mathfrak{U}_t$ be an orientation preserving edge cycle $(e_1, \cdots, e_t)$ of length t in σ. If ϕ sends the elements of $\mathfrak{U}_t$ to one of the $[z_1^{k_1}, \cdots, z_r^{k_r}]P(\boldsymbol{z})$ elements of $\mathcal{Z}$ of weight $(k_1, \cdots, k_r)$, then the contribution to the weight of ϕ is $(tk_1, \cdots, tk_r)$ since the value of ϕ on one edge, say e_1, determines the values of $e_2, \cdots, e_t$, and these are consistent (mapping back to e_1) since $\mathfrak{U}_t$ is orientation preserving. Thus it can be seen that the series $P(\boldsymbol{z}^t)$ has as its coefficient of $z_1^{tk_1}, \cdots, z_r^{tk_r}$ the number of ways ϕ can be defined on the elements of $\mathfrak{U}_t$ so that ϕ is fixed by $\boldsymbol{\sigma}$ and the contribution to $w(\phi)$ is $(tk_1, \cdots, tk_r)$. It follows that $P(\boldsymbol{z}^t)^{\beta_t(\sigma)}$ enumerates by weight the ways of defining functions which are fixed on all orientation preserving edge cycles of length t in σ.

Similarly, let $\mathfrak{V}_t$ be an orientation reversing edge cycle $(e_1, \cdots, e_t)$ of length t in σ. The critical thing here is that $\sigma^t(e_1) = e_1^R$, so mapping ϕ not left fixed unless $\phi(e_1) \in R$. If ϕ sends the elements of $\mathfrak{V}_t$ to one of the $[z_1^{k_1}, \cdots, z_r^{k_r}]R(\boldsymbol{z})$ elements of $\mathcal{Z}$ of weight $(k_1, \cdots, k_r)$, then the contribution to the weight of ϕ is $(tk_1, \cdots, tk_r)$. Thus it can be seen that the series $R(\boldsymbol{z}^t)$ has as its coefficient of $z_1^{tk_1}, \cdots, z_r^{tk_r}$, the number of ways ϕ can be defined on the elements of $\mathfrak{V}_t$ so that ϕ is fixed by $\boldsymbol{\sigma}$ and the contribution to $w(\phi)$ is $(tk_1, \cdots, tk_r)$. It follows that $R(\boldsymbol{z}^t)^{\gamma_r(\sigma)}$ enumerates by weight the ways of defining functions which are fixed on all orientation reversing edge cycles of length t in σ.

On considering all cycles of σ, we can then express the series for fixed functions as the product

$$\sum_{k\geq 0} \mathfrak{n}(\alpha, \boldsymbol{k})\boldsymbol{z}^{\boldsymbol{k}} = \prod_t F(\boldsymbol{z}^t)^{\alpha_t(\sigma)} \prod_t P(\boldsymbol{z}^t)^{\beta_t(\sigma)} \prod_t R(\boldsymbol{z}^t)^{\gamma_t(\sigma)}.$$

Equation (3.13) follows from this, equation (3.14) and the definition of $\boldsymbol{Z}(\mathcal{A})$, since the choices for cycles are independent, and the product of the generating functions generates all of these choices due to distributive law for multiplication over addition.

□

As a consequence of this generalization of Pólya's Theorem, in Pólya's Theorem, $\boldsymbol{Z}(\mathcal{A})[a_k \leftarrow F(x^k)]$ generalizes to $\boldsymbol{Z}(\mathcal{A})[a_k \leftarrow F(x^k, y^k), b_k \leftarrow P(x^k, y^k), c_k \leftarrow Q(x^k, y^k)]$, which is abbreviated to $\boldsymbol{Z}(\mathcal{A})[F(x,y), P(x,y), Q(x,y)]$. For us, when counting graphs, the resulting power series is the counting series that lie in the ring $\mathbb{Z}[y][[x]]$, where $\mathbb{Z}$ is the integers, while the triple $F(x,y), P(x,y), Q(x,y)$ of series in $\mathbb{Q}[y][[x]]$ defines a more general homomorphism from the cycle index ring to the counting series ring. In the following, for a power series $P(x,y)$, when it is not misleading, we might denote $P(x^k, y^k)$ simply by P_k.

Now, let $\boldsymbol{P}$ denote a set of 2–pole networks and $\boldsymbol{R}$ denote the subset (possibly empty) of it consisting of those that are pole–reversible. Let $\boldsymbol{F}$ denote a set of node–rooted graphs. Let $\mathfrak{G}$ denote the set of all graphs which can be obtained by selecting a graph from a set $\boldsymbol{A}$, replacing each node with a graph from a set $\boldsymbol{F}$, and replacing each edge with a network from a set $\boldsymbol{P}$. Then

Generalized Composition Theorem. *The cycle index sum of the* $\boldsymbol{A}$*–inequivalent functions in* $Z_1^X \times Z_2^Y$ *is* $\boldsymbol{Z}(\mathfrak{G}) = \boldsymbol{Z}(\boldsymbol{A})\Big[\boldsymbol{Z}(\boldsymbol{F}), \boldsymbol{Z}(\boldsymbol{P}), \boldsymbol{Z}(\boldsymbol{R})\Big]$.

In order to prove the Generalized Composition Theorem by applying the Generalized Pólya's Theorem, we consider "decorating" each graph in $\mathfrak{G}$ by assigning a positive integer (not necessarily unique) to each node and edge and orienting some (perhaps none) of the edges.

Let f_i be the weight that is associated with the assignment of i to a node of $\mathfrak{G}$, let g_i be the weight that is associated with the assignment of i to an oriented edge of $\mathfrak{G}$ and let h_i be the weight that is associated with the assignment of i to an edge of $\mathfrak{G}$ that has been left unoriented.

Let $\alpha = \sum_i f_i$, $\beta = \sum_i g_i$ and $\gamma = \sum_i h_i$. In the cycle index sums, c's are substituted with γ, to denote that those edges are left unoriented and b's are substituted with $\gamma + 2\beta$ to denote that those edges are either not oriented or oriented in one of the two ways. Then by the Generalized Pólya's Theorem

$$\left[\prod f_i^{k_i} \prod g_i^{n_i} \prod h_i^{m_i}\right] \boldsymbol{Z}(\mathfrak{G})[\alpha, 2\beta + \gamma, \gamma]$$

is the number of non-isomorphic decorated graphs in $\mathfrak{G}$ having exactly k_i nodes assigned i, for $i = 1, 2, \cdots$, n_i edges that are oriented and assigned i, for $i = 1, 2, \cdots$, and m_i edges that are not oriented and assigned i, for $i = 1, 2, \cdots$.

Then

$$\left(\boldsymbol{Z}(\boldsymbol{A})\Big[\boldsymbol{Z}(\boldsymbol{F}), \boldsymbol{Z}(\boldsymbol{P}), \boldsymbol{Z}(\boldsymbol{R})\Big]\right)[\alpha, 2\beta + \gamma, \gamma] \overset{\text{by algebraic associativity}}{=}$$

$$\boldsymbol{Z}(\boldsymbol{A})\Big[\boldsymbol{Z}(\boldsymbol{F})[\alpha, 2\beta + \gamma, \gamma], \boldsymbol{Z}(\boldsymbol{P})[\alpha, 2\beta + \gamma, \gamma], \boldsymbol{Z}(\boldsymbol{R})[\alpha, 2\beta + \gamma, \gamma]\Big]$$

$$\overset{\text{by combinatorial associativity}}{=} \boldsymbol{Z}(\mathfrak{G})[\alpha, 2\beta + \gamma, \gamma].$$

This gives

$$\left(\boldsymbol{Z}(\mathfrak{G}) - \boldsymbol{Z}(\boldsymbol{A})\Big[\boldsymbol{Z}(\boldsymbol{F}), \boldsymbol{Z}(\boldsymbol{P}), \boldsymbol{Z}(\boldsymbol{R})\Big]\right)[\alpha, 2\beta + \gamma, \gamma] = 0.$$

The result of substitution of α, $2\beta + \gamma$ and γ are algebraic combinations of power sums in the sets of variables $\Big\{f_i\Big\}$, $\Big\{g_i\Big\}$ and $\Big\{h_i\Big\}$. Since these power sums are algebraically independent, as noted by Pólya in [41, pp. 26] in a similar but simpler context, it follows that,

$$Z(\mathfrak{G}) = Z(A)\Big[Z(F), Z(P), Z(R)\Big] \tag{3.15}$$

§3.2 Deriving basic equations

In the rest of this chapter and in Chapters 4 through 6, applying the decomposition characterization Theorems, which are based on Tutte's decomposition Theorem and developed in the previous chapter of this dissertation, the defining relations for minimally 2–connected, 3–edge–connected blocks and minimally 2–edge–connected blocks are expressed in terms of cycle index sums. The basic tool used for expressing the decomposition characterizations as cycle index sum relations is Generalized Composition Theorem. These relations are then transformed by an appropriate choice of ring homomorphism so that they involve only counting series. The fundamental algebraic facts upon which this approach depends are that homomorphisms may be composed, and are associative under composition. Then the unknown counting series in these relations are eliminated, so that the desired counting series is related only to the counting series of those that can be directly computed using explicitly known cycle index sums. The resulting equations are solved recursively to any desired order by composing term by term and extracting the coefficients iteration by iteration.

We initiate the process of deriving the counting equations of the three classes of graphs that we would like to count by developing appropriate terminology and reviewing some important equations which form the basis for our calculations.

Often graphs decompose into tree like structures where each node of the tree corresponds to a smaller graph. Tutte decomposition also does this. So we need tools to count the number of graphs in a set $\mathfrak{G}$, where each graph G can be obtained by attaching some graphs from another set of graphs $\mathfrak{M}$ in a tree like structure, $\Gamma(G)$. The tool that we will use for this purpose is a dissimilarity characteristic equation which was originally discovered by Otter [40]. The automorphisms of $\Gamma(\boldsymbol{G})$ are all induced from the automorphisms of $\boldsymbol{G}$. If m is a component of $\boldsymbol{G}$, then $\boldsymbol{G}_m$ is a graph which results from distinguishing m in $\boldsymbol{G}$. Thus the automorphisms of $\boldsymbol{G}_m$ consists of the automorphisms of $\boldsymbol{G}$ which map m on to itself. If e is an edge of $\Gamma(\boldsymbol{G})$ then $\boldsymbol{G}_e$ is a graph which results from distinguishing e and considering it irreversible. Thus the automorphisms of $\boldsymbol{G}_e$ are the automorphisms of $\boldsymbol{G}$ which fix e and each of the components at the ends of e. When $\Gamma(\boldsymbol{G})$ has an automorphism which fixes an edge $e = s$ but switches the components at the end of e, then it is called a symmetry edge. Here the automorphisms of $\boldsymbol{G}$ that reverse the end components of s are included in $\boldsymbol{G}_s$. Hence the dissimilarity characteristic equation giving cycle index of $\boldsymbol{G}$ is written as,

$$\boldsymbol{Z}(\boldsymbol{G}) = \sum_m \boldsymbol{Z}(\boldsymbol{G}_m) - \sum_e \boldsymbol{Z}(\boldsymbol{G}_e) + \sum_s \boldsymbol{Z}(\boldsymbol{G}_s), \tag{3.16}$$

where m ranges over all dissimilar components, in $\mathfrak{M}$, of $\boldsymbol{G}$, e ranges over all edges of $\Gamma(\boldsymbol{G})$ and s ranges over symmetry edges of $\Gamma(\boldsymbol{G})$. A reader wishing a rigorous proof of (3.16) is referred to [12] where it is proved for the decomposition of connected graphs into bridgeless graphs.

Counting graphs with prescribed properties usually involves decomposing a graph into core and components. For unlabeled graphs it also involves keeping track of the number of automorphisms of the core with a given cycle decomposition by means of a cycle index. In the case of counting all n-node graphs, the core is the complete graph on n nodes, in which each edge is replaced by a component with 2 nodes and 0 or 1 edge to obtain the graphs to be counted. From now on, let $\boldsymbol{G}$ denote the cycle index sum of all graphs, $\boldsymbol{K}$ denote the cycle index sum of complete graphs. Then the above statement can be written in symbols as,

$$\boldsymbol{G} = \boldsymbol{K}[a_i \leftarrow a_i, b_i \leftarrow (1 + b_i), c_i \leftarrow (1 + c_i)]. \tag{3.17}$$

In the case of connected graphs, the core is a set, in which each element is replaced by a component, which is a connected graph, to obtain an arbitrary graph. From now on, let $\boldsymbol{C}$ denote the cycle index sum of all connected graphs. It follows from Composition Theorem that $\boldsymbol{Z}(S_n)[\boldsymbol{C}]$ denotes the cycle index sum of graphs with exactly n components. Summing over all n and using the well known identity,

$$\sum_{n\geq 0} \boldsymbol{Z}(S_n) = \exp\left(\sum_{k\geq 1} \frac{a_k}{k}\right) \tag{3.18}$$

the above statement, can be written in symbols as,

$$1 + \boldsymbol{G} = \exp\left(\sum_{k\geq 1} \frac{a_k}{k}[\boldsymbol{C}]\right). \tag{3.19}$$

Note that the 1 on left hand side of the above equation denote the empty graph, a graph with no nodes or edges. Robinson solved for $\boldsymbol{C}$ using *Möbius inversion* as follows. On both sides taking log, composing on a_i, multiplying by $\dfrac{\mu(i)}{i}$ and summing over all $i \geq 1$ give,

$$\begin{aligned}
\sum_{i\geq 1} \frac{\mu(i)}{i} a_i[\log(1+\boldsymbol{G})] &= \sum_{i\geq 1} \frac{\mu(i)}{i} a_i \left[\sum_{k\geq 1} \frac{a_k}{k}[\boldsymbol{C}]\right] \\
&= \sum_{i\geq 1} \mu(i) \left\{\sum_{k\geq 1} \frac{a_{ki}}{ki}[\boldsymbol{C}]\right\} \\
&= \sum_{m\geq 1} \left(\frac{a_m}{m}[\boldsymbol{C}] \sum_{i|m} \mu(i)\right) = \boldsymbol{C}, \qquad (3.20) \\
\text{since } \sum_{i|m} \mu(i) &= \begin{cases} 0, & m > 1, \\ 1, & m = 1. \end{cases}
\end{aligned}$$

We will use this method of inversion several times later in this dissertation on different cycle index sums.

In both cases, the cycle structures of the automorphisms of the core are known. Thus one can compute the cycle indices of the automorphism groups of all possible cores and then use Pólya's Theorem, which relates the cycle index sum and the counting series for components and compositions.

Recall that any graph in class of all *Blocks*, from now on denoted by $\boldsymbol{B}$, is either a 2–connected graph or an edge. In the case of counting all blocks, the core is a set of node–rooted blocks joined at the root; every other node has a node–rooted connected graph attached to it, to obtain another node–rooted connected graph. Then this can be written in symbols as,

$$(a_1\boldsymbol{C}') = a_1 \exp\left(\sum_{i\geq 1} \frac{a_i}{i}\left[\boldsymbol{B}'[a_1\boldsymbol{C}']\right]\right). \tag{3.21}$$

As before, rooting by nodes is denoted by a prime. In this case, the unknown is the cycle index sum for cores, from which the counting series can be obtained. However, this cycle index sum cannot be obtained from the counting series for components and composition; so Robinson[49] generalized Pólya's Theorem to relate not counting series but cycle index sums for components and compositions to those of cores.

Generalizing Norman's solution to count graphs with given blocks[39], using calculus Robinson[49] obtained indirectly another equation relating $\boldsymbol{B}$ and $\boldsymbol{C}$,

$$\boldsymbol{C} = (a_1 + \boldsymbol{B} - a_1\boldsymbol{B}')[a_1\boldsymbol{C}']. \tag{3.22}$$

This can be more directly derived using an argument involving the Otter's dissimilarity characteristic equation (3.16) on the *block–cut node tree* of connected graphs. For any connected graph G, the block–cut node tree $\Gamma(G)$ associated with G has a node for every block b and every cut node v of G. Every edge e it has is between a b and a v, indicating that v joins b to G in such a way that $G - v$ is disconnected.

In this context, the first term in the equation (3.16) corresponds to distinguishing block nodes in $\Gamma(G)$ or cut nodes of G. Distinguishing a block node corresponds to a block with every node replaced by a node rooted connected graph, which is denoted as $\boldsymbol{B}[a_1\boldsymbol{C}']$. An arbitrary node of G can be either a cut node of G or not a cut node of G. Distinguishing a node of G that is not a cut node amounts to rooting a block and replacing every node of it with a node rooted connected graph, which is denoted as $a_1\boldsymbol{B}'[a_1\boldsymbol{C}']$. Distinguishing an arbitrary node G corresponds to $a_1\boldsymbol{C}'$. Subtracting the former from the latter gives the contribution of distinguishing a cut node of G. Thus the first term in the equation (3.16), in this case, becomes, $\boldsymbol{B}[a_1\boldsymbol{C}'] + a_1\boldsymbol{C}' - a_1\boldsymbol{B}'[a_1\boldsymbol{C}']$.

Now, the second term in the equation (3.16) corresponds to distinguishing an edge e of $\Gamma(G)$. Recall that any edge in $\Gamma(G)$ is between a cut node and a block node. Since a cut node is an arbitrary node of G but not a node that is not a cut node, we have $a_1\boldsymbol{C}' - a_1$ for one side of the edge and a rooted block with every node replaced by a node rooted connected graph, denoted by $\boldsymbol{B}'[a_1\boldsymbol{C}']$ for the other side of the edge. Thus, the second term in the equation (3.16), in this case, becomes, $(a_1\boldsymbol{C}' - a_1)\boldsymbol{B}'[a_1\boldsymbol{C}']$.

Since every edge in $\Gamma(G)$ is between a cut node and a block node, there is no edge with both ends isomorphic. That is, there is no symmetric edge in $\Gamma(G)$. Hence no contribution to the third term in this case.

Putting together (3.16) for this case, we have

$$
\begin{aligned}
\boldsymbol{C} &= \boldsymbol{B}[a_1\boldsymbol{C}'] + a_1\boldsymbol{C}' - a_1\boldsymbol{B}'[a_1\boldsymbol{C}'] - (a_1\boldsymbol{C}' - a_1)\boldsymbol{B}'[a_1\boldsymbol{C}'] \\
&= \boldsymbol{B}[a_1\boldsymbol{C}'] + a_1\boldsymbol{C}' - (a_1\boldsymbol{C}')\boldsymbol{B}'[a_1\boldsymbol{C}'], \text{ cancelling} \\
&= \boldsymbol{B}[a_1\boldsymbol{C}'] + a_1\boldsymbol{C}' - (a_1\boldsymbol{B}')[a_1\boldsymbol{C}'] \\
&= (a_1 + \boldsymbol{B} - a_1\boldsymbol{B}')[a_1\boldsymbol{C}'].
\end{aligned}
$$

Now, an edge–rooted connected graph can be decomposed into an edge–rooted block along with node–rooted connected branches at the nodes of the block. Putting this in symbols,

$$\frac{\partial \boldsymbol{B}}{\partial b_1}[a_1C', b_1, c_1] = \frac{\partial \boldsymbol{C}}{\partial b_1}. \tag{3.23}$$

$$\frac{\partial \boldsymbol{B}}{\partial c_1}[a_1C', b_1, c_1] = \frac{\partial \boldsymbol{C}}{\partial c_1}. \tag{3.24}$$

The above two equations can also be derived from equations (3.21) and (3.22) by appropriate differentiation and simplification.

Now, taking $\frac{\partial}{\partial a_1}$ and then multiplying by a_1, equation (3.19) yields,

$$a_1\boldsymbol{G}' = \boldsymbol{G} \cdot a_1\boldsymbol{C}'. \tag{3.25}$$

Rewriting this using equation (3.17) we get

$$(a_1\boldsymbol{C}')\boldsymbol{K}[a_1, 1+b_1, 1+c_1] = \left(a_1\frac{\partial \boldsymbol{K}}{\partial a_1}\right)[a_1, 1+b_1, 1+c_1]. \tag{3.26}$$

Similarly, differentiating equations (3.17) and (3.19) appropriately, one can get,

$$\left(\frac{\partial \boldsymbol{C}}{\partial b_1}\right)\boldsymbol{K}[a_1, 1+b_1, 1+c_1] = \left(\frac{\partial \boldsymbol{K}}{\partial b_1}\right)[a_1, 1+b_1, 1+c_1]. \tag{3.27}$$

$$\left(\frac{\partial \boldsymbol{C}}{\partial c_1}\right)\boldsymbol{K}[a_1, 1+b_1, 1+c_1] = \left(\frac{\partial \boldsymbol{K}}{\partial c_1}\right)[a_1, 1+b_1, 1+c_1]. \tag{3.28}$$

Now, expression for $\boldsymbol{K}$ is explicitly available in [51] which can be differentiated appropriately to get explicit expressions for $\frac{\partial \boldsymbol{K}}{\partial a_1}$, $\frac{\partial \boldsymbol{K}}{\partial b_1}$, and for $\frac{\partial \boldsymbol{K}}{\partial c_1}$ so that all of them together will look like,

$$\boldsymbol{K} = \sum_{\substack{\bar{\sigma}=(\sigma_1,\sigma_2,\cdots,\sigma_n)\\ \sigma_i\geq 0}} \left\{\prod_{i<j} b_{lcm(i,j)}^{gcd(i,j)\sigma_i\sigma_j}\right\}\left\{\prod_{i\geq 1}\frac{a_i^{\sigma_i}}{\sigma_i!i^{\sigma_i}} b_i^{\frac{i\sigma_i(\sigma_i-1)}{2}+\lfloor\frac{i-1}{2}\rfloor\sigma_i} c_i^{\sigma_{2i}}\right\}. \tag{3.29}$$

$$a_1\frac{\partial \boldsymbol{K}}{\partial a_1} = \sum_{\substack{\bar{\sigma}=(\sigma_1=\sigma_1'+1,\sigma_2,\cdots,\sigma_{n-1})\\ \sigma_i\geq 0}} \left\{\prod_{i<j} b_{lcm(i,j)}^{gcd(i,j)\sigma_i\sigma_j}\right\}\sigma_1\left\{\prod_{i\geq 1}\frac{a_i^{\sigma_i}}{\sigma_i!i^{\sigma_i}} b_i^{\frac{i\sigma_i(\sigma_i-1)}{2}+\lfloor\frac{i-1}{2}\rfloor\sigma_i} c_i^{\sigma_{2i}}\right\}. \tag{3.30}$$

$$b_1\frac{\partial \boldsymbol{K}}{\partial b_1} = \sum_{\substack{\bar{\sigma}=(\sigma_1=\sigma_1'+2,\sigma_2,\cdots,\sigma_{n-2})\\ \sigma_i\geq 0}} \left\{\prod_{i<j} b_{lcm(i,j)}^{gcd(i,j)\sigma_i\sigma_j}\right\}\left\{\frac{1}{2(\sigma_1-2)!}a_1^{\sigma_1} b_1^{\frac{\sigma_1(\sigma_1-1)}{2}} c_1^{\sigma_2}\right\}$$
$$\left\{\prod_{i>1}\frac{a_i^{\sigma_i}}{\sigma_i!i^{\sigma_i}} b_i^{\frac{i\sigma_i(\sigma_i-1)}{2}+\lfloor\frac{i-1}{2}\rfloor\sigma_i} c_i^{\sigma_{2i}}\right\}. \tag{3.31}$$

$$c_1\frac{\partial \boldsymbol{K}}{\partial c_1} = \sum_{\substack{\bar{\sigma}=(\sigma_1,\sigma_2=\sigma_2'+1,\sigma_3,\cdots,\sigma_{n-2})\\ \sigma_i\geq 0}} \left\{\prod_{i<j} b_{lcm(i,j)}^{gcd(i,j)\sigma_i\sigma_j}\right\}\left\{\frac{\sigma_2}{\sigma_1!}a_1^{\sigma_1} b_1^{\frac{\sigma_1(\sigma_1-1)}{2}} c_1^{\sigma_2}\right\}$$

$$\left\{\frac{1}{2^{\sigma_2}\sigma_2!}a_2^{\sigma_2}b_2^{\sigma_2(\sigma_2-1)}c_2^{\sigma_4}\right\}\left\{\prod_{i>2}\frac{a_i^{\sigma_i}}{\sigma_i!i^{\sigma_i}}b_i^{\frac{i\sigma_i(\sigma_i-1)}{2}+\lfloor\frac{i-1}{2}\rfloor\sigma_i}c_i^{\sigma_{2i}}\right\}. \tag{3.32}$$

Note that in equations (3.30), (3.31), (3.32) σ_i' denotes the corresponding σ_i in the corresponding partition $\bar{\sigma}$ of equation (3.29).

Recalling Tutte's decomposition Theorem, as mentioned in the previous chapter of this dissertation, for any 2–connected graph the indecomposable components are 3–connected graphs, bonds with at least 3 edges and cycles with at least 3 edges. From now on, let T denote the cycle index sum of all 3–connected graphs, M denote the cycle index sum of all cycles with at least 3 edges and N denote the cycle index sum of all bonds with at least 3 edges.

When composing cycle index sums of components with the cycle index sum of a core, at times we have to keep an edge of the core from being replaced in substitution by a component. Such edges are denoted by b_1^* and c_1^*. When all the composition solving is over when we get to counting series which are which are obtained by composing power series, these are recovered into the counting series. In other words, b_1^* and c_1^* are defined so that when composed with power series they yield y and when composed with cycle index sums, they yield b_1 and c_1 respectively. The cycle index sum of all cycles with exactly one special edge is denoted by M_0 and the cycle index sum of all cycles with at most one special edge is denoted by M_1. Also, the cycle index sum of all bonds with at most one special edge is denoted by N_1. The corresponding edge–rooted classes are denoted with a superfix $+$ or $-$ according to whether it is positive or negative.

The cyclic group of degree n, denoted C_n, is generated by the cycle $(123\cdots n)$. Redfield provided the following formula for $\boldsymbol{Z}(C_n)$ using the Euler ϕ–function.

Theorem 13. *The cycle index of the cyclic group C_n is given by*

$$\boldsymbol{Z}(C_n) = n^{-1} \sum_{k|n} \phi(k) a_k^{n/k} b_k^{n/k}.$$

The *dihedral group of degree n*, denoted by D_n, is generated by the cycle $(1\ 2\ 3\ \cdots n)$ and the reflection $(1\ n)(2\ n-1)(3\ n-2)\cdots$. Its cycle index can be expressed in terms of $\boldsymbol{Z}(C_n)$.

Corollary 2. *The cycle index of the dihedral group D_n is given by*

$$\boldsymbol{Z}(D_n) = \frac{1}{2}\boldsymbol{Z}(C_n) + \begin{cases} \frac{1}{2} a_1 c_1 a_2^{(n-1)/2} b_2^{(n-1)/2}, & n \text{ odd} \\ \frac{1}{4}\left\{ a_1^2 a_2^{(n-2)/2} b_2^{n/2} + a_2^{n/2} c_1^2 b_2^{(n-2)/2} \right\}, & n \text{ even.} \end{cases} \tag{3.33}$$

Since the cycle index of a cycle with n nodes is $\boldsymbol{Z}(D_n)$, the cycle index sum of all the cycles with at least three edges is obtained by summing equation (3.33) over all n. With correction terms $-\frac{a_1 b_1}{2} - \frac{a_1^2 b_1^2}{4} - \frac{a_2 b_2}{4}$, we then get

$$\begin{aligned} M &= -\frac{1}{2} \sum_{d \geq 1} \frac{\phi(d)}{d} \log(1 - a_d b_d) - \frac{a_1 b_1}{2} - \frac{a_1^2 b_1^2}{4} - \frac{a_2 b_2}{4} \\ &\quad + \frac{a_1 c_1 a_2 b_2}{2(1 - a_2 b_2)} + \frac{a_1^2 a_2 b_2^2}{4(1 - a_2 b_2)} + \frac{a_2^2 c_1^2 b_2}{4(1 - a_2 b_2)}. \end{aligned} \tag{3.34}$$

The cycle index sums of all other aforementioned types of Ms can be explicitly represented as

$$M^+ = \frac{2\partial M}{a_1^2 \partial b_1} = \frac{a_1 b_1^2}{1 - a_1 b_1}. \tag{3.35}$$

$$M^- = \frac{2\partial M}{a_2 \partial c_1} = \frac{a_1 b_2 + a_2 b_2 c_1}{1 - a_2 b_2}. \tag{3.36}$$

$$M_0 = \frac{a_1^2}{2} b_1^* M^+ + \frac{a_2}{2} c_1^* M^- = \frac{a_1^3 b_1^* b_1^2}{2(1 - a_1 b_1)} + \frac{a_2 c_1^*(a_1 b_2 + a_2 b_2 c_1)}{2(1 - a_2 b_2)}. \tag{3.37}$$

$$M_0^+ = \frac{2\partial M_0}{a_1^2 \partial b_1} = \frac{b_1^* \partial M^+}{\partial b_1} = \frac{b_1^*(2a_1 b_1 - a_1^2 b_1^2)}{(1 - a_1 b_1)^2}. \tag{3.38}$$

$$M_0^- = \frac{2\partial M_0}{a_2 \partial c_1} = \frac{c_1^* \partial M^-}{\partial c_1} = \frac{c_1^* a_2 b_2}{1 - a_2 b_2}. \tag{3.39}$$

$$M_1 = M + M_0. \tag{3.40}$$

$$M_1^+ = M^+ + M_0^+ = \frac{a_1b_1(1-a_1b_1)(b_1+b_1^*)+b_1^*a_1b_1}{(1-a_1b_1)^2}$$
$$= \frac{a_1b_1^2 - a_1^2b_1^3 + 2a_1b_1^*b_1 - a_1^2b_1^*b_1^2}{(1-a_1b_1)^2}. \tag{3.41}$$
$$M_1^- = M^- + M_0^- = \frac{a_1b_2 + a_2b_2(c_1+c_1^*)}{1-a_2b_2}. \tag{3.42}$$

On the other hand, a bond with n edges has two types of automorphisms; one which leave both the nodes fixed and the other type which swaps them. When the nodes are swapped edge cycles of odd length reverse the orientation and edge cycles of even length preserve the orientation. Since the edges are unordered, the edge automorphisms form S_n. Summing over all n, recalling (3.18), and subtracting the terms that correspond to bonds with less than three edges, the cycle index sum of all bonds with at least three edges is

$$N = \frac{a_1^2}{2}\left\{\exp\left(\sum_{i\geq 1}\frac{b_i}{i}\right) - 1 - b_1 - \frac{b_1^2}{2} - \frac{b_2}{2}\right\}$$
$$+ \frac{a_2}{2}\left\{\exp\left(\sum_{i \text{ odd}}\frac{c_i}{i} + \sum_{i \text{ even}}\frac{b_i}{i}\right) - 1 - c_1 - \frac{c_1^2}{2} - \frac{b_2}{2}\right\}$$
$$= \frac{a_1^2}{2}\exp\left(\sum_{i\geq 1}\frac{b_i}{i}\right) + \frac{a_2}{2}\exp\left(\sum_{i \text{ odd}}\frac{c_i}{i} + \sum_{i \text{ even}}\frac{b_i}{i}\right)$$
$$- \frac{a_1^2}{2}\left\{(1+b_1) + \frac{b_1^2}{2} + \frac{b_2}{2}\right\} - \frac{a_2}{2}\left\{(1+c_1) + \frac{c_1^2}{2} + \frac{b_2}{2}\right\}. \tag{3.43}$$

The cycle index sums of all aforementioned types of Ns can be explicitly represented as,

$$N^+ = \frac{2\partial N}{a_1^2\partial b_1} = \exp\left(\sum_{i\geq 1}\frac{b_i}{i}\right) - (1+b_1). \tag{3.44}$$
$$N^- = \frac{2\partial N}{a_2\partial c_1} = \exp\left(\sum_{i \text{ odd}}\frac{c_i}{i} + \sum_{i \text{ even}}\frac{b_i}{i}\right) - (1+c_1). \tag{3.45}$$
$$N_1 = N + \frac{a_1^2}{2}b_1^*N^+ + \frac{a_2}{2}c_1^*N^-$$
$$= \frac{a_1^2}{2}\exp\left(\sum_{i\geq 1}\frac{b_i}{i}\right) + \frac{a_2}{2}\exp\left(\sum_{i \text{ odd}}\frac{c_i}{i} + \sum_{i \text{ even}}\frac{b_i}{i}\right)$$

$$
\begin{aligned}
&- \frac{a_1^2}{2}\left\{(1+b_1) + \frac{b_1^2}{2} + \frac{b_2}{2}\right\} - \frac{a_2}{2}\left\{(1+c_1) + \frac{c_1^2}{2} + \frac{b_2}{2}\right\} \\
&+ \frac{a_1^2}{2} b_1^* \exp\left(\sum_{i \geq i} \frac{b_i}{i}\right) - \frac{a_1^2}{2} b_1^*(1+b_1) \\
&+ \frac{a_2}{2} c_1^* \exp\left(\sum_{i \text{ odd}} \frac{c_i}{i} + \sum_{i \text{ even}} \frac{b_i}{i}\right) - \frac{a_2}{2} c_1^*(1+c_1). \\
&= \frac{a_1^2}{2}(1+b_1^*)\exp\left(\sum_{i \geq 1} \frac{b_i}{i}\right) + \frac{a_2}{2}(1+c_1^*)\exp\left(\sum_{i \text{ odd}} \frac{c_i}{i} + \sum_{i \text{ even}} \frac{b_i}{i}\right) \\
&- \frac{a_1^2}{2}\left\{(1+b_1^*)(1+b_1) + \frac{b_1^2}{2} + \frac{b_2}{2}\right\} - \frac{a_2}{2}\left\{(1+c_1^*)(1+c_1) + \frac{c_1^2}{2} + \frac{b_2}{2}\right\}.
\end{aligned} \tag{3.46}
$$

$$
\begin{aligned}
N_1^+ &= N^+ + b_1^*\left\{\exp\left(\sum_{i \geq 1} \frac{b_i}{i}\right) - 1\right\} \\
&= (1+b_1^*)\exp\left(\sum_{i \geq 1} \frac{b_i}{i}\right) - (1+b_1^*) - b_1.
\end{aligned} \tag{3.47}
$$

$$
\begin{aligned}
N_1^- &= N^- + c_1^*\left\{\exp\left(\sum_{i \text{ odd}} \frac{c_i}{i} + \sum_{i \text{ even}} \frac{b_i}{i}\right) - 1\right\} \\
&= (1+c_1^*)\exp\left(\sum_{i \text{ odd}} \frac{c_i}{i} + \sum_{i \text{ even}} \frac{b_i}{i}\right) - (1+c_1^*) - c_1.
\end{aligned} \tag{3.48}
$$

Let the pair D^+ and D^- denote all non–empty positively and negatively edge–rooted 2–connected graphs (or 2–pole networks). In these, root edge may belong to any of the three types of the components. Further, since we are actually looking for blocks, a single edge also has to be included. If the rooted edge belongs to a 3–connected component, the positive and negative networks is denoted by T^+ and T^- respectively. Thus, all 2–pole networks can be symbolically written as follows (This is a recursive definition. When explaining these, one has to start from somewhere and define one using the other. So many symbols in this will be clear as the reader continues to read next couple of paragraphs.)

$$
D^+ = b_1 + T^+[a_1, D^+, D^-] + N_1^+[a_1, D^+ - P^+ - b_1, D^- - P^- - c_1]
$$

$$+ M^+[a_1, D^+ - S^+, D^- - S^-]. \tag{3.49}$$

$$D^- = c_1 + T^-[a_1, D^+, D^-] + N_1^-[a_1, D^+ - P^+ - b_1, D^- - P^- - c_1]$$
$$+ M^-[a_1, D^+ - S^+, D^- - S^-]. \tag{3.50}$$

Let the pair $\{S^+, S^-\}$ denote all *Series networks*. In these networks, the rooted edge is a free edge belonging to a cyclic component. In other words, to get one of these, one has to edge-root a cycle and replace each of the other cycle edges with other components. Since in the Tutte decomposition, a cyclic component cannot be next to a cyclic component, we should substitute $D^+ - S^+$ and $D^- - S^-$ for b_k and c_k, respectively in the edge–rooted cycles. In 2–connected graphs, there is no restrictions on the number of free edges in a cyclic component. The single edge already included in D^+ and D^- takes care of this case. Thus,

$$S^+ = M^+[a_1, D^+ - S^+, D^- - S^-]. \tag{3.51}$$

$$S^- = M^-[a_1, D^+ - S^+, D^- - S^-]. \tag{3.52}$$

Let $\{P^+, P^-\}$ denote all *Parallel networks*, In these networks, the rooted edge is a free edge belonging to bond component. In other words, to get one of these, one has to edge-root a bond and replace each of the other bond edges with other components. Since in the Tutte decomposition, a bond component cannot be next to a bond component, we should substitute $D^+ - P^+$ and $D^- - P^-$ for positive and negative edges in the edge–rooted bonds. There cannot be more than one free edge in a bond component, as otherwise, the resulting graph will be a multi-graph. So we remove the single edge from D^+ and D^- and then have an optional special edge in the edge–rooted bond in the form of b^* or c^* to which no composition is supposed to happen. That explains the use of N_1^+ and N_1^- instead of plain N^+ and N^-. Thus,

$$P^+ = N_1^+[a_1, D^+ - P^+ - b_1, D^- - P^- - c_1]. \tag{3.53}$$

$$P^- = N_1^-[a_1, D^+ - P^+ - b_1, D^- - P^- - c_1]. \tag{3.54}$$

The corresponding minimally 2–connected counter parts will be distinguished using a hat. For example, $\hat{\boldsymbol{B}}$ denote minimally 2–connected graphs; 3–edge–connected counter parts will be distinguished using a tilde. For example, $\tilde{\boldsymbol{B}}$ denote 3–edge–connected graphs and the corresponding minimally 2–edge–connected counter parts will be distinguished using a check. For example, $\check{\boldsymbol{B}}$ denote minimally 2–edge–connected blocks.

A single edge is not a minimally 2–connected graph. So there is no b_1 in $\hat{D}^+$ and no c_1 in $\hat{D}^-$. By the decomposition Theorem given in previous chapter, for minimally 2–connected graphs, free edges can only be in a cyclic component. So we have to explicitly add a free edge in the components that make $\hat{S}^+$ and $\hat{S}^-$ from the cores that are edge–rooted cycles. Since there cannot be a free edge in a bond component, we use plain N^+ and N^- for the cores of $\hat{P}^+$ and $\hat{P}^-$, respectively. Thus, as before,

$$\begin{aligned}\hat{D}^+ &= T^+[a_1, \hat{D}^+, \hat{D}^-] + N^+[a_1, \hat{D}^+ - \hat{P}^+, \hat{D}^- - \hat{P}^-] \\ &\quad + M^+[a_1, b_1 + \hat{D}^+ - \hat{S}^+, c_1 + \hat{D}^- - \hat{S}^-]. \end{aligned} \tag{3.55}$$

$$\begin{aligned}\hat{D}^- &= T^-[a_1, \hat{D}^+, \hat{D}^-] + N^-[a_1, \hat{D}^+ - \hat{P}^+, \hat{D}^- - \hat{P}^-] \\ &\quad + M^-[a_1, b_1 + \hat{D}^+ - \hat{S}^+, c_1 + \hat{D}^- - \hat{S}^-]. \end{aligned} \tag{3.56}$$

$$\hat{S}^+ = M^+[a_1, b_1 + \hat{D}^+ - \hat{S}^+, c_1 + \hat{D}^- - \hat{S}^-]. \tag{3.57}$$

$$\hat{S}^- = M^-[a_1, b_1 + \hat{D}^+ - \hat{S}^+, c_1 + \hat{D}^- - \hat{S}^-]. \tag{3.58}$$

$$\hat{P}^+ = N^+[a_1, \hat{D}^+ - \hat{P}^+, \hat{D}^- - \hat{P}^-]. \tag{3.59}$$

$$\hat{P}^- = N^-[a_1, \hat{D}^+ - \hat{P}^+, \hat{D}^- - \hat{P}^-]. \tag{3.60}$$

A single edge is not a 3–edge–connected block. So there is no b_1 in $\tilde{D}^+$ and no c_1 in $\tilde{D}^-$. By the decomposition Theorem given in the previous chapter, for 3–edge–connected blocks, there can be at most one free edge in a cyclic component. Thus, we use M_1^+ and M_1^- as cores to make $\tilde{S}^+$ and $\tilde{S}^-$. Similarly, since at most one free edge is allowed in a bond component, we use N_1^+ and N_1^- as cores to make $\tilde{P}^+$ and $\tilde{P}^-$. Also, there is no restriction on the number of free edges in a 3–connected component. So we add a single edge with the components that gets composed with T^+ and T^-. Thus, as before,

$$\tilde{D}^+ = T^+[a_1, b_1 + \tilde{D}^+, c_1 + \tilde{D}^-] + N_1^+[a_1, \tilde{D}^+ - \tilde{P}^+, \tilde{D}^- - \tilde{P}^-] + M_1^+[a_1, \tilde{D}^+ - \tilde{S}^+, \tilde{D}^- - \tilde{S}^-]. \tag{3.61}$$

$$\tilde{D}^- = T^-[a_1, b_1 + \tilde{D}^+, c_1 + \tilde{D}^-] + N_1^-[a_1, \tilde{D}^+ - \tilde{P}^+, \tilde{D}^- - \tilde{P}^-] + M_1^-[a_1, \tilde{D}^+ - \tilde{S}^+, \tilde{D}^- - \tilde{S}^-]. \tag{3.62}$$

$$\tilde{S}^+ = M_1^+[a_1, \tilde{D}^+ - \tilde{S}^+, \tilde{D}^- - \tilde{S}^-]. \tag{3.63}$$

$$\tilde{S}^- = M_1^-[a_1, \tilde{D}^+ - \tilde{S}^+, \tilde{D}^- - \tilde{S}^-]. \tag{3.64}$$

$$\tilde{P}^+ = N_1^+[a_1, \tilde{D}^+ - \tilde{P}^+, \tilde{D}^- - \tilde{P}^-]. \tag{3.65}$$

$$\tilde{P}^- = N_1^-[a_1, \tilde{D}^+ - \tilde{P}^+, \tilde{D}^- - \tilde{P}^-]. \tag{3.66}$$

A single edge is not a minimally 2–edge–connected block. So there is no b_1 in $\check{D}^+$ and no c_1 in $\check{D}^-$. By the decomposition Theorem given in the previous chapter, for minimally 2–edge–connected blocks, each free edge belongs to a cyclic component that contains at least one other free edge. This means, when making $\check{S}^+$ and $\check{S}^-$ we allow a single edge to be composed with M^+ and M^- and then remove the case of having exactly one free edge in a cyclic component. Since there cannot be a free edge in a bond component, we use plain N^+ and N^- as cores when making $\check{P}^+$ and $\check{P}^-$. Also, 3–connected components cannot have a free edge. Thus, as before,

$$\check{D}^+ = T^+[a_1, \check{D}^+, \check{D}^-] + N^+[a_1, \check{D}^+ - \check{P}^+, \check{D}^- - \check{P}^-] + M^+[a_1, b_1 + \check{D}^+ - \check{S}^+, c_1 + \check{D}^- - \check{S}^-] - M_0^+[a_1, \check{D}^+ - \check{S}^+, \check{D}^- - \check{S}^-]. \tag{3.67}$$

$$\check{D}^- = T^-[a_1, \check{D}^+, \check{D}^-] + N^-[a_1, \check{D}^+ - \check{P}^+, \check{D}^- - \check{P}^-]$$

$$+ M^-[a_1, b_1 + \check{D}^+ - \check{S}^+, c_1 + \check{D}^- - \check{S}^-] - M_0^-[a_1, \check{D}^+ - \check{S}^+, \check{D}^- - \check{S}^-]. \tag{3.68}$$

$$\check{S}^+ = M^+[a_1, b_1 + \check{D}^+ - \check{S}^+, c_1 + \check{D}^- - \check{S}^-] - M_0^+[a_1, \check{D}^+ - \check{S}^+, \check{D}^- - \check{S}^-]. \tag{3.69}$$

$$\check{S}^- = M^-[a_1, b_1 + \check{D}^+ - \check{S}^+, c_1 + \check{D}^- - \check{S}^-] - M_0^-[a_1, \check{D}^+ - \check{S}^+, \check{D}^- - \check{S}^-]. \tag{3.70}$$

$$\check{P}^+ = N^+[a_1, \check{D}^+ - \check{P}^+, \check{D}^- - \check{P}^-]. \tag{3.71}$$

$$\check{P}^- = N^-[a_1, \check{D}^+ - \check{P}^+, \check{D}^- - \check{P}^-]. \tag{3.72}$$

In equations (3.49) through (3.72) we have represented the corresponding decomposition Theorems in symbols. These play a central role in deriving the counting equations. Equations (3.49) through (3.54) are used with corresponding equations for each class of the graphs. The parts that are unknown are solved within each set with the ones that are explicitly known or can be solved for an explicit expression.

For a cycle index sum X, let $\vec{X}$ denote the ordered pair (X^+, X^-). We define an inner product $\diamond$ for two cycle index sums X and Y to be

$$\vec{X} \diamond \vec{Y} = (X^+, X^-) \diamond (Y^+, Y^-) = \frac{a_1^2}{2} X^+ Y^+ + \frac{a_2}{2} X^- Y^-. \tag{3.73}$$

Using Otter's dissimilarity characteristic equation (3.16), the decomposition of blocks can be written as :

$$\begin{aligned}
\boldsymbol{B} &= T[a_1, D^+, D^-] + N_1[a_1, D^+ - P^+ - b_1, D^- - P^- - c_1] \\
&\quad + M[a_1, D^+ - S^+, D^- - S^-] \\
&\quad - \Big\{ \vec{T}[a_1, D^+, D^-] \diamond \vec{N}_1[a_1, D^+ - P^+ - b_1, D^- - P^- - c_1] \\
&\quad + \vec{N}_1[a_1, D^+ - P^+ - b_1, D^- - P^- - c_1] \diamond \vec{M}[a_1, D^+ - S^+, D^- - S^-] \\
&\quad + \vec{T}[a_1, D^+, D^-] \diamond \vec{M}[a_1, D^+ - S^+, D^- - S^-]
\end{aligned}$$

$$+\frac{1}{2}\overrightarrow{T}[a_1, D^+, D^-] \diamond \overrightarrow{T}[a_1, D^+, D^-]$$
$$+\left(\frac{a_1^2+a_2}{4}\right) a_2[T^+[a_1, D^+, D^-]]\Big\} + \left(\frac{a_1^2+a_2}{2}\right) a_2\left[T^+[a_1, D^+, D^-]\right]$$
$$+\frac{1}{2}(a_1^2 b_1 + a_2 c_1). \tag{3.74}$$

This is equivalent to equation (23) in Walsh's paper [56] The first term of (3.16) corresponds to distinguishing a node corresponding to a 3–connected or a bond or a cycle component. This accounts for the terms,

$T[a_1, D^+, D^-] + N_1[a_1, D^+ - P^+ - b_1, D^- - P^- - c_1] + M[a_1, D^+ - S^+, D^- - S^-]$ in the above equation. The second term of (3.16) corresponds to distinguishing an edge between a 3–connected component and a bond component or a bond component and a cyclic component or a cyclic component and a 3–connected component or a two 3–connected components. This accounts for all the terms with in curly brackets in the above equation. The third term of (3.16) corresponds to distinguishing a symmetric edge. A symmetric edge in this decomposition can only be between two 3–connected components. This accounts for the term $\left(\frac{a_1^2+a_2}{2}\right) a_2\left[T^+[a_1, D^+, D^-]\right]$ in the above equation. Also, since $\boldsymbol{B}$ denotes all blocks and a single edge is a block but not a 2–connected graph, we have the last term $\frac{1}{2}(a_1^2 b_1 + a_2 c_1)$ to balance both sides of the equation.

The handling of the cycle index sums becomes computationally too difficult for both storing and computing. This calls for inversion of cycle index sum technique. We choose $\mu(x, y)$, $\nu(x, y)$, $\epsilon(x, y)$ and $\delta(x, y)$ in such a way that

$$D^+[x, \mu(x, y), \nu(x, y)] = \delta(x, y) = \hat{D}^+[x, y, y] = \tilde{D}^+[x, y, y] + y = \check{D}^+[x, y, y]$$
$$\text{and} \tag{3.75}$$
$$D^-[x, \mu(x, y), \nu(x, y)] = \epsilon(x, y) = \hat{D}^-[x, y, y] = \tilde{D}^-[x, y, y] + y = \check{D}^-[x, y, y].$$

Composing (3.74) on both sides with $[x, \mu, \nu]$ and using (3.75), we get,

$$\boldsymbol{B}[x, \mu, \nu] = T[x, \delta, \epsilon] + N_1\left[x, (D^+ - P^+)[x, \mu, \nu] - \mu, (D^- - P^-)[x, \mu, \nu] - \nu\right]$$

$$
\begin{aligned}
&+ M\left[x, \delta - S^{+}[x,\mu,\nu], \epsilon - S^{-}[x,\mu,\nu]\right] \\
&- \Big\{ \vec{T}[x,\delta,\epsilon] \diamond \vec{N}_1\left[x, (D^{+} - P^{+})[x,\mu,\nu] - \mu, (D^{-} - P^{-})[x,\mu,\nu] - \nu\right] \\
&+ \vec{N}_1\left[x, (D^{+} - P^{+})[x,\mu,\nu] - \mu, (D^{-} - P^{-})[x,\mu,\nu] - \nu\right] \\
&\diamond \vec{M}\left[x, \delta - S^{+}[x,\mu,\nu], \epsilon - S^{-}[x,\mu,\nu]\right] \\
&+ \vec{T}[x,\delta,\epsilon] \diamond \vec{M}\left[x, \delta - S^{+}[x,\mu,\nu], \epsilon - S^{-}[x,\mu,\nu]\right] + \frac{1}{2}\vec{T}[x,\delta,\epsilon] \diamond \vec{T}[x,\delta,\epsilon] \\
&+ \frac{x^2}{2} a_2[T^{+}[x,\delta,\epsilon]] \Big\} + x^2 a_2\left[T^{+}[x,\delta,\epsilon]\right] + \frac{x^2}{2}(\mu(x,y) + \nu(x,y)).
\end{aligned}
$$

We note that by virtue of our choice of power series for inversion, in each equation similar to this for all other three classes that we wish to count, there will be terms involving T which will be exactly like the ones we have here. So we would like to rearrange this to write as,

$$
\begin{aligned}
&T[x,\delta,\epsilon] - \frac{1}{2}\vec{T}[x,\delta,\epsilon] \diamond \vec{T}[x,\delta,\epsilon] + \frac{x^2}{2} a_2[T^{+}[x,\delta,\epsilon]] \\
&= \boldsymbol{B}[x,\mu,\nu] - N_1\left[x, (D^{+} - P^{+})[x,\mu,\nu] - \mu, (D^{-} - P^{-})[x,\mu,\nu] - \nu\right] \\
&- M\left[x, \delta - S^{+}[x,\mu,\nu], \epsilon - S^{-}[x,\mu,\nu]\right] \\
&+ \vec{T}[x,\delta,\epsilon] \diamond \vec{N}_1\left[x, (D^{+} - P^{+})[x,\mu,\nu] - \mu, (D^{-} - P^{-})[x,\mu,\nu] - \nu\right] \\
&+ \vec{N}_1\left[x, (D^{+} - P^{+})[x,\mu,\nu] - \mu, (D^{-} - P^{-})[x,\mu,\nu] - \nu\right] \\
&\diamond \vec{M}\left[x, \delta - S^{+}[x,\mu,\nu], \epsilon - S^{-}[x,\mu,\nu]\right] \\
&+ \vec{T}[x,\delta,\epsilon] \diamond \vec{M}\left[x, \delta - S^{+}[x,\mu,\nu], \epsilon - S^{-}[x,\mu,\nu]\right] - \frac{x^2}{2}(\mu(x,y) + \nu(x,y)). \quad (3.76)
\end{aligned}
$$

For each class of graphs, first $\mu(x,y)$ and $\nu(x,y)$ will be computed. Then corresponding $\eta(x,y)$, $\delta(x,y)$, $\epsilon(x,y)$, $\boldsymbol{B}[x,\mu,\nu]$ are computed and supplied in the equation (3.76) and in the equation corresponding to this for the class of graph. Since the LHS of both the resulting equations look the same, as mentioned above, by rearranging, we get a relation for the counting series for the graph. If we get $\eta(x,y)$, $\delta(x,y)$, $\epsilon(x,y)$ and $\boldsymbol{B}[x,\mu,\nu]$ in terms of $\boldsymbol{K}$ and its derivatives, since (3.29) through

(3.32) are available for explicit computations, given $\mu(x, y)$ and $\nu(x, y)$, it is possible for one to compute each of the four power series.

§3.3 Deriving reversion power series

§3.3.1 $\delta(x, y)$

Since D^+ contains all non-empty positively edge–rooted blocks, we can write $D^+ = (1 + b_1)\frac{2}{a_1^2}\frac{\partial \boldsymbol{B}}{\partial b_1} - 1.$

Composing $[x, \mu, \nu]$ on both sides of this gives,
$D^+[x, \mu, \nu] = \delta(x, y) = (1 + \mu(x, y))\frac{2}{x^2}\frac{\partial \boldsymbol{B}}{\partial b_1}[x, \mu, \nu] - 1.$
This can be written as $1 + \delta(x, y) = (1 + \mu(x, y))\frac{2}{x^2}\frac{\partial \boldsymbol{B}}{\partial b_1}[x, \mu, \nu].$

Let $\eta(x, y)$ be such that $a_1\boldsymbol{C}'[\eta, \mu, \nu] = x.$ (3.77)

To see that such a series exists, define the *order* of a term in a cycle index sum to be the exponent of x resulting from the substitution $a_j \leftarrow x^j$; then $a_1\boldsymbol{C}' = a_1 + a_1^2 b_1 +$ (terms of higher order).

Now (3.77) is equivalent to $\eta(x, y) = x - (a_1\boldsymbol{C}' - a_1)[\eta, \mu, \nu]$, which can be seen to define $\eta(x, y)$ iteratively in powers of x. One has $\eta(0, y) = 0$ to start; having found $\eta(x, y)$ through powers of x^{n-1}, substitution in the right side gives $\eta(x, y)$ correctly through the coefficient of x^n.

Then composing $[\eta, \mu, \nu]$ on both sides of equation (3.23), we get,
$\frac{\partial \boldsymbol{C}}{\partial b_1}[\eta, \mu, \nu] = \frac{\partial \boldsymbol{B}}{\partial b_1}[a_1\boldsymbol{C}', b_1, c_1][\eta, \mu, \nu] = \frac{\partial \boldsymbol{B}}{\partial b_1}[x, \mu, \nu].$
Therefore, $(1 + \delta(x, y)) = (1 + \mu(x, y))\frac{2}{x^2}\frac{\partial \boldsymbol{C}}{\partial b_1}[\eta, \mu, \nu].$
Rearranging, $\frac{\partial \boldsymbol{C}}{\partial b_1}[\eta, \mu, \nu] = \frac{x^2(1 + \delta(x, y))}{2(1 + \mu(x, y))}.$

Composing, $[\eta, \mu, \nu]$ on both sides of (3.27), we get

$$\frac{\partial \boldsymbol{K}}{\partial b_1}[\eta, 1+\mu, 1+\nu] = \frac{\partial \boldsymbol{C}}{\partial b_1}[\eta, \mu, \nu]\boldsymbol{K}[\eta, 1+\mu, 1+\nu]$$
$$= \frac{x^2(1+\delta(x,y))}{2(1+\mu(x,y))}\boldsymbol{K}[\eta, 1+\mu, 1+\nu].$$

i.e., $\frac{2}{x^2}\left(b_1\frac{\partial \boldsymbol{K}}{\partial b_1}\right)[\eta, 1+\mu, 1+\nu] = (1+\delta(x,y))\boldsymbol{K}[\eta, 1+\mu, 1+\nu].$

Adding and subtracting 1 on right hand side $\boldsymbol{K}$ we get,

$$\frac{2}{x^2}\left(b_1\frac{\partial \boldsymbol{K}}{\partial b_1}\right)[\eta, 1+\mu, 1+\nu] = (1+\delta(x,y)) + (1+\delta(x,y))(\boldsymbol{K}-1)[\eta, 1+\mu, 1+\nu].$$

$$\delta(x,y)+1 = \frac{2}{x^2}\left(b_1\frac{\partial \boldsymbol{K}}{\partial b_1}\right)[\eta, 1+\mu, 1+\nu] - (1+\delta(x,y))(\boldsymbol{K}-1)[\eta, 1+\mu, 1+\nu].$$

Let $F(x,y) = (\boldsymbol{K}-1)[\eta, 1+\mu, 1+\nu]$.

Then $\delta(x,y)+1 = \frac{2}{x^2}\left(b_1\frac{\partial \boldsymbol{K}}{\partial b_1}\right)[\eta, 1+\mu, 1+\nu] - (1+\delta(x,y))F(x,y).$

This can be written as

$$\delta(x,y)+1 = \left(\frac{\eta(x,y)}{x}\right)^2\frac{2}{a_1^2}\left(b_1\frac{\partial \boldsymbol{K}}{\partial b_1}\right)[\eta, 1+\mu, 1+\nu] - (1+\delta(x,y))F(x,y). \tag{3.78}$$

§3.3.2 $\epsilon(x,y)$

Similarly, since D^- contains all non-empty negatively edge–rooted blocks,

we can write,

$$D^- = (1+c_1)\frac{2}{a_2}\frac{\partial \boldsymbol{B}}{\partial c_1} - 1.$$

Composing $[x, \mu, \nu]$ on both sides gives,

$$D^-[x,\mu,\nu] = \epsilon(x,y) = (1+\nu(x,y))\frac{2}{x^2}\frac{\partial \boldsymbol{B}}{\partial c_1}[x,\mu,\nu] - 1.$$

i.e., $1+\epsilon(x,y) = (1+\nu(x,y))\frac{2}{x^2}\frac{\partial \boldsymbol{B}}{\partial c_1}[x,\mu,\nu].$

Then composing $[\eta, \mu, \nu]$ on both sides of (3.24), and using (3.77), we get

$$\frac{\partial \boldsymbol{C}}{\partial c_1}[\eta,\mu,\nu] = \frac{\partial \boldsymbol{B}}{\partial c_1}[a_1\boldsymbol{C}', b_1, c_1][\eta,\mu,\nu] = \frac{\partial \boldsymbol{B}}{\partial c_1}[x,\mu,\nu].$$

Therefore, $1+\epsilon(x,y) = (1+\nu(x,y))\frac{2}{x^2}\frac{\partial \boldsymbol{C}}{\partial c_1}[\eta,\mu,\nu].$

Rearranging, $\dfrac{\partial \boldsymbol{C}}{\partial c_1}[\eta, \mu, \nu] = \dfrac{x^2(1+\epsilon(x,y))}{2(1+\nu(x,y))}$.

Composing $[\eta, \mu, \nu]$ on both sides of (3.28), we get

$$\begin{aligned}\frac{\partial \boldsymbol{K}}{\partial c_1}[\eta, 1+\mu, 1+\nu] &= \frac{\partial \boldsymbol{C}}{\partial c_1}[\eta, \mu, \nu]\boldsymbol{K}[\eta, 1+\mu, 1+\nu] \\ &= \frac{x^2(1+\epsilon(x,y))}{2(1+\nu(x,y))}\boldsymbol{K}[\eta, 1+\mu, 1+\nu].\end{aligned}$$

i.e., $\dfrac{2}{x^2}\left(c_1\dfrac{\partial \boldsymbol{K}}{\partial c_1}\right)[\eta, 1+\mu, 1+\nu] = (1+\epsilon(x,y))\boldsymbol{K}[\eta, 1+\mu, 1+\nu]$.

Adding and subtracting 1 on right hand side $\boldsymbol{K}$ we get,

$$\frac{2}{x^2}\left(c_1\frac{\partial \boldsymbol{K}}{\partial c_1}\right)[\eta, 1+\mu, 1+\nu] = (1+\epsilon(x,y)) + (1+\epsilon(x,y))(\boldsymbol{K}-1)[\eta, 1+\mu, 1+\nu].$$

$$\epsilon(x,y)+1 = \frac{2}{x^2}\left(c_1\frac{\partial \boldsymbol{K}}{\partial c_1}\right)[\eta, 1+\mu, 1+\nu] - (1+\epsilon(x,y))(\boldsymbol{K}-1)[\eta, 1+\mu, 1+\nu].$$

Then $\epsilon(x,y)+1 = \dfrac{2}{x^2}\left(c_1\dfrac{\partial \boldsymbol{K}}{\partial c_1}\right)[\eta, 1+\mu, 1+\nu] - (1+\epsilon(x,y))F(x,y)$.

This can be written as

$$\epsilon(x,y)+1 = \left(\frac{\eta(x^2,y^2)}{x^2}\right)\frac{2}{a_2}\left(c_1\frac{\partial \boldsymbol{K}}{\partial c_1}\right)[\eta, 1+\mu, 1+\nu] - (1+\epsilon(x,y))F(x,y). \tag{3.79}$$

§3.3.3 $\eta(x,y)$

Now, composing $[\eta, \mu, \nu]$ on both sides of (3.26), and using equation (3.77) we get, $x\boldsymbol{K}[\eta, 1+\mu, 1+\nu] = \left(a_1\dfrac{\partial \boldsymbol{K}}{\partial a_1}\right)[\eta, 1+\mu, 1+\nu]$. Adding and subtracting 1 on LHS $\boldsymbol{K}$ and adding and subtracting a_1 on RHS, we get,

$$x + x(\boldsymbol{K}-1)[\eta, 1+\mu, 1+\nu] = \eta(x,y) + a_1\left(\frac{\partial \boldsymbol{K}}{\partial a_1} - 1\right)[\eta, 1+\mu, 1+\nu].$$

This on rearranging,

$$\eta(x,y) = x + x(\boldsymbol{K}-1)[\eta, 1+\mu, 1+\nu] - a_1\left(\frac{\partial \boldsymbol{K}}{\partial a_1} - 1\right)[\eta, 1+\mu, 1+\nu].$$

This can be written as,

$$\eta(x,y) = x + xF(x,y) - a_1\left(\frac{\partial \boldsymbol{K}}{\partial a_1} - 1\right)[\eta, 1+\mu, 1+\nu]. \tag{3.80}$$

§3.3.4 $B(x,y)$

Now, expanding equation (3.22), $\boldsymbol{C} = a_1\boldsymbol{C}' + \boldsymbol{B}[a_1\boldsymbol{C}'] - (a_1\boldsymbol{C}')\boldsymbol{B}'[a_1\boldsymbol{C}']$.

Rearranging this, $\boldsymbol{B}[a_1\boldsymbol{C}'] = -a_1\boldsymbol{C}' + \boldsymbol{C} + (a_1\boldsymbol{C}')\boldsymbol{B}'[a_1\boldsymbol{C}']$.

Composing $[\eta, \mu, \nu]$ on both sides and using (3.77),

$$\boldsymbol{B}[x,\mu,\nu] = -x + \boldsymbol{C}[\eta,\mu,\nu] + x\boldsymbol{B}'[x,\mu,\nu]. \tag{3.81}$$

Now, composing $[\eta, \mu, \nu]$ on both sides of equation (3.21) and rearranging

we get, $\dfrac{\eta(x,y)}{x} = \exp\left(-\sum_{i\geq 1}\dfrac{a_i}{i}[\boldsymbol{B}'[x,\mu,\nu]]\right)$.

Then by Möbius inversion (3.2) $\sum_{k\geq 1}\dfrac{\mu(k)}{k}a_k\left[\log\left(\dfrac{\eta(x,y)}{x}\right)\right] = -\boldsymbol{B}'[x,\mu,\nu]$.

Also, composing $[\eta, \mu, \nu]$ on both sides of equation (3.19) gives

$$\boldsymbol{K}[\eta, 1+\mu, 1+\nu] = G[\eta,\mu,\nu] = \exp\left(\sum_{i\geq 1}\frac{a_i}{i}[\boldsymbol{C}[\eta,\mu,\nu]]\right).$$

Then by Möbius inversion (3.2), $\sum_{k\geq 1}\dfrac{\mu(k)}{k}a_k[\log(\boldsymbol{K}[\eta,1+\mu,1+\nu])] = \boldsymbol{C}[\eta,\mu,\nu]$.

So that equation (3.81) becomes,

$$\begin{aligned}\boldsymbol{B}[x,\mu,\nu] = -x &+ \sum_{k\geq 1}\frac{\mu(k)}{k}a_k[\log(\boldsymbol{K}[\eta,1+\mu,1+\nu])] \\ &- x\sum_{k\geq 1}\frac{\mu(k)}{k}a_k\left[\log\left(\frac{\eta(x,y)}{x}\right)\right].\end{aligned}$$

This can be written as

$$\boldsymbol{B}[x,\mu,\nu] = -x + \sum_{k\geq 1}\frac{\mu(k)}{k}\left\{a_k[\log(\boldsymbol{K}[\eta,1+\mu,1+\nu])] - xa_k\left[\log\left(\frac{\eta(x,y)}{x}\right)\right]\right\}.$$

Let $\log(\boldsymbol{K}[\eta,1+\mu,1+\nu]) = \sum_{i\geq 0} f_i(y)x^i$ and $\log\left(\dfrac{\eta(x,y)}{x}\right) = \sum_{i\geq 0} e_i(y)x^i$.

Then $a_k\left[\sum_{0\leq i\leq n} f_i(y)x^i\right] - xa_k\left[\sum_{0\leq i\leq n} e_i(y)x^i\right] = \sum_{0\leq i\leq n} f_k(y^k)x^{ik} - \sum_{0\leq i\leq n} e_i(y^k)x^{ki+1}$.

Thus,

$$\boldsymbol{B}[x,\mu,\nu] = -x + \sum_{k\geq 0} \frac{\mu(k)}{k} f_0(y^k) + \sum_{n\geq 1} \left\{ \sum_{k\cdot i=n} \frac{\mu(k)}{k} f_i(y^k) - \sum_{k\cdot i=n-1} \frac{\mu(k)}{k} e_i(y^k) \right\} x^n.$$

Since $f_0(y) = 0$, $\boldsymbol{B}[x,\mu,\nu] + x = \sum_{n\geq 1} \left\{ \sum_{k\cdot i=n} \frac{\mu(k)}{k} f_i(y^k) - \sum_{k\cdot i=n-1} \frac{\mu(k)}{k} e_i(y^k) \right\} x^n.$

(3.82)

§3.4 Deriving counting series for 2–connected graphs

Expressions for $S^+[x,\mu,\nu]$, $S^-[x,\mu,\nu]$, $(D^+ - P^+)[x,\mu,\nu] - \mu(x,y)$ and $(D^- - P^-)[x,\mu,\nu] - \nu(x,y)$ are needed to substitute in the final counting series of each of the classes of graphs. Further, $S^+[x,\mu,\nu]$ and $S^-[x,\mu,\nu]$ are used along with their counter parts in each class of graphs to compute the corresponding $\mu(x,y)$ and $\nu(x,y)$. Since all these four power series are common to all three classes graphs that we wish to count, we will list their derivations here.

§3.4.1 $S^+[x,\mu,\nu] = p(x,y)$

Now, from (3.51), $S^+ = M^+[a_1, D^+ - S^+, D^- - S^-]$.

Substituting for M^+, from (3.35), we get, $S^+ = \frac{a_1 b_1^2}{1 - a_1 b_1}[a_1, D^+ - S^+, D^- - S^-]$.

Expanding this we get, $S^+ = \frac{a_1(D^+ - S^+)^2}{1 - a_1(D^+ - S^+)}$.

Or, $S^+ = a_1 S^+(D^+ - S^+) + a_1(D^+ - S^+)^2 = a_1(D^+ - S^+)(S^+ + D^+ - S^+)$.

Thus, $S^+ = a_1(D^+ - S^+)D^+$.

This on composition on both sides with $[x,\mu,\nu]$ becomes,

$p(x,y) = x\delta(x,y)\left\{\delta(x,y) - p(x,y)\right\}$ where, $p(x,y) = \sum_{i\geq 0} p_i(y)x^i = S^+[x,\mu,\nu]$.

Solving for the nth term of $p(x,y)$, we get,

$$p_n(y) = \sum_{1\leq k\leq n} \delta_{k-1}(y)\left\{\delta_{n-k}(y) - p_{n-k}(y)\right\},$$

with $p_0(y) = 0$ and $p_1(y) = \delta_0(y)(\delta_0(y) - p_0(y)) = \delta_0(y)^2$. (3.83)

§3.4.2 $S^-[x,\mu,\nu] = s(x,y)$

On the other hand, from (3.52), $S^- = M^-[a_1, D^+ - S^+, D^- - S^-]$.

Substituting for M^-, from (3.36), we get,

$$S^- = \frac{a_1 b_2 + a_2 b_2 c_1}{1 - a_2 b_2}[a_1, D^+ - S^+, D^- - S^-].$$

Expanding this,

$$\begin{aligned} S^- &= \frac{a_1(D_2^+ - S_2^+) + a_2(D_2^+ - S_2^+)(D^- - S^-)}{1 - a_2(D_2^+ - S_2^+)} \\ &= a_2(D_2^+ - S_2^+)S^- + a_1(D_2^+ - S_2^+) + a_2(D_2^+ - S_2^+)(D^- - S^-). \end{aligned}$$

i.e., $S^- = a_1(D_2^+ - S_2^+) + a_2(D_2^+ - S_2^+)D^-$.

This on composition on both sides with $[x,\mu,\nu]$ becomes,

$$s(x,y) = x\Big\{\delta(x^2,y^2) - p(x^2,y^2)\Big\} + x^2\epsilon(x,y)\Big\{\delta(x^2,y^2) - p(x^2,y^2)\Big\}$$

where, $s(x,y) = \sum_{i\geq 0} s_i(y)x^i = S^-[x,\mu,\nu]$.

Then for $n \geq 1$,

$$s_{2n}(y) = \sum_{1\leq k\leq n}\Big\{\delta_{k-1}(y^2) - p_{k-1}(y^2)\Big\}\epsilon_{2n-2k}(y) \text{ and}$$

$$s_{2n+1}(y) = \delta_n(y^2) - p_n(y^2) + \sum_{1\leq k\leq n}\Big\{\delta_{k-1}(y^2) - p_{k-1}(y^2)\Big\}\epsilon_{2n+1-2k}(y),$$

with $s_0(y) = 0$ and $s_1(y) = \delta_0(y^2) - p_0(y^2) = \delta_0(y^2)$. (3.84)

§3.4.3 $D^+ - P^+$

Now, from (3.53), $P^+ = N_1^+[a_1, D^+ - P^+ - b_1, D^- - P^- - c_1]$.

Substituting the expression for N_1^+ from equation (3.47) in this,

$$P^+ = \left\{(1 + b_1^*)\exp\left(\sum_{i\geq 1}\frac{b_i}{i}\right) - (1 + b_1^*) - b_1\right\}\left[a_1, D^+ - P^+ - b_1, D^- - P^- - c_1\right].$$

This becomes,

$$P^+ = (1+b_1)\exp\left(\sum_{i\geq 1}\frac{b_i}{i}[D^+ - P^+ - b_1]\right) - (1+b_1) - (D^+ - P^+ - b_1).$$

Then $\exp\left(\sum_{i\geq 1}\frac{b_i}{i}[D^+ - P^+ - b_1]\right) = \dfrac{P^+ + (1+b_1) + D^+ - P^+ - b_1}{1+b_1} = \dfrac{1+D^+}{1+b_1}.$

Then by Möbius inversion (3.2), $D^+ - P^+ - b_1 = \sum_{i\geq 1}\frac{\mu(i)}{i}a_i\left[\log\left(\frac{1+D^+}{1+b_1}\right)\right].$

This on composition with $[x,\mu,\nu]$ becomes,

$$(D^+ - P^+)[x,\mu,\nu] - \mu(x,y) = \sum_{i\geq 1}\frac{\mu(i)}{i}a_i\left[\log\left(\frac{1+\delta(x,y)}{1+\mu(x,y)}\right)\right]. \tag{3.85}$$

§3.4.4 $D^- - P^-$

Now, from (3.54), $P^- = N_1^-[a_1, D^+ - P^+ - b_1, D^- - P^- - c_1]$.

Substituting the expression for N_1^- from equation (3.48) in this gives

$$P^- = \left\{(1+c_1^*)\exp\left(\sum_{i \text{ odd}}\frac{c_i}{i} + \sum_{i \text{ even}}\frac{b_i}{i}\right) - (1+c_1+c_1^*)\right\}$$
$$\left[a_1, D^+ - P^+ - b_1, D^- - P^- - c_1\right].$$

This on expansion becomes,

$$P^- = (1+c_1)\left\{\exp\left(\sum_{i \text{ odd}}\frac{c_i}{i}[D^- - P^- - c_1] + \sum_{i \text{ even}}\frac{b_i}{i}[D^+ - P^+ - b_1]\right)\right\}$$
$$-(1+D^- - P^-).$$

This on rearranging becomes,

$$\exp\left(\sum_{i \text{ odd}}\frac{c_i}{i}[D^- - P^- - c_1] + \sum_{i \text{ even}}\frac{b_i}{i}[D^+ - P^+ - b_1]\right)$$
$$= \frac{P^- + 1 + D^- - P^-}{1+c_1} = \frac{1+D^-}{1+c_1}.$$

Then by Möbius inversion (3.2), $\sum_{k \text{ odd}}\frac{\mu(k)}{k}a_k\left[\log\left(\frac{1+D^-}{1+c_1}\right)\right]$

$$= \sum_{k \text{ odd}}\frac{\mu(k)}{1}\sum_{i \text{ odd}}\frac{a_{ik}}{ik}[D^- - P^- - c_1] + \sum_{k \text{ odd}}\frac{\mu(k)}{1}\sum_{i \text{ even}}\frac{a_{ik}}{ik}[D^+ - P^+ - b_1]$$

$$= \sum_{m \text{ odd}} \frac{a_m}{m}[D^- - P^- - c_1] \sum_{i|m} \frac{\mu(i)}{i} + \sum_{m \text{ even}} \frac{a_m}{m}[D^+ - P^+ - b_1] \sum_{i|m} \frac{\mu(i)}{i}$$

$$= a_1[D^- - P^- - c_1] + \sum_{k \geq 1} \frac{a_{2^k}}{2^k}[D^+ - P^+ - b_1].$$

Therefore, $D^- - P^- - c_1 =$

$$\sum_{k \text{ odd}} \frac{\mu(k)}{k} a_k \left[\log\left(\frac{1+D^-}{1+c_1}\right)\right] - \sum_{m=2^{1+e},\, e\geq 0} \frac{a_m}{m}[D^+ - P^+ - b_1]$$

This on composition on both sides with $[x, \mu, \nu]$ and using equation (3.85),

$$(D^- - P^-)[x,\mu,\nu] - \nu(x,y) =$$

$$\sum_{k \text{ odd}} \frac{\mu(k)}{k} a_k \left[\log\left(\frac{1+\epsilon(x,y)}{1+\nu(x,y)}\right)\right] - \sum_{m=2^{1+e},\, e\geq 0} \frac{1}{m} \sum_{i \geq 1} \frac{\mu(i)}{i} a_{im} \left[\log\left(\frac{1+\delta(x,y)}{1+\mu(x,y)}\right)\right]. \tag{3.86}$$

§3.5 Some routines

In later chapters we will assume the existence of some pieces of code. These are listed here.

§3.5.1 $E1(H(x,y), \beta(x,y))$

Let $H(x,y) = \exp\left(\sum_{i\geq 1} \frac{b_i}{i}[\beta(x,y)]\right)$.

Taking $x\frac{\partial}{\partial x}$ on both sides,

$$x\frac{\partial}{\partial x}(H(x,y)) = H(x,y) \sum_{i \geq 1} \frac{b_i}{i} \left[x\frac{\partial}{\partial x}(\beta(x,y))\right]$$

$$= H(x,y) \sum_{i\geq 1} \frac{1}{i} \sum_{j \geq 1} \beta_j(y^i)(i \cdot j) x^{i \cdot j}.$$

i.e., $$\sum_{n \geq 1} n H_n(y) x^n = \left(\sum_{n\geq 0} H_n(y) x^n\right) * \left(\sum_{i \geq 1}\sum_{j\geq 1} j\beta_j(y^i) x^{i\cdot j}\right)$$

$$= \left(\sum_{n\geq 0} H_n(y) x^n\right) * \left(\sum_{n \geq 1}\left(\sum_{i \cdot j = n} j\beta_j(y^i)\right) x^n\right).$$

Solving for the nth term of $H(x,y)$, we get,

$$H_n(y) = \frac{1}{n} \sum_{1 \le k \le n} H_{n-k}(y) \left\{ \sum_{i \cdot j = k} j\beta_j(y^i) \right\},$$

with $H_0(y) = 1; H_1(y) = H_0(y)\beta_1(y) = \beta_1(y).$ (3.87)

We call this as routine $E1(H(x,y), \beta(x,y))$.

§3.5.2 $E2(M(x,y), \beta(x,y), \gamma(x,y))$

Let $M(x,y) = \exp\left(\sum_{i \text{ even}} \frac{b_i}{i}[\beta(x,y)] + \sum_{i \text{ odd}} \frac{c_i}{i}[\gamma(x,y)] \right).$

Taking $x\frac{\partial}{\partial x}$ on both sides,

$$x\frac{\partial}{\partial x}(M(x,y)) = M(x,y) \left(\sum_{i \text{ even}} \sum_{j \ge 1} j\beta_j(y^i)x^{i \cdot j} + \sum_{i \text{ odd}} \sum_{j \ge 1} j\gamma_j(y^i)x^{i \cdot j} \right).$$

Solving for the nth term of $M(x,y)$, we get,

$$M_n(y) = \frac{1}{n} \sum_{1 \le k \le n} M_{n-k}(y) \sum_{i \cdot j = k} \begin{Bmatrix} j\beta_j(y^i), & \text{if } i \text{ is even} \\ j\gamma_j(y^i), & \text{if } i \text{ is odd} \end{Bmatrix},$$

with $M_0(y) = 1; M_1(y) = M_0(y)\gamma_1(y) = \gamma_1(y).$ (3.88)

We call this as routine $E2(M(x,y), \beta(x,y), \gamma(x,y))$.

§3.5.3 $MP(A(x,y), \beta(x,y))$

Let $A(x,y) = M^+[x, \beta, \gamma] = \frac{a_1 b_1^2}{1 - a_1 b_1}[x, \beta, \gamma] = \frac{x\beta(x,y)^2}{1 - x\beta(x,y)}$

$$= xA(x,y)\beta(x,y) + x\beta(x,y)^2.$$

Solving for the nth term of $M(x,y)$, we get,

$$A_n(y) = \sum_{1 \le k \le n} \left\{ A_{k-1}(y)\beta_{n-k}(y) + \beta_{k-1}(y)\beta_{n-k}(y) \right\},$$

with $A_0(y) = 0; A_1(y) = \beta_0(y)\beta_0(y).$ (3.89)

We call this as routine $MP(A(x,y), \beta(x,y))$.

§3.5.4 $MN(A(x,y),\beta(x,y),\gamma(x,y))$

Let $A(x,y) = M^-[x,\beta,\gamma] = \dfrac{a_1b_2 + a_2b_2c_1}{1-a_2b_2}[x,\beta,\gamma]$.

$$\text{Then}\quad A(x,y) = x^2A(x,y)\beta(x^2,y^2) + x\beta(x^2,y^2) + x^2\beta(x^2,y^2)\gamma(x,y)$$
$$= x\beta(x^2,y^2) + x^2\beta(x^2,y^2)\Big\{A(x,y) + \gamma(x,y)\Big\}.$$

Solving for the $2n$th term of $A(x,y)$,

$$A_{2n}(y) = \sum_{1\le k\le n} \beta_{k-1}(y^2)\Big\{A_{2n-2k}(y) + \gamma_{2n-2k}(y)\Big\}.$$

Solving for the $(2n+1)$th term of $A(x,y)$, we get,

$$A_{2n+1}(y) = \beta_n(y^2) + \sum_{1\le k\le n} \beta_{k-1}(y^2)\Big\{A_{2n+1-2k}(y) + \gamma_{2n+1-2k}(y)\Big\}, \tag{3.90}$$

with $A_0(y) = 0$; $A_1(y) = \beta_0(y^2)$.

We call this as routine $MN(A(x,y),\beta(x,y),\gamma(x,y))$.

§3.5.5 $M0P(R(x,y),\beta(x,y))$

Let $R(x,y) = M_0^+[x,\beta,\gamma] = \dfrac{b_1^*(2a_1b_1 - a_1^2b_1^2)}{(1-a_1b_1)^2}[x,\beta,\gamma]$.

$$\text{Then } R(x,y) = \frac{y(2x\beta(x,y) - x^2\beta(x,y)^2)}{(1-x\beta(x,y))^2}$$
$$= R(x,y)(2x\beta(x,y) - x^2\beta(x,y)^2) + y(2x\beta(x,y) - x^2\beta(x,y)^2)$$
$$= 2xy\beta(x,y) + 2xR(x,y)\beta(x,y) - x^2R(x,y)\beta(x,y)^2 - yx^2\beta(x,y)^2.$$

So that, for $n \ge 2$,

$$R_n(y) = 2y\beta_{n-1}(y) + 2\sum_{1\le k\le n} R_{n-k}(y)\beta_{k-1}(y)$$
$$- \sum_{2\le k\le n} R_{n-k}(y) \sum_{0\le r\le k-2} \beta_r(y)\beta_{k-2-r}(y) - y\sum_{2\le k\le n} \beta_{n-k}(y)\beta_{k-2}(y),$$

with $R_0(y) = 0$ and $R_1(y) = 2\beta_0(y)(R_0(y) + y) = 2\beta_0(y)y$. (3.91)

§3.5.6 $M0N(R(x,y),\beta(x,y))$

Let $R(x,y) = M_0^-[x,\beta,\gamma] = \dfrac{c_1^*a_2b_2}{1-a_2b_2}[x,\beta,\gamma] = \dfrac{yx^2\beta(x^2,y^2)}{1-x^2\beta(x^2,y^2)}$

$$= R(x,y)x^2\beta_2 + yx^2\beta_2.$$

So that, for $n \geq 1$, $R_{2n}(y) = y\beta_{n-1}(y^2) + \sum_{1 \leq k \leq n} \beta_{k-1}(y^2)R_{2n-2k}(y)$

and $R_{2n+1}(y) = \sum_{1 \leq k \leq n} \beta_{k-1}(y^2)R_{2n+1-2k}(y)$; with $R_0(y) = R_1(y) = 0$. (3.92)

§3.5.7 $M0(R(x,y), \beta(x,y), \gamma(x,y))$

Let $R(x,y) = M_0[x, \beta, \gamma]$.

$$= \left\{ \frac{a_1^2 b_1^* a_1 b_1^2}{2(1 - a_1 b_1)} + \frac{a_2 c_1^*(a_1 b_2 + a_2 b_2 c_1)}{2(1 - a_2 b_2)} \right\}[x, \beta, \gamma]$$

$$= \frac{x^3 y \beta(x,y)^2}{2(1 - x\beta(x,y))} + \frac{x^2 y (x\beta(x^2,y^2) + x^2\beta(x^2,y^2)\gamma(x,y))}{2(1 - x^2\beta(x^2,y^2))}.$$

i.e., $R(1 - x\beta)(1 - x^2\beta_2) = \frac{1}{2}\left\{(1 - x^2\beta_2)x^3\beta^2 y + (1 - x\beta)x^2 y(x\beta_2 + x^2\beta_2\gamma)\right\}$

Or, $$R = \frac{1}{2}\Big\{2x^2\beta_2 R + 2xR\beta(1 - x^2\beta_2)$$
$$+ x^3\beta^2 y(1 - x^2\beta_2) + (x^2 y - x^3 y\beta)(x\beta_2 + x^2\beta_2\gamma)\Big\}.$$

Expanding and collecting coefficients,

$$R(x,y) = \frac{1}{2}\Big\{2x\beta R + 2x^2\beta_2 R + x^3(y\beta^2 - 2\beta_2\beta R + y\beta_2)$$
$$+ x^4 y\beta_2(\gamma - \beta) - x^5 y\beta_2\beta(\beta + \gamma)\Big\}.$$

Then $R_0(y) = 0$; $\qquad R_1(y) = \frac{1}{2}R_0(y)\beta_0(y) = 0.$

$$R_{2n}(y) = \frac{1}{2}\Big\{2\sum_{1 \leq k \leq 2n} \beta_{k-1}(y)R_{2n-k}(y) + 2\sum_{1 \leq k \leq n} \beta_{k-1}(y^2)R_{2n-2k}(y)$$
$$+ \sum_{3 \leq k \leq 2n} y\beta_{k-3}(y)\beta_{2n-k}(y) - 2\sum_{1 \leq k \leq n} \beta_{k-1}(y^2) \sum_{0 \leq r \leq 2n-1-2k} \beta_r(y)R_{2n-1-2k-r}(y)$$
$$+ \sum_{2 \leq k \leq n} y\beta_{k-2}(y^2)\Big\{\gamma_{2n-2k}(y) - \beta_{2n-2k}(y)\Big\}$$
$$- \sum_{2 \leq k \leq n} y\beta_{k-2}(y^2) \sum_{0 \leq r \leq 2n-1-2k} \beta_{2n-1-2k-r}(y)\Big\{\gamma_r(y) + \beta_r(y)\Big\}\Big\}.$$

$$R_{2n+1}(y) = \frac{1}{2}\Bigg\{2\sum_{1\le k\le 2n+1}\beta_{k-1}(y)R_{2n+1-k}(y) + 2\sum_{1\le k\le n}\beta_{k-1}(y^2)R_{2n+1-2k}(y)$$

$$+\sum_{3\le k\le 2n+1} y\beta_{k-3}(y)\beta_{2n+1-k}(y)$$

$$-2\sum_{1\le k\le n}\beta_{k-1}(y^2)\sum_{0\le r\le 2n-2k}\beta_r(y)R_{2n-2k-r}(y) + y\beta_{n-1}(y^2)$$

$$+\sum_{2\le k\le n} y\beta_{k-2}(y^2)\Big\{\gamma_{2n+1-2k}(y) - \beta_{2n+1-2k}(y)\Big\}$$

$$-\sum_{2\le k\le n} y\beta_{k-2}(y^2)\sum_{0\le r\le 2n-2k}\beta_{2n-2k-r}(y)\Big\{\gamma_r(y) + \beta_r(y)\Big\}\Bigg\}.$$

Let $m = \left\{\begin{matrix} \frac{n}{2} & \text{if } 2|n \\ \frac{n-1}{2} & \text{otherwise}\end{matrix}\right\}$.

Then for $n \ge 2$,

$$R_n(y) = \frac{1}{2}\Bigg\{2\beta_0(y)R_{n-1}(y) + 2\beta_1(y)R_{n-2}(y)$$

$$+\sum_{3\le k\le n}\Big\{2\beta_{k-1}(y)R_{n-k}(y) + y\beta_{k-3}(y)\beta_{n-k}(y)\Big\}$$

$$+2\beta_0(y^2)\Big\{R_{n-2}(y) - \sum_{0\le r\le n-3}\beta_r(y)R_{n-3-r}(y)\Big\}$$

$$+\sum_{2\le k\le m}\Bigg\{2\beta_{k-1}(y^2)\Big\{R_{n-2k}(y) - \sum_{0\le r\le n-1-2k}\beta_r(y)R_{n-1-2k-r}(y)\Big\}$$

$$+y\beta_{k-2}(y^2)\Big\{\gamma_{n-2k}(y) - \beta_{n-2k}(y) - \sum_{0\le r\le n-1-2k}\beta_{n-1-2k-r}(y)\{\gamma_r(y)+\beta_r(y)\}\Big\}\Bigg\}$$

$$+\left\{\begin{matrix} y\beta_{m-1}(y^2) & \text{if } 2\nmid n \\ 0 & \text{otherwise}\end{matrix}\right\}\Bigg\}. \tag{3.93}$$

§3.5.8 $M1P(R(x,y),\beta(x,y))$

Let $R(x,y) = M_1^+[x,\beta,\gamma] = \left\{\dfrac{a_1b_1^2 - a_1^2b_1^3 + 2a_1b_1^*b_1 - a_1^2b_1^*b_1^2}{(1-a_1b_1)^2}\right\}[x,\beta,\gamma]$.

i.e., $R(x,y) = R(x,y)(2x\beta(x,y) - x^2\beta(x,y)^2)$

$$+ x\beta(x,y)^2 - x^2\beta(x,y)^3 + 2xy\beta(x,y) - x^2y\beta(x,y)^2$$

$$= 2xy\beta(x,y) - x^2y\beta(x,y)^2 + x\beta(x,y)(2R(x,y) + \beta(x,y))$$
$$- x^2\beta(x,y)^2(R(x,y) + \beta(x,y)).$$

So that, for $n \geq 2$,

$$R_n(y) = 2y\beta_{n-1}(y) - y \sum_{1\leq k\leq n-1} \beta_{k-1}(y)\beta_{n-k-1}(y) +$$
$$+ \sum_{1\leq k\leq n} \beta_{k-1}(y)(2R_{n-k}(y) + \beta_{n-k}(y))$$
$$- \sum_{1\leq k\leq n-1} (R_{n-k-1}(y) + \beta_{n-k-1}(y))\left\{ \sum_{0\leq r\leq k-1} \beta_r(y)\beta_{k-1-r}(y)\right\}$$
$$= 2y\beta_{n-1}(y) + \beta_{n-1}(y)(2R_0(y) + \beta_0(y))$$
$$+ \sum_{1\leq k\leq n-1} \Bigg\{ -y\beta_{k-1}(y)\beta_{n-k-1}(y) + \beta_{k-1}(y)(2R_{n-k}(y) + \beta_{n-k}(y))$$
$$- (R_{n-k-1}(y) + \beta_{n-k-1}(y))\left\{ \sum_{0\leq r\leq k-1} \beta_r(y)\beta_{k-1-r}(y)\right\}\Bigg\}.$$

Since $R_0(y) = 0$,

$$R_n(y) = \beta_{n-1}(y)(2y + \beta_0(y))$$
$$+ \sum_{1\leq k\leq n-1} \Bigg\{ \beta_{k-1}(y)(-y\beta_{n-k-1}(y) + 2R_{n-k}(y) + \beta_{n-k}(y))$$
$$- (R_{n-k-1}(y) + \beta_{n-k-1}(y))\left\{ \sum_{0\leq r\leq k-1} \beta_r(y)\beta_{k-1-r}(y)\right\}\Bigg\},$$

with $R_1(y) = 2y\beta_0(y) + \beta_0(y)(2R_0(y) + \beta_0(y)) = 2y\beta_0(y) + \beta_0(y)^2$. (3.94)

§3.5.9 $M1N(R(x,y), \beta(x,y), \gamma(x,y))$

Let $R(x,y) = M_1^-[x,\beta,\gamma]\left\{\dfrac{a_1b_2 + a_2b_2(c_1 + c_1^*)}{1 - a_2b_2}\right\}[x,\beta,\gamma]$.

i.e., $R(x,y) = x^2R(x,y)\beta(x^2,y^2) + x\beta(x^2,y^2) + x^2\beta(x^2,y^2)(\gamma(x,y) + y)$

$$= x^2\beta(x^2,y^2)(R(x,y) + \gamma(x,y)) + x\beta(x^2,y^2) + x^2y\beta(x^2,y^2).$$

Then $R_0(y) = 0$ and $R_1(y) = \beta_0(y^2)$.

Then for $n \geq 1$, $R_{2n}(y) = y\beta_{n-1}(y^2) + \sum_{1\leq k\leq n} \beta_{k-1}(y^2)\Big\{ R_{2n-2k}(y) + \gamma_{2n-2k}(y)\Big\}$.

$$R_{2n+1}(y) = \beta_n(y^2) + \sum_{1\leq k\leq n} \beta_{k-1}(y^2)\Big\{ R_{2n+1-2k}(y) + \gamma_{2n+1-2k}(y)\Big\}. \qquad (3.95)$$

§3.5.10 $V(g(x,y), \alpha(x,y), \beta(x,y))$

Let $g(x,y) = \log\left(\dfrac{1 - x\alpha(x,y)}{1 - x\beta(x,y)}\right)$.

Taking $x\dfrac{\partial}{\partial x}$ on both sides,

$$xg_x(x,y) = \left(\frac{1-x\beta}{1-x\alpha}\right)\left\{\frac{(1-x\beta)(-\alpha - x\alpha_x)x - (1-x\alpha)(-\beta - x\beta_x)x}{(1-x\beta)^2}\right\}.$$

Cancelling $(1-x\beta)$, we get $xg_x = \dfrac{(x - x^2\alpha)(\beta + x\beta_x) - (x - x^2\beta)(\alpha + x\alpha_x)}{(1-x\alpha)(1-x\beta)}$.

Expanding, $xg_x = \dfrac{x\beta + x^2\beta_x - x^2\alpha\beta - x^3\alpha\beta_x - x\alpha - x^2\alpha_x + x^2\alpha\beta + x^3\beta\alpha_x}{1 - x(\alpha+\beta) + x^2\alpha\beta}$.

Cancelling $x^2\alpha\beta$ and rearranging,

$$xg_x = x(xg_x(\alpha+\beta)) - x^2(xg_x\alpha\beta) + x(\beta - \alpha) + x(x\beta_x - x\alpha_x) + x^2(\beta x\alpha_x - \alpha x\beta_x).$$

Solving for the nth term of $g(x,y)$

$$ng_n(y) = \sum_{2\leq k\leq n} (k-1)g_{k-1}\Big\{\alpha_{n-k} + \beta_{n-k}\Big\} - \sum_{3\leq k\leq n} (k-2)g_{k-2} \sum_{0\leq r\leq (n-k)} \alpha_r\beta_{n-k-r}$$
$$+ \beta_{n-1}(y) - \alpha_{n-1}(y) + (n-1)\beta_{n-1}(y) - (n-1)\alpha_{n-1}(y)$$
$$+ \sum_{2\leq k\leq n} \Big\{\beta_{k-2}(n-k)\alpha_{n-k}(y) - \alpha_{k-2}(n-k)\beta_{n-k}(y)\Big\}.$$

Gathering, for $n \geq 2$,

$$g_n(y) = \frac{1}{n}\Bigg\{ \sum_{2\leq k\leq n} \Big\{(k-1)g_{k-1}(\alpha_{n-k} + \beta_{n-k}) - (k-2)g_{k-2} \sum_{0\leq r\leq (n-k)} \alpha_r\beta_{n-k-r}$$
$$+ \beta_{k-2}(n-k)\alpha_{n-k}(y) - \alpha_{k-2}(n-k)\beta_{n-k}(y)\Big\} + n(\beta_{n-1} - \alpha_{n-1})\Bigg\}, \qquad (3.96)$$

with $g_0(y) = 0; g_1(y) = \beta_0(y) - \alpha_0(y)$.

We call this as routine $V(g(x,y), \alpha(x,y), \beta(x,y))$.

§3.5.11 $W(P(x,y),\beta(x,y),\gamma(x,y))$

Let $P(x,y) = \left\{ \frac{a_1a_2b_2c_1}{2(1-a_2b_2)} + \frac{a_1^2a_2b_2^2}{4(1-a_2b_2)} + \frac{a_2^2b_2c_1^2}{4(1-a_2b_2)} \right\}[x,\beta,\gamma].$

Then $P(x,y) =$

$$\frac{1}{4(1-x^2\beta(x^2,y^2))}\left\{2x^3\gamma(x,y)\beta(x^2,y^2) + x^4\beta(x^2,y^2)^2 + x^4\gamma(x,y)^2\beta(x^2,y^2)\right\}.$$

Bringing the RHS denominator to LHS and rearranging, we get,

$$P(x,y) = x^2P(x,y)\beta(x^2,y^2) + \frac{1}{2}x^3\gamma(x,y)\beta(x^2,y^2)$$
$$+\frac{1}{4}x^4\beta(x^2,y^2)^2 + \frac{x^4}{4}\gamma(x,y)^2\beta(x^2,y^2).$$

Collecting the common terms,

$$P(x,y) = x^2\beta(x^2,y^2)\left\{P(x,y) + \frac{1}{2}x\gamma(x,y) + \frac{1}{4}x^2\beta(x^2,y^2) + \frac{x^2}{4}\gamma(x,y)^2\right\}.$$

Solving for $2n$th coeff of $P(x,y)$,

$$P_{2n}(y) = \sum_{1\le k\le n} \beta_{k-1}(y^2)$$
$$*\left\{P_{2n-2k}(y) + \frac{1}{2}\gamma_{2n-2k-1}(y) + \frac{1}{4}\beta_{n-k-1}(y^2) + \frac{1}{4}\sum_{2\le r\le 2(n-k)}\gamma_{r-2}(y)\gamma_{2(n-k)-r}(y)\right\}.$$

Solving for $(2n+1)$th coeff of $P(x,y)$,

$$P_{2n+1}(y) = \sum_{1\le k\le n} \beta_{k-1}(y^2)$$
$$*\left\{P_{2n+1-2k}(y) + \frac{1}{2}\gamma_{2(n-k)}(y) + \frac{1}{4}0 + \frac{1}{4}\sum_{2\le r\le 2(n-k)+1}\gamma_{r-2}(y)\gamma_{2(n-k)-r+1}(y)\right\}.$$

Letting $m = \begin{cases} \frac{n}{2}, & \text{if } 2|n \\ \frac{n-1}{2}, & \text{otherwise} \end{cases}$

allows us to rewrite both these equations into single equation as,

$$P_n(y) = \sum_{1\le k\le m} \beta_{k-1}(y^2)\left\{ P_{n-2k}(y) + \frac{1}{2}\gamma_{n-1-2k}(y)\right.$$

$$+ \frac{1}{4}\left(\sum_{2 \le r \le n-2k} \gamma_{r-2}(y)\gamma_{n-2k-r}(y) \right) + \begin{pmatrix} \frac{1}{4}\beta_{\frac{n}{2}-k-1}(y^2), & \text{if } 2|n \\ 0, & \text{otherwise} \end{pmatrix} \Bigg\}, \quad (3.97)$$

with $P_0(y) = P_1(y) = 0; P_2(y) = \beta_0(y^2)P_0(y) = 0.$

We call this as routine $W(P(x,y), \beta(x,y), \gamma(x,y))$.

CHAPTER 4

Counting unlabeled minimally 2–connected graphs

§4.1 Initial derivations

In this section, we first derive $q(x,y)$ and $t(x,y)$. Then using $p(x,y)$ and $s(x,y)$ derived in 3rd chapter, we derive equations to express $\mu(x,y)$ in terms of $p(x,y)$ and $q(x,y)$ and $\nu(x,y)$ in terms of $s(x,y)$ and $t(x,y)$.

§4.1.1 $\hat{S}^+[x,y,y] = q(x,y)$

Recall (3.57), $\hat{S}^+ = M^+[a_1, b_1 + \hat{D}^+ - \hat{S}^+, c_1 + \hat{D}^- - \hat{S}^-]$.

Substituting for M^+ from (3.35), $\hat{S}^+ = \dfrac{a_1 b_1^2}{1 - a_1 b_1}[a_1, b_1 + \hat{D}^+ - \hat{S}^+, c_1 + \hat{D}^- - \hat{S}^-]$.

Expanding this gives, $\hat{S}^+ = \dfrac{a_1(b_1 + \hat{D}^+ - \hat{S}^+)^2}{1 - a_1(b_1 + \hat{D}^+ - \hat{S}^+)}$.

So that,
$$\begin{aligned}\hat{S}^+ &= a_1(b_1 + \hat{D}^+ - \hat{S}^+)\hat{S}^+ + a_1(b_1 + \hat{D}^+ - \hat{S}^+)^2 \\ &= a_1(b_1 + \hat{D}^+ - \hat{S}^+)(\hat{S}^+ + b_1 + \hat{D}^+ - \hat{S}^+) \\ &= a_1(b_1 + \hat{D}^+ - \hat{S}^+)(b_1 + \hat{D}^+).\end{aligned}$$

Composing both sides with $[x,y,y]$ this becomes,

$$q(x,y) = x\Big\{y + \delta(x,y) - q(x,y)\Big\} * \Big\{y + \delta(x,y)\Big\},$$

where, $q(x,y) = \sum_{i \geq 0} q_i(y)x^i = \hat{S}^+[x,y,y]$.

Thus, $q(x,y) = xy^2 + 2xy\delta(x,y) - xyq(x,y) + x\delta(x,y)\Big\{\delta(x,y) - q(x,y)\Big\}$.

Solving for the nth term of $q(x,y)$, gives, for $n \geq 2$,

$$q_n(y) = y\Big\{2\delta_{n-1}(y) - q_{n-1}(y)\Big\} + \sum_{1\le k\le n} \delta_{k-1}(y)\Big\{\delta_{n-k}(y) - q_{n-k}(y)\Big\}, \qquad (4.1)$$

with $q_0(y) = 0$

and $q_1(y) = y^2 + 2y\delta_0(y) - yq_0(y) + \delta_0(y)\Big\{\delta_0(y) - q_0(y)\Big\} = y^2.$

§4.1.2 $\hat{S}^-[x, y, y] = t(x, y)$

Also, recall (3.58), $\hat{S}^- = M^-[a_1, b_1 + \hat{D}^+ - \hat{S}^+, c_1 + \hat{D}^- - \hat{S}^-]$.

Substituting for M^- from (3.36),

$$\hat{S}^- = \frac{a_1b_2 + a_2b_2c_1}{1 - a_2b_2}[a_1, b_1 + \hat{D}^+ - \hat{S}^+, c_1 + \hat{D}^- - \hat{S}^-].$$

Expanding, $\hat{S}^- = \dfrac{a_1(b_2 + \hat{D}_2^+ - \hat{S}_2^+) + a_2(b_2 + \hat{D}_2^+ - \hat{S}_2^+)(c_1 + \hat{D}^- - \hat{S}^-)}{1 - a_2(b_2 + \hat{D}_2^+ - \hat{S}_2^+)}.$

So that,
$$\hat{S}^- = a_2(b_2 + \hat{D}_2^+ - \hat{S}_2^+)\hat{S}^- + a_1(b_2 + \hat{D}_2^+ - \hat{S}_2^+)$$
$$+ a_2(b_2 + \hat{D}_2^+ - \hat{S}_2^+)(c_1 + \hat{D}^- - \hat{S}^-)$$
$$= a_1(b_2 - \hat{D}_2^+ - \hat{S}_2^+) + a_2(b_2 + \hat{D}_2^+ - \hat{S}_2^+)(c_1 + \hat{D}^-)$$
$$= (a_1 + a_2(c_1 + \hat{D}^-))(b_2 - \hat{D}_2^+ - \hat{S}_2^+).$$

Composing both sides with $[x, y, y]$ this becomes,

$$t(x, y) = \Big\{x + x^2(y + \epsilon(x, y))\Big\} * \Big\{y^2 - \delta(x^2, y^2) - q(x^2, y^2)\Big\}$$

where, $t(x, y) = \sum_{i\ge 0} t_i(y)x^i = \hat{S}^-[x, y, y]$.

Or,
$$t(x, y) = xy^2 + x^2y^3 + x^2y^2\epsilon(x, y)$$
$$+ \Big\{x + x^2y + x^2\epsilon(x, y)\Big\} * \Big\{\delta(x^2, y^2) - q(x^2, y^2)\Big\}.$$

Then collecting the odd and even terms, for $n \ge 2$,

$$t_{2n}(y) = \epsilon_{2n-2}(y)y^2 + y\Big\{\delta_{n-1}(y^2) - q_{n-1}(y^2)\Big\}$$
$$+ \sum_{1\le k\le n} \Big\{\delta_{k-1}(y^2) - q_{k-1}(y^2)\Big\}\epsilon_{2n-2k}(y).$$

$$t_{2n+1}(y) = \epsilon_{2n-1}(y)y^2 + \delta_n(y^2) - q_n(y^2)$$

$$+ \sum_{1 \le k \le n} \left\{ \delta_{k-1}(y^2) - q_{k-1}(y^2) \right\} \epsilon_{2n+1-2k} \tag{4.2}$$

with $t_0(y) = 0; t_1(y) = y^2 + \delta_0(y^2) - q_0(y^2) = y^2$;

and $t_2(y) = y^3 + \epsilon_0(y)y^2 + (y + \epsilon_0(y)) \left\{ \delta_0(y^2) - q_0(y^2) \right\} = y^3$.

§4.1.3 $\hat{D}^+ - \hat{P}^+$

Recall (3.59), $\hat{P}^+ = N^+[a_1, \hat{D}^+ - \hat{P}^+, \hat{D}^- - \hat{P}^-]$.

Substituting for N^+ from (3.44) gives,

$$\hat{P}^+ = \left\{ \exp\left(\sum_{i \ge 1} \frac{b_i}{i} \right) - (1 + b_1) \right\} [a_1, \hat{D}^+ - \hat{P}^+, \hat{D}^- - \hat{P}^-].$$

On expansion this becomes, $\hat{P}^+ = \exp\left(\sum_{i \ge 1} \frac{b_i}{i} [\hat{D}^+ - \hat{P}^+] \right) - \left\{ 1 + (\hat{D}^+ - \hat{P}^+) \right\}$.

Or, $\exp\left(\sum_{i \ge 1} \frac{b_i}{i} [\hat{D}^+ - \hat{P}^+] \right) = \hat{P}^+ + 1 + \hat{D}^+ - \hat{P}^+ = 1 + \hat{D}^+$.

Then by Möbius inversion (3.2), $\hat{D}^+ - \hat{P}^+ = \sum_{i \ge 1} \frac{\mu(i)}{i} a_i[\log(1 + \hat{D}^+)]$.

Composing both sides with $[x, y, y]$, gives

$$(\hat{D}^+ - \hat{P}^+)[x, y, y] = \sum_{i \ge 1} \frac{\mu(i)}{i} a_i[\log(1 + \delta(x, y))]. \tag{4.3}$$

§4.1.4 $\hat{D}^- - \hat{P}^-$

Recall (3.60), $\hat{P}^- = N^-[a_1, \hat{D}^+ - \hat{P}^+, \hat{D}^- - \hat{P}^-]$.

Substituting the expression for N^- from (3.45) in this gives

$$\hat{P}^- = \left\{ \exp\left(\sum_{i \text{ odd}} \frac{c_i}{i} + \sum_{i \text{ even}} \frac{b_i}{i} \right) - (1 + c_1) \right\} [a_1, \hat{D}^+ - \hat{P}^+, \hat{D}^- - P^-].$$

On expansion this becomes,

$$\hat{P}^- = \exp\left(\sum_{i \text{ odd}} \frac{c_i}{i} [\hat{D}^- - \hat{P}^-] + \sum_{i \text{ even}} \frac{b_i}{i} [\hat{D}^+ - \hat{P}^+] \right) - (1 + \hat{D}^- - \hat{P}^-).$$

This on rearranging becomes,

$$\exp\left(\sum_{i\text{ odd}} \frac{c_i}{i}[\hat{D}^- - \hat{P}^-] + \sum_{i\text{ even}} \frac{b_i}{i}[\hat{D}^+ - \hat{P}^+]\right) = \hat{P}^- + 1 + \hat{D}^- - \hat{P}^- = 1 + \hat{D}^-.$$

Then by Möbius inversion (3.2),

$$\begin{aligned}\sum_{k\text{ odd}} \frac{\mu(k)}{k} a_k[\log(1+\hat{D}^-)] &= \sum_{k\text{ odd}} \frac{\mu(k)}{1} \sum_{i\text{ odd}} \frac{a_{ik}}{ik}[\hat{D}^- - \hat{P}^-] \\ &+ \sum_{k\text{ odd}} \frac{\mu(k)}{1} \sum_{i\text{ even}} \frac{a_{ik}}{ik}[\hat{D}^+ - \hat{P}^+] \\ &= \sum_{m\text{ odd}} \frac{a_m}{m}[\hat{D}^- - \hat{P}^-] \sum_{i|m} \frac{\mu(i)}{i} \\ &+ \sum_{m\text{ even}} \frac{a_m}{m}[\hat{D}^+ - \hat{P}^+] \sum_{i|m;\, i\text{ odd}} \frac{\mu(i)}{i} \\ &= a_1[\hat{D}^- - \hat{P}^-] + \sum_{k\geq 1} \frac{a_{2^k}}{2^k}[\hat{D}^+ - \hat{P}^+].\end{aligned}$$

Therefore, $\hat{D}^- - \hat{P}^- = \sum_{k\text{ odd}} \frac{\mu(k)}{k} a_k[\log(1+\hat{D}^-)] - \sum_{m=2^{1+e},\, e\geq 0} \frac{a_m}{m}[\hat{D}^+ - \hat{P}^+].$

Composing both sides with $[x,y,y]$ and using equation (4.3)

$$\begin{aligned}(\hat{D}^- - \hat{P}^-)[x,y,y] &= \sum_{k\text{ odd}} \frac{\mu(k)}{k} a_k[\log(1+\epsilon(x,y))] \\ &- \sum_{m=2^{1+e},\, e\geq 0} \frac{1}{m} \sum_{i\geq 1} \frac{\mu(i)}{i} a_{im}[\log(1+\delta(x,y))]. \end{aligned} \tag{4.4}$$

§4.1.5 $\mu(x,y)$

Now, rearranging, (3.49), gives

$$\begin{aligned}T^+[a_1, D^+, D^-] &= D^+ - N_1^+[a_1, D^+ - P^+ - b_1, D^- - P^- - c_1] - b_1 \\ &- M^+[a_1, D^+ - S^+, D^- - S^-].\end{aligned}$$

Applying (3.53) and (3.51), on RHS of this gives

$$T^+[a_1, D^+, D^-] = D^+ - P^+ - b_1 - S^+.$$

Composing with $[x,\mu,\nu]$ and using (3.75) gives

$$T^+[x,\delta,\epsilon] = (D^+ - P^+)[x,\mu,\nu] - \mu(x,y) - S^+[x,\mu,\nu]. \tag{4.5}$$

Also, rearranging (3.55) gives

$$T^+[a_1, \hat{D}^+, \hat{D}^-] = \hat{D}^+ - N^+[a_1, \hat{D}^+ - \hat{P}^+, \hat{D}^- - \hat{P}^-]$$
$$- M^+[a_1, b_1 + \hat{D}^+ - \hat{S}^+, c_1 + \hat{D}^- - \hat{S}^-].$$

Applying (3.59) and (3.57), on RHS of this gives $T^+[a_1, \hat{D}^+, \hat{D}^-] = \hat{D}^+ - \hat{P}^+ - \hat{S}^+$.

Composing this with $[x, y, y]$, and using (3.75), gives

$$T^+[x, \delta, \epsilon] = (\hat{D}^+ - \hat{P}^+)[x, y, y] - \hat{S}^+[x, y, y]. \tag{4.6}$$

This along with (4.5), gives

$$(D^+ - P^+)[x, \mu, \nu] - \mu(x, y) - S^+[x, \mu, \nu] = (\hat{D}^+ - \hat{P}^+)[x, y, y] - \hat{S}^+[x, y, y]. \tag{4.7}$$

$$\therefore \hat{S}^+[x, y, y] - S^+[x, \mu, \nu] = (\hat{D}^+ - \hat{P}^+)[x, y, y] - \left\{(D^+ - P^+)[x, \mu, \nu] - \mu(x, y)\right\}$$
$$= \sum_{i \geq 1} \frac{\mu(i)}{i} a_i[\log(1 + \delta(x, y))]$$
$$- \sum_{i \geq 1} \frac{\mu(i)}{i} a_i \left[\log\left(\frac{1 + \delta(x, y)}{1 + \mu(x, y)}\right)\right], \text{ from (4.3) \& (3.85).}$$
$$= \sum_{i \geq 1} \frac{\mu(i)}{i} a_i[\log(1 + \mu(x, y))].$$

Then by Möbius inversion (3.2),

$$1 + \mu(x, y) = \exp\left(\sum_{i \geq 1} \frac{a_i}{i}[\hat{S}^+[x, y, y] - S^+[x, \mu, \nu]]\right)$$
$$= \exp\left(\sum_{i \geq 1} \frac{a_i}{i}[q(x, y) - p(x, y)]\right). \tag{4.8}$$

Thus, $\mu(x, y)$ can be computed using $E1(1 + \mu(x, y), q(x, y) - p(x, y))$

with $\mu_0(y) = 0; \mu_1(y) = q_1(y) - p_1(y) = y^2 - 0 = y^2$.

§4.1.6 $\nu(x, y)$

Now, rearranging, (3.50), gives

$$T^-[a_1, D^+, D^-] = D^- - N_1^+[a_1, D^+ - P^+ - b_1, D^- - P^- - c_1] - c_1$$
$$- M^-[a_1, D^+ - S^+, D^- - S^-].$$

Applying (3.54) and (3.52), on RHS of this gives,

$T^-[a_1, D^+, D^-] = D^- - P^- - c_1 - S^-$.

Composing this with $[x, \mu, \nu]$ and using (3.75) gives,

$$T^-[x, \delta, \epsilon] = (D^- - P^-)[x, \mu, \nu] - \nu(x, y) - S^-[x, \mu, \nu]. \tag{4.9}$$

Also, rearranging (3.56) gives

$$T^-[a_1, \hat{D}^+, \hat{D}^-] = \hat{D}^- - N^-[a_1, \hat{D}^+ - \hat{P}^+, \hat{D}^- - \hat{P}^-]$$
$$- M^-[a_1, b_1 + \hat{D}^+ - \hat{S}^+, c_1 + \hat{D}^- - \hat{S}^-].$$

Applying (3.60) and (3.58), on RHS of this gives $T^-[a_1, \hat{D}^+, \hat{D}^-] = \hat{D}^- - \hat{P}^- - \hat{S}^-$.

Composing this with $[x, y, y]$, and using (3.75), gives

$$T^-[x, \delta, \epsilon] = (\hat{D}^- - \hat{P}^-)[x, y, y] - \hat{S}^-[x, y, y]. \tag{4.10}$$

This along with (4.9), gives

$$(D^- - P^-)[x, \mu, \nu] - \nu(x, y) - S^-[x, \mu, \nu] = (\hat{D}^- - \hat{P}^-)[x, y, y] - \hat{S}^-[x, y, y]. \tag{4.11}$$

$$\therefore \hat{S}^-[x, y, y] - S^-[x, \mu, \nu] = (\hat{D}^- - \hat{P}^-)[x, y, y] - \left\{(D^- - P^-)[x, \mu, \nu] - \nu\right\}$$
$$= \sum_{k \text{ odd}} \frac{\mu(k)}{k} a_k[\log(1 + \epsilon(x, y))] - \sum_{m=2^{1+e},\, e\geq 0} \frac{1}{m} \sum_{i\geq 1} \frac{\mu(i)}{i} a_{im}[\log(1 + \delta(x, y))]$$
$$- \sum_{k \text{ odd}} \frac{\mu(k)}{k} a_k \left[\log\left(\frac{1 + \epsilon(x, y)}{1 + \nu(x, y)}\right)\right] + \sum_{m=2^{1+e},\, e\geq 0} \frac{1}{m} \sum_{i\geq 1} \frac{\mu(i)}{i} a_{im} \left[\log\left(\frac{1 + \delta(x, y)}{1 + \mu(x, y)}\right)\right],$$

from (3.86) and (4.4).

i.e., $t(x, y) - s(x, y)$

$$= \sum_{k \text{ odd}} \frac{\mu(k)}{k} a_k[\log(1 + \nu(x, y))] - \sum_{m=2^{1+e},\, e\geq 0} \frac{1}{m} \sum_{i\geq 1} \frac{\mu(i)}{i} a_{im}[\log(1 + \mu(x, y))]$$
$$= \sum_{k \text{ odd}} \frac{\mu(k)}{k} a_k[\log(1 + \nu(x, y))] - \sum_{m=2^{1+e},\, e\geq 0} \frac{a_m}{m}[q(x, y) - p(x, y)].$$

$$\therefore \sum_{k \text{ odd}} \frac{\mu(k)}{k} a_k[\log(1 + \nu(x, y))]$$

$$= a_1[t(x,y) - s(x,y)] + \sum_{m=2^{1+e},\, e\geq 0} \frac{a_m}{m}[q(x,y) - p(x,y)]$$
$$= \sum_{m \text{ odd}} \frac{a_m}{m}[t(x,y) - s(x,y)] \sum_{i|m} \frac{\mu(i)}{i} + \sum_{m \text{ odd}} \frac{a_m}{m}[q(x,y) - p(x,y)] \sum_{i|m,\, i \text{ odd}} \frac{\mu(i)}{i}$$
$$= \sum_{k \text{ odd}} \frac{\mu(k)}{1} \sum_{i \text{ odd}} \frac{a_{ik}}{ik}[t(x,y) - s(x,y)] + \sum_{k \text{ odd}} \frac{\mu(k)}{1} \sum_{i \text{ even}} \frac{a_{ik}}{ik}[q(x,y) - p(x,y)].$$

Then by Möbius inversion (3.2),

$$1 + \nu(x,y) = \exp\left(\sum_{i \text{ odd}} \frac{a_i}{i}[t(x,y) - s(x,y)] + \sum_{i \text{ even}} \frac{a_i}{i}[q(x,y) - p(x,y)]\right). \quad (4.12)$$

Thus, $\nu(x,y)$ can be computed using

$E2(1 + \nu(x,y), q(x,y) - p(x,y), t(x,y) - s(x,y))$

with $\nu_0(y) = 0$; $\quad \nu_1(y) = t_1(y) - s_1(y) = y^2 - 0 = y^2$.

§4.2 Main counting equation

Now, using Otter's dissimilarity characteristic equation(3.16) as in the case of 2–connected graphs (3.74), the decomposition of minimally 2–connected graphs can be written as :

$$\begin{aligned}
\hat{B} &= T[a_1, \hat{D}^+, \hat{D}^-] + N[a_1, \hat{D}^+ - \hat{P}^+, \hat{D}^- - \hat{P}^-] \\
&+ M[a_1, b_1 + \hat{D}^+ - \hat{S}^+, c_1 + \hat{D}^- - \hat{S}^-] \\
&- \Big\{ \vec{T}[a_1, \hat{D}^+, \hat{D}^-] \diamond \vec{N}[a_1, \hat{D}^+ - \hat{P}^+, \hat{D}^- - \hat{P}^-] \\
&+ \vec{N}[a_1, \hat{D}^+ - \hat{P}^+, \hat{D}^- - \hat{P}^-] \diamond \vec{M}[a_1, b_1 + \hat{D}^+ - \hat{S}^+, c_1 + \hat{D}^- - \hat{S}^-] \\
&+ \vec{T}[a_1, \hat{D}^+, \hat{D}^-] \diamond \vec{M}[a_1, b_1 + \hat{D}^+ - \hat{S}^+, c_1 + \hat{D}^- - \hat{S}^-] \\
&+ \frac{1}{2}\vec{T}[a_1, \hat{D}^+, \hat{D}^-] \diamond \vec{T}[a_1, \hat{D}^+, \hat{D}^-] \\
&+ \left(\frac{a_1^2 + a_2}{4}\right) a_2 \left[T^+[a_1, \hat{D}^+, \hat{D}^-]\right] \Big\} + \left(\frac{a_1^2 + a_2}{2}\right) a_2 \left[T^+[a_1, \hat{D}^+, \hat{D}^-]\right].
\end{aligned}$$

Composing on both sides with $[x, y, y]$ and using (3.75) gives,

$$
\begin{aligned}
\hat{B}[x,y,y] &= T[x,\delta,\epsilon] + N\left[x,(\hat{D}^{+}-\hat{P}^{+})[x,y,y],(\hat{D}^{-}-\hat{P}^{-})[x,y,y]\right] \\
&+ M\left[x,y+\delta-\hat{S}^{+}[x,y,y],y+\epsilon-\hat{S}^{-}[x,y,y]\right] \\
&- \Big\{\overrightarrow{T}[x,\delta,\epsilon]\diamond\overrightarrow{N}\left[x,(\hat{D}^{+}-\hat{P}^{+})[x,y,y],(\hat{D}^{-}-\hat{P}^{-})[x,y,y]\right] \\
&+ \overrightarrow{N}\left[x,(\hat{D}^{+}-\hat{P}^{+})[x,y,y],(\hat{D}^{-}-\hat{P}^{-})[x,y,y]\right] \\
&\diamond\overrightarrow{M}\left[x,y+\delta-\hat{S}^{+}[x,y,y],y+\epsilon-\hat{S}^{-}[x,y,y]\right] \\
&+ \overrightarrow{T}[x,\delta,\epsilon]\diamond\overrightarrow{M}\left[x,y+\delta-\hat{S}^{+}[x,y,y],y+\epsilon-\hat{S}^{-}[x,y,y]\right] \\
&+ \frac{1}{2}\overrightarrow{T}[x,\delta,\epsilon]\diamond\overrightarrow{T}[x,\delta,\epsilon] + \frac{x^{2}}{2}a_{2}\left[T^{+}[x,\delta,\epsilon]\right]\Big\} + x^{2}a_{2}\left[T^{+}[x,\delta,\epsilon]\right].
\end{aligned}
$$

Now, using (3.76) on RHS of this gives

$$
\begin{aligned}
\hat{B}[x,y,y] &= B[x,\mu,\nu] - N_{1}\left[x,(D^{+}-P^{+})[x,\mu,\nu]-\mu,(D^{-}-P^{-})[x,\mu,\nu]-\nu\right] \\
&- M\left[x,\delta-S^{+}[x,\mu,\nu],\epsilon-S^{-}[x,\mu,\nu]\right] \\
&+ \overrightarrow{T}[x,\delta,\epsilon]\diamond\overrightarrow{N}_{1}\left[x,(D^{+}-P^{+})[x,\mu,\nu]-\mu,(D^{-}-P^{-})[x,\mu,\nu]-\nu\right] \\
&+ \overrightarrow{N}_{1}\left[x,(D^{+}-P^{+})[x,\mu,\nu]-\mu,(D^{-}-P^{-})[x,\mu,\nu]-\nu\right] \\
&\diamond\overrightarrow{M}\left[x,\delta-S^{+}[x,\mu,\nu],\epsilon-S^{-}[x,\mu,\nu]\right] \\
&+ \overrightarrow{T}[x,\delta,\epsilon]\diamond\overrightarrow{M}\left[x,\delta-S^{+}[x,\mu,\nu],\epsilon-S^{-}[x,\mu,\nu]\right] - \frac{x^{2}}{2}(\mu(x,y)+\nu(x,y)) \\
&+ N\left[x,(\hat{D}^{+}-\hat{P}^{+})[x,y,y],(\hat{D}^{-}-\hat{P}^{-})[x,y,y]\right] \\
&+ M\left[x,y+\delta-\hat{S}^{+}[x,y,y],y+\epsilon-\hat{S}^{-}[x,y,y]\right] \\
&- \overrightarrow{T}[x,\delta,\epsilon]\diamond\overrightarrow{N}\left[x,(\hat{D}^{+}-\hat{P}^{+})[x,y,y],(\hat{D}^{-}-\hat{P}^{-})[x,y,y]\right] \\
&- \overrightarrow{N}\left[x,(\hat{D}^{+}-\hat{P}^{+})[x,y,y],(\hat{D}^{-}-\hat{P}^{-})[x,y,y]\right] \\
&\diamond\overrightarrow{M}\left[x,y+\delta-\hat{S}^{+}[x,y,y],y+\epsilon-\hat{S}^{-}[x,y,y]\right] \\
&- \overrightarrow{T}[x,\delta,\epsilon]\diamond\overrightarrow{M}\left[x,y+\delta-\hat{S}^{+}[x,y,y],y+\epsilon-\hat{S}^{-}[x,y,y]\right].
\end{aligned}
\tag{4.13}
$$

Rearranging (4.13) gives

$$\begin{aligned}
\hat{B}[x,y,y] = {} & B[x,\mu,\nu] + N\left[x,(\hat{D}^+ - \hat{P}^+)[x,y,y],(\hat{D}^- - \hat{P}^-)[x,y,y]\right] \\
& - N_1\left[x,(D^+ - P^+)[x,\mu,\nu] - \mu,(D^- - P^-)[x,\mu,\nu] - \nu\right] \\
& + M\left[x,y+\delta - \hat{S}^+[x,y,y],y+\epsilon - \hat{S}^-[x,y,y]\right] \\
& - M\left[x,\delta - S^+[x,\mu,\nu],\epsilon - S^-[x,\mu,\nu]\right] \\
& + \vec{T}[x,\delta,\epsilon] \diamond \vec{N}_1\left[x,(D^+ - P^+)[x,\mu,\nu] - \mu,(D^- - P^-)[x,\mu,\nu]\right] \\
& - \vec{T}[x,\delta,\epsilon] \diamond \vec{N}\left[x,(\hat{D}^+ - \hat{P}^+)[x,y,y],(\hat{D}^- - \hat{P}^-)[x,y,y]\right] \\
& + \vec{N}_1\left[x,(D^+ - P^+)[x,\mu,\nu] - \mu,(D^- - P^-)[x,\mu,\nu] - \nu\right] \\
& \diamond \vec{M}\left[x,\delta - S^+[x,\mu,\nu],\epsilon - S^-[x,\mu,\nu]\right] \\
& - \vec{N}\left[x,(\hat{D}^+ - \hat{P}^+)[x,y,y],(\hat{D}^- - \hat{P}^-)[x,y,y]\right] \\
& \diamond \vec{M}\left[x,y+\delta - \hat{S}^+[x,y,y],y+\epsilon - \hat{S}^-[x,y,y]\right] \\
& + \vec{T}[x,\delta,\epsilon] \diamond \vec{M}\left[x,\delta - S^+[x,\mu,\nu],\epsilon - S^-[x,\mu,\nu]\right] \\
& - \vec{T}[x,\delta,\epsilon] \diamond \vec{M}\left[x,y+\delta - \hat{S}^+[x,y,y],y+\epsilon - \hat{S}^-[x,y,y]\right] \\
& - \frac{x^2}{2}(\mu(x,y) + \nu(x,y)). \qquad (4.14)
\end{aligned}$$

§4.3 Final counting series

Now, let $u(x,y) = (\hat{D}^+ - \hat{P}^+)[x,y,y]$.

Then (4.6) becomes, $T^+[x,\delta,\epsilon] = u(x,y) - q(x,y)$. (4.15)

Also by (4.3), $u(x,y) = \sum_{i\geq 1} u_i(y)x^i = \sum_{i\geq 1} \frac{\mu(i)}{i} a_i[\log(1+\delta(x,y))]$.

Letting $w(x,y) = \log(1+\delta(x,y))$, gives, $u_n(y) = \sum_{i\cdot j=n} \frac{\mu(i)}{i} w_j(y^i)$. (4.16)

Moreover, rearranging (4.7) gives,

$(D^+ - P^+)[x,\mu,\nu] - \mu(x,y) = p(x,y) - q(x,y) + u(x,y)$. (4.17)

Now, let $v(x,y) = (\hat{D}^- - \hat{P}^-)[x,y,y]$.

Then (4.10) becomes, $T^-[x,\delta,\epsilon] = v(x,y) - t(x,y)$. (4.18)

Also by (4.4), $v(x,y) = \sum_{i\geq 1} v_i(y)x^i = \sum_{k \text{ odd}} \frac{\mu(k)}{k} a_k[\log(1+\epsilon(x,y))]$

$$- \sum_{m=2^{1+e},\ e\geq 0} \frac{a_m}{m} [u(x,y)].$$

Letting $h(x,y) = \log(1+\epsilon(x,y))$ gives,

$$v_n(y) = \sum_{i\cdot j=n,\ i \text{ odd}} \frac{\mu(i)}{i} h_j(y^i) - \sum_{2|n, m\cdot i=n} \frac{1}{m} u_i(y^m), \text{where, } m = 2^{1+e}, \text{for } e \geq 0. \tag{4.19}$$

Moreover, rearranging (4.11), gives,

$$(D^- - P^-)[x,\mu,\nu] - \nu(x,y) = s(x,y) - t(x,y) + v(x,y). \tag{4.20}$$

Recall from previous equations,

$$\delta(x,y) - S^+[x,\mu,\nu] = \delta(x,y) - p(x,y),$$

$$\epsilon(x,y) - S^-[x,\mu,\nu] = \epsilon(x,y) - s(x,y),$$

$$y + \delta(x,y) - \hat{S}^+[x,y,y] = y + \delta(x,y) - q(x,y),$$

$$y + \epsilon(x,y) - \hat{S}^-[x,y,y] = y + \epsilon(x,y) - t(x,y).$$

Substituting all of these in (4.14), gives,

$$\begin{aligned}
\hat{B}[x,y,y] = {} & B[x,\mu,\nu] \text{ (say, } B(x,y) \text{)} \\
& + N[x,u,v] - N_1[x,p-q+u,s-t+v] \text{ (say, } A1(x,y) \text{)} \\
& + M[x,y+\delta-q,y+\epsilon-t] - M[x,\delta-p,\epsilon-s] \text{ (say, } A2(x,y) \text{)} \\
& + \overrightarrow{T}[x,\delta,\epsilon] \diamond \overrightarrow{N}_1[x,p-q+u,s-t+v] \\
& - \overrightarrow{T}[x,\delta,\epsilon] \diamond \overrightarrow{N}[x,u,v] \text{ (say, } A3(x,y) \text{)} \\
& + \overrightarrow{N}_1[x,p-q+u,s-t+v] \diamond \overrightarrow{M}[x,\delta-p,\epsilon-s] \\
& - \overrightarrow{N}[x,u,v] \diamond \overrightarrow{M}[x,y+\delta-q,y+\epsilon-t] \text{ (say, } A4(x,y)) \\
& + \overrightarrow{T}[x,\delta,\epsilon] \diamond \overrightarrow{M}[x,\delta-p,\epsilon-s] \\
& - \overrightarrow{T}[x,\delta,\epsilon] \diamond \overrightarrow{M}[x,y+\delta-q,y+\epsilon-t] \text{ (say, } A5(x,y))
\end{aligned}$$

$$- \frac{x^2}{2}(\mu(x,y) + \nu(x,y)).$$

In terms of nth coefficients, for $n \geq 2$, this is,

$$\hat{B}_n(y) = B_n(y) + A1_n(y) + A2_n(y) + A3_n(y) + A4_n(y) + A5_n(y)$$
$$- \frac{1}{2}\Big\{ \mu_{n-2}(y) + \nu_{n-2}(y) \Big\}. \tag{4.21}$$

§4.3.1 $A1(x,y)$

$$A1(x,y) = N[x,u,v] - N_1[x, p-q+u, s-t+v].$$

Let pqu to denote $p-q+u$ and stv to denote $s-t+v$.

Then from (3.43) and (3.46), we get

$$A1(x,y) = \frac{x^2}{2}\Bigg\{ \exp\left(\sum_{i\geq 1}\frac{u_i}{i}\right) + \exp\left(\sum_{i \text{ even}}\frac{u_i}{i} + \sum_{i \text{ odd}}\frac{v_i}{i}\right)$$
$$- \left(1 + u + \frac{u^2}{2} + \frac{u_2}{2}\right) - \left(1 + v + \frac{v^2}{2} + \frac{u_2}{2}\right)\Bigg\}$$
$$- \frac{x^2}{2}\Bigg\{ (1+\mu)\exp\left(\sum_{i\geq 1}\frac{pqu_i}{i}\right) + (1+\nu)\exp\left(\sum_{i \text{ even}}\frac{pqu_i}{i} + \sum_{i \text{ odd}}\frac{stv_i}{i}\right)$$
$$- \left\{ (1+\mu)(1+pqu) + \frac{pqu^2}{2} + \frac{pqu_2}{2} + (1+\nu)(1+stv) + \frac{stv^2}{2} + \frac{pqu_2}{2} \right\}\Bigg\}.$$

Applying (3.87) and (3.88), let $T1$ come from $E1(T1, pqu)$, $T2$ come from $E2(T2, pqu, stv)$, $T3$ come from $E1(T3, u)$ and $T4$ come from $E2(T4, u, v)$.

Then, $$A1(x,y) = \frac{x^2}{2}\Bigg\{ T3 + T4 - \left(1 + u + \frac{u^2}{2}\right) - \left(1 + v + \frac{v^2}{2}\right) - u_2$$
$$- (1+\mu)T1 - (1+\nu)T2 + (1+\mu)(1+pqu) + (1+\nu)(1+stv)$$
$$+ \frac{pqu^2}{2} + \frac{stv^2}{2} + pqu_2 \Bigg\}$$
$$= \frac{x^2}{2}\Bigg\{ (pqu - T1)(1+\mu) + (stv - T2)(1+\nu)$$
$$+ \frac{1}{2}(pqu^2 + stv^2 - u^2 - v^2) - 2 + 2$$
$$+ (T3 + T4 + \mu + \nu - u - v) + pqu_2 - u_2 \Bigg\}.$$

Thus, for $n \geq 2$,

$$A1_n(y) = \frac{1}{2}\Bigg\{ \sum_{2\leq k\leq n} \Big\{ (pqu_{n-k} - T1_{n-k})\mu_{k-2} + (stv_{n-k} - T2_{n-k})\nu_{k-2} \Big\}$$
$$+ \frac{1}{2} \sum_{2\leq k\leq n} \Big\{ pqu_{n-k}pqu_{k-2} + stv_{n-k}stv_{k-2} - u_{n-k}u_{k-2} - v_{n-k}v_{k-2} \Big\}$$
$$+ T3_{n-2} + T4_{n-2} + \mu_{n-2} + \nu_{n-2} - u_{n-2} - v_{n-2}$$
$$+ \begin{Bmatrix} pqu_m(y^2) - u_m(y^2), & \text{if } 2|n \text{ for } m = \left(\frac{n-2}{2}\right) \\ 0, & \text{otherwise} \end{Bmatrix} \Bigg\}. \tag{4.22}$$

§4.3.2 $A2(x, y)$

$A2(x, y) = M[x, y + \delta - q, y + \epsilon - t] - M[x, \delta - p, \epsilon - s].$

Let ydq to denote $y + \delta - q$, yet to denote $y + \epsilon - t$,

dp to denote $\delta - p$ and es to denote $\epsilon - s$.

Then using (3.34), gives

$$A2(x, y) = \frac{1}{2}\Bigg\{ \sum_{d\geq 1} \frac{\phi(d)}{d} a_d \left[\log\left(\frac{1 - x\cdot dp}{1 - x \cdot ydq} \right) \right] \Bigg\}$$
$$+ \frac{x}{2}(dp - ydq) + \frac{x^2}{4}(dp^2 - ydq^2) + \frac{x^2}{4}(dp_2 - ydq_2)$$
$$+ \frac{1}{4(1 - x^2 \cdot ydq_2)} \Big\{ 2x^3 \cdot yet \cdot ydq_2 + x^4 \cdot ydq_2^2 + x^4 yet^2 \cdot ydq_2 \Big\}$$
$$- \frac{1}{4(1 - x^2 \cdot dp_2)} \Big\{ 2x^3 \cdot es \cdot dp_2 + x^4 \cdot dp_2^2 + x^4 \cdot es^2 \cdot dp_2 \Big\}.$$

Applying (3.96) and (3.97), let $T9$ come from $V(T9, dp, ydq)$,

$T10$ come from $W(T10, ydq, yet)$, $T11$ come from $W(T11, dp, es)$,

$$A2(x, y) = \frac{1}{2}\Bigg\{ \sum_{n\geq 1} \left(\sum_{i\cdot d = n} \frac{\phi(d)}{d} T9_i(y^d) \right) x^n \Bigg\} + \frac{x}{2}(dp - ydq) + \frac{x^2}{4}(dp^2 - ydq^2)$$
$$+ \frac{x^2}{4}(dp_2 - ydq_2) + T10(x, y) - T11(x, y).$$

Then for $n \geq 2$,

$$A2_n(y) = \sum_{i\cdot d=n} \frac{\phi(d)}{2d} T9_i(y^d) + \frac{1}{2}(dp_{n-1} - ydq_{n-1})$$
$$+ \frac{1}{4} \sum_{2\le k\le n} (dp_{k-2} \cdot dp_{n-k} - ydq_{k-2} \cdot ydq_{n-k}) + T10_n(y) - T11_n(y)$$
$$+ \frac{1}{4} \begin{Bmatrix} dp_m(y^2) - ydq_m(y^2), & \text{if } 2|n \text{ for } m = (\frac{n-2}{2}) \\ 0, & \text{otherwise} \end{Bmatrix} \tag{4.23}$$

with $A2_0(y) = 0$ and

$$A2_1(y) = \frac{\phi(1)}{2} T9_1(y) + \frac{1}{2}(dp_0 - ydq_0)$$
$$= \frac{\phi(1)}{2}(-dp_0 + ydq_0) + \frac{1}{2}(dp_0 - ydq_0) = \frac{1}{2}y - \frac{1}{2}y = 0.$$

§4.3.3 $A3(x, y)$

$$A3(x,y) = \vec{T}[x,\delta,\epsilon] \diamond \vec{N}_1[x, pqu, stv] - \vec{T}[x,\delta,\epsilon] \diamond \vec{N}[x,u,v].$$

Applying the definition of $\diamond$ from (3.73),

$$A3(x,y) = \frac{x^2}{2} T^+[x,\delta,\epsilon] N_1^+[x, pqu, stv] + \frac{x^2}{2} T^-[x,\delta,\epsilon] N_1^-[x, pqu, stv]$$
$$- \frac{x^2}{2} T^+[x,\delta,\epsilon] N^+[x,u,v] - \frac{x^2}{2} T^-[x,\delta,\epsilon] N^-[x,u,v].$$

Applying (4.15), (4.18), (3.47), (3.48), (3.44) and (3.45), gives

$$A3(x,y) = \frac{x^2}{2} \Bigg\{ (u-q) \Bigg\{ (1+\mu) \exp\left(\sum_{i\ge 1} \frac{pqu_i}{i} \right) - (1+\mu) - pqu \Bigg\}$$
$$+ (v-t) \Bigg\{ (1+\nu) \exp\left(\sum_{i \text{ even}} \frac{pqu_i}{i} + \sum_{i \text{ odd}} \frac{stv_i}{i} \right) - (1+\nu) - stv \Bigg\}$$
$$- (u-q) \Bigg\{ \exp\left(\sum_{i\ge 1} \frac{u_i}{i} \right) - (1+u) \Bigg\}$$
$$- (v-t) \Bigg\{ \exp\left(\sum_{i \text{ even}} \frac{u_i}{i} + \sum_{i \text{ odd}} \frac{v_i}{i} \right) - (1+v) \Bigg\} \Bigg\}.$$

As before, let $T1$ come from $E1(T1, pqu)$, $T2$ come from $E2(T2, pqu, stv)$, $T3$ come from $E1(T3, u)$, $T4$ come from $E2(T4, u, v)$. Then,

$$A3(x,y) = \frac{x^2}{2}\Bigg\{(u-q)\Big\{(1+\mu)T1 - (1+\mu) - pqu\Big\}$$
$$+ (v-t)\Big\{(1+\nu)T2 - (1+\nu) - stv\Big\}$$
$$- (u-q)\Big\{T3 - (1+u)\Big\} - (v-t)\Big\{T4 - (1+v)\Big\}\Bigg\}$$
$$= \frac{x^2}{2}\Bigg\{(u-q)\Big\{(1+\mu)T1 - (1+\mu) - pqu - T3 + 1 + u\Big\}$$
$$+ (v-t)\Big\{(1+\nu)T2 - (1+\nu) - stv - T4 + 1 + v\Big\}\Bigg\}.$$

Thus, for $n \geq 2$,

$$A3_n(y) = \frac{1}{2}\Bigg\{\sum_{0\leq k\leq n-2}\Bigg\{\Big(u_{n-k-2}(y) - q_{n-k-2}(y)\Big)$$
$$* \Bigg\{\Big(\sum_{0\leq r\leq k}\mu_r(y)T1_{k-r}(y)\Big) - \mu_k(y) - pqu_k(y) - T3_k(y) + u_k(y)\Bigg\}$$
$$+ \Big(v_{n-k-2}(y) - t_{n-k-2}(y)\Big)$$
$$* \Bigg\{\Big(\sum_{0\leq r\leq k}\nu_r(y)T2_{k-r}(y)\Big) - \nu_k(y) - stv_k(y) - T4_k(y) + v_k(y)\Bigg\}\Bigg\}$$
$$+ u_{n-2}(y) - q_{n-2}(y) + v_{n-2}(y) - t_{n-2}(y)\Bigg\}. \tag{4.24}$$

§4.3.4 $A4(x,y)$

$$A4(x,y) = \vec{N}_1[x, pqu, stv] \diamond \vec{M}[x, dp, es] - \vec{N}[x,u,v] \diamond \vec{M}[x, ydq, yet].$$

Applying the definition of $\diamond$ from (3.73),

$$A4(x,y) = \frac{x^2}{2}N_1^+[x, pqu, stv]M^+[x, dp, es] + \frac{x^2}{2}N_1^-[x, pqu, stv]M^-[x, dp, es]$$
$$- \frac{x^2}{2}N^+[x,u,v]M^+[x, ydq, yet] - \frac{x^2}{2}N^-[x,u,v]M^-[x, ydq, yet].$$

Applying (3.47), (3.48), (3.44) and (3.45) gives

$$A4(x,y) = \frac{x^2}{2}\Bigg\{\Bigg\{(1+\mu)\exp\Big(\sum_{i\geq 1}\frac{pqu_i}{i}\Big) - (1+\mu) - pqu\Bigg\}M^+[x, dp, es]$$

$$+\left\{(1+\nu)\exp\left(\sum_{i\text{ even}}\frac{pqu_i}{i}+\sum_{i\text{ odd}}\frac{stv_i}{i}\right)-(1+\nu)-stv\right\}M^-[x,dp,es]$$

$$-\left\{\exp\left(\sum_{i\geq 1}\frac{u_i}{i}\right)-(1+u)\right\}M^+[x,ydq,yet]$$

$$-\left\{\exp\left(\sum_{i\text{ even}}\frac{u_i}{i}+\sum_{i\text{ odd}}\frac{v_i}{i}\right)-(1+v)\right\}M^-[x,ydq,yet]\Bigg\}.$$

As before, let $T1$ come from $E1(T1,pqu)$, $T2$ come from $E2(T2,pqu,stv)$, $T3$ come from $E1(T3,u)$, $T4$ come from $E2(T4,u,v)$.

Also, using (3.89) and (3.90), let $T5$ come from $MP(T5,dp)$, $T6$ come from $MP(T6,ydq)$, $T7$ come from $MN(T7,dp,es)$; $T8$ come from $MN(T8,ydq,yet)$.

$$A4(x,y)=\frac{x^2}{2}\Bigg\{\left\{(1+\mu)T1-(1+\mu)-pqu\right\}T5$$

$$+\left\{(1+\nu)T2-(1+\nu)-stv\right\}T7$$

$$-\left\{T3-(1+u)\right\}T6-\left\{T4-(1+v)\right\}T8\Bigg\}.$$

Thus, for $n\geq 2$,

$$A4_n(y)=\frac{1}{2}\Bigg\{\sum_{0\leq k\leq n-2}\Bigg\{\left\{\sum_{0\leq r\leq k}\mu_r(y)\cdot T1_{k-r}(y)-\mu_k(y)-pqu_k(y)\right\}*T5_{n-k-2}(y)$$

$$+\left\{\sum_{0\leq r\leq k}\nu_r(y)\cdot T2_{k-r}(y)-\nu_k(y)-stv_k(y)\right\}*T7_{n-k-2}(y)$$

$$-(T3_{n-k-2}(y)-u_{n-k-2}(y))*T6_k(y)$$

$$-(T4_{n-k-2}(y)-v_{n-k-2}(y))*T8_k(y)\Bigg\}+T6_{n-2}(y)+T8_{n-2}(y)\Bigg\}. \tag{4.25}$$

§4.3.5 $A5(x,y)$

$$A5(x,y)=\overrightarrow{T}[x,\delta,\epsilon]\diamond\overrightarrow{M}[x,dp,es]-\overrightarrow{T}[x,\delta,\epsilon]\diamond\overrightarrow{M}[x,ydq,yet].$$

Applying the definition of $\diamond$ from (3.73),

$$A5(x,y) = \frac{x^2}{2}T^+[x,\delta,\epsilon]M^+[x,dp,es] + \frac{x^2}{2}T^-[x,\delta,\epsilon]M^-[x,dp,es]$$
$$- \frac{x^2}{2}T^+[x,\delta,\epsilon]M^+[x,ydq,yet] - \frac{x^2}{2}T^-[x,\delta,\epsilon]M^-[x,ydq,yet].$$

Again, using (4.15) and (4.18), gives

$$A5(x,y) = \frac{x^2}{2}\Bigg\{(u-q)\Big\{M^+[x,dp,es] - M^+[x,ydq,yet]\Big\}$$
$$+ (v-t)\Big\{M^-[x,dp,es] - M^-[x,yeq,yet]\Big\}\Bigg\}.$$

As before let $T5$ come from $MP(T5,dp)$, $T6$ come from $MP(T6,ydq)$,

$T7$ come from $MN(T7,dp,es)$, $T8$ come from $M1P(T8,ydq,yet)$.

$$A5(x,y) = \frac{x^2}{2}\Big\{(u-q)(T5-T6) + (v-t)(T7-T8)\Big\}.$$

Thus, for $n \geq 2$,

$$A5_n(y) = \frac{1}{2}\Bigg\{\sum_{2\leq k\leq n}\Big\{(u_{k-2}(y) - q_{k-2}(y)) * (T5_{n-k}(y) - T6_{n-k}(y))$$
$$+ (v_{k-2}(y) - t_{k-2}(y)) * (T7_{n-k}(y) - T8_{n-k}(y))\Big\}\Bigg\}. \qquad (4.26)$$

§4.4 Summary of the counting algorithm

In the computations below, power series are computed term by term. At any given time order n term of each of the power series (except in the case of $\eta(x,y)$, for which order $(n+1)$ term is computed) is computed one after the other in the sequence given here, utilizing all its low order terms and all the available terms of the power series computed before it. Steps are numbered for easy perusal.

1. From (3.83), compute, for $n \geq 2$, $p_n(y) = \sum_{1\leq k\leq n} \delta_{k-1}(y)\Big\{\delta_{n-k}(y) - p_{n-k}(y)\Big\}$.

Then from (4.1), compute, for $n \geq 2$,

$$q_n(y) = y\Big\{2\delta_{n-1}(y) - q_{n-1}(y)\Big\} + \sum_{1\leq k\leq n} \delta_{k-1}(y)\Big\{\delta_{n-k}(y) - q_{n-k}(y)\Big\}.$$

Then from (4.8), compute, for $n \geq 2$, nth term of $\mu(x,y)$ using

these in $E1(1+\mu, q-p)$.

2. From (3.84), compute, for $n \geq 2$, $s_n(y)$ using,

$$s_{2n}(y) = \sum_{1 \leq k \leq n} \left\{ \delta_{k-1}(y^2) - p_{k-1}(y^2) \right\} \epsilon_{2n-2k}(y) \text{ and}$$

$$s_{2n+1}(y) = \delta_n(y^2) - p_n(y^2) + \sum_{1 \leq k \leq n} \left\{ \delta_{k-1}(y^2) - p_{k-1}(y^2) \right\} \epsilon_{2n+1-2k}(y).$$

From (4.2) compute, for $n \geq 2$, $t_n(y)$ using

$$t_{2n}(y) = \epsilon_{2n-2}(y)y^2 + y\left\{ \delta_{n-1}(y^2) - q_{n-1}(y^2) \right\}$$
$$+ \sum_{1 \leq k \leq n} \left\{ \delta_{k-1}(y^2) - q_{k-1}(y^2) \right\} \epsilon_{2n-2k}(y) \text{ and}$$

$$t_{2n+1}(y) = \epsilon_{2n-1}(y)y^2 + \delta_n(y^2) - q_n(y^2)$$
$$+ \sum_{1 \leq k \leq n} \left\{ \delta_{k-1}(y^2) - q_{k-1}(y^2) \right\} \epsilon_{2n+1-2k}.$$

Then from (4.12), compute, for $n \geq 2$, nth term of $\nu(x,y)$ using

these in $E2(1+\nu, q-p, t-s)$.

3. From (3.80), $\eta(x,y) - x = xF(x,y) - a_1\left(\frac{\partial \boldsymbol{K}}{\partial a_1} - 1\right)[\eta, 1+\mu, 1+\nu]$,

where $F(x,y) = (\boldsymbol{K}-1)[\eta, 1+\mu, 1+\nu]$. We compute the nth term $F(x,y)$ by

doing the composition on the terms generated for $\boldsymbol{K}$.

Let $\boldsymbol{K}_a$ denote $a_1\left(\frac{\partial \boldsymbol{K}}{\partial a_1} - 1\right)$. Compute the power series

$R(x,y) = \boldsymbol{K}_a[\eta, 1+\mu, 1+\nu]$ by doing the composition on the terms

generated for $\boldsymbol{K}_a$. Compute, for $n \geq 2$, the $(n+1)$th term

of $\eta(x,y)$, using $\eta_{n+1}(y) = F_n(y) - R_{n+1}(y)$.

4. From (3.78),

$$\delta(x,y) + 1 = \left(\frac{\eta(x,y)}{x}\right)^2 \frac{2}{a_1^2}\left(b_1 \frac{\partial \boldsymbol{K}}{\partial b_1}\right)[\eta, 1+\mu, 1+\nu] - (1+\delta(x,y))F(x,y).$$

Let $\boldsymbol{K}_b$ denote $\dfrac{2}{a_1^2}\left(b_1\dfrac{\partial \boldsymbol{K}}{\partial b_1}\right)$. Compute the power series

$S(x,y) = \boldsymbol{K}_b[\eta, 1+\mu, 1+\nu]$ by composing on the terms generated for $\boldsymbol{K}_b$.

Then $\delta(x,y)+1 = \left(\dfrac{\eta(x,y)}{x}\right)^2 S(x,y) - (1+\delta(x,y))F(x,y)$.

Let $E(x,y) = \sum_{i\geq 0} E_i(y)x^i = \left(\dfrac{\eta(x,y)}{x}\right)$. Then,

$$\delta_n(y) = \sum_{0\leq k\leq n} S_{n-k}(y)\left(\sum_{0\leq r\leq k} E_r(y)E_{k-r}(y)\right)$$
$$- F_n(y) - \sum_{0\leq k\leq n} \delta_k(y)F_{n-k}(y).$$

Since $E_0(y) = 1$ and $F_0(y) = \delta_0(y) = 0$, we have,

$$\delta_n(y) = \sum_{1\leq k\leq n} S_{n-k}(y)\left(\sum_{0\leq r\leq k} E_r(y)E_{k-r}(y)\right)$$
$$+ S_n(y) - F_n(y) - \sum_{1\leq k\leq n-1} \delta_k(y)F_{n-k}(y).$$

This can be written as

$$\delta_n(y) = \sum_{1\leq k\leq n}\left\{S_{n-k}(y)\left(\sum_{0\leq r\leq k} E_r(y)E_{k-r}(y)\right) - \delta_k(y)F_{n-k}(y)\right\} + S_n(y) - F_n(y).$$

5. From (3.79),

$$\epsilon(x,y)+1 = \left(\frac{\eta(x^2,y^2)}{x^2}\right)\frac{2}{a_2}\left(c_1\frac{\partial \boldsymbol{K}}{\partial c_1}\right)[\eta, 1+\mu, 1+\nu] - (1+\epsilon(x,y))F(x,y).$$

Let $\boldsymbol{K}_c$ denote $\dfrac{2}{a_2}\left(c_1\dfrac{\partial \boldsymbol{K}}{\partial c_1}\right)$. Compute the power series

$T(x,y) = \boldsymbol{K}_c[\eta, 1+\mu, 1+\nu]$ by composing on the terms generated for $\boldsymbol{K}_c$.

Then, $\epsilon(x,y)+1 = \left(\dfrac{\eta(x^2,y^2)}{x^2}\right)T(x,y) - (1+\epsilon(x,y))F(x,y)$. Then,

$$\epsilon_n(y) = \sum_{1\leq k\leq \lfloor \frac{n}{2}\rfloor} E_k(y^2)T_{n-2k}(y) - \sum_{1\leq k\leq n-1}\epsilon_k(y)F_{n-k} + T_n(y) - F_n(y).$$

6. Let $B(x,y) = \boldsymbol{B}[x,\mu,\nu]$. Then, from (3.82), for $n\geq 2$,

$$B_n(y) = \sum_{k \cdot i = n} \frac{\mu(k)}{k} f_i(y^k) - \sum_{k \cdot i = n-1} \frac{\mu(k)}{k} e_i(y^k).$$

7. From (4.3), $u(x,y) = \sum_{i \geq 1} u_i(y) x^i = \sum_{i \geq 1} \frac{\mu(i)}{i} a_i[\log(1 + \delta(x,y))]$.

Let $w(x,y) = \log(1 + \delta(x,y))$, so that, from (4.16) $u_n(y) = \sum_{i \cdot j = n} \frac{\mu(i)}{i} w_j(y^i)$.

8. From (4.4),

$$v(x,y) = \sum_{i \geq 1} v_i(y) x^i = \sum_{k \text{ odd}} \frac{\mu(k)}{k} a_k[\log(1 + \epsilon(x,y))] - \sum_{m = 2^{1+e},\, e \geq 0} \frac{a_m}{m} [u(x,y)].$$

Let $h(x,y) = \log(1 + \epsilon(x,y))$, so that, from (4.19)

$$v_n(y) = \sum_{i \cdot j = n,\, i \text{ odd}} \frac{\mu(i)}{i} h_j(y^i) - \sum_{2|n,\, m \cdot i \cdot j = n} \frac{1}{m} u_i(y^m), \text{where, } m = 2^{1+e}, \text{for, } e \geq 0.$$

9. Having computed the essential power series, we now compute some auxiliary power series that are used in the remaining computation. For all $n \geq 2$,

$pqu_n(y) = p_n(y) - q_n(y) + u_n(y)$,

$stv_n(y) = s_n(y) - t_n(y) + v_n(y)$,

$dp_n(y) = \delta_n(y) - p_n(y)$,

$es_n(y) = \epsilon_n(y) - s_n(y)$,

$ydq_n(y) = \delta_n(y) - q_n(y)$ and

$yet_n(y) = \epsilon_n(y) - t_n(y)$.

Now, we pass on these into routines $E1$, $E2$, MP, MN, V, W to compute the nth term of power series $T1(x,y)$ through $T11(x,y)$.

$E1(T1, pqu)$, $E2(T2, pqu, stv)$, $E1(T3, u)$, $E2(T4, u, v)$,

$MP(T5, dp)$, $MP(T6, ydq)$, $MN(T7, dp, es)$, $MN(T8, ydq, yet)$,

$V(T9, dp, ydq)$, $W(T10, ydq, yet)$, and $W(T11, dp, es)$.

10. From (4.22), for $n \geq 2$,

$$A1_n(y) = \frac{1}{2}\Bigg\{ \sum_{2\le k\le n} \Big\{ (pqu_{n-k} - T1_{n-k})\mu_{k-2} + (stv_{n-k} - T2_{n-k})\nu_{k-2} \Big\}$$

$$+ \frac{1}{2} \sum_{2\le k\le n} \Big\{ pqu_{n-k}pqu_{k-2} + stv_{n-k}stv_{k-2} - u_{n-k}u_{k-2} - v_{n-k}v_{k-2} \Big\}$$

$$+ T3_{n-2} + T4_{n-2} + \mu_{n-2} + \nu_{n-2} - u_{n-2} - v_{n-2}$$

$$+ \begin{Bmatrix} pqu_m(y^2) - u_m(y^2), & \text{if } 2|n \text{ for } m = \left(\frac{n-2}{2}\right) \\ 0, & \text{otherwise} \end{Bmatrix} \Bigg\}.$$

11. From (4.23), for $n \geq 2$,

$$A2_n(y) = \sum_{i\cdot d=n} \frac{\phi(d)}{2d} T9_i(y^d) + \frac{1}{2}(dp_{n-1} - ydq_{n-1})$$

$$+ \frac{1}{4} \sum_{2\le k\le n} (dp_{k-2} \cdot dp_{n-k} - ydq_{k-2} \cdot ydq_{n-k}) + T10_n(y) - T11_n(y)$$

$$+ \frac{1}{4} \begin{Bmatrix} dp_m(y^2) - ydq_m(y^2) & \text{if } 2|n \text{ for } m = \left(\frac{n-2}{2}\right) \\ 0 & \text{otherwise} \end{Bmatrix}$$

with $A2_0(y) = 0 = A2_1(y)$.

12. From (4.24), for $n \geq 2$,

$$A3_n(y) = \frac{1}{2}\Bigg\{ \sum_{0\le k\le n-2} \Bigg\{ \Big(u_{n-k-2}(y) - q_{n-k-2}(y) \Big)$$

$$* \Bigg\{ \Big(\sum_{0\le r\le k} \mu_r(y) T1_{k-r}(y) \Big) - \mu_k(y) - pqu_k(y) - T3_k(y) + u_k(y) \Bigg\}$$

$$+ \Big(v_{n-k-2}(y) - t_{n-k-2}(y) \Big)$$

$$* \Bigg\{ \Big(\sum_{0\le r\le k} \nu_r(y) T2_{k-r}(y) \Big) - \nu_k(y) - stv_k(y) - T4_k(y) + v_k(y) \Bigg\} \Bigg\}$$

$$+ u_{n-2}(y) - q_{n-2}(y) + v_{n-2}(y) - t_{n-2}(y) \Bigg\}.$$

13. From (4.25), for $n \geq 2$,

$$A4_n(y) = \frac{1}{2}\Bigg\{ \sum_{0\le k\le n-2} \Bigg\{ \Bigg\{ \sum_{0\le r\le k} \mu_r(y) \cdot T1_{k-r}(y) - \mu_k(y) - pqu_k(y) \Bigg\} * T5_{n-k-2}(y)$$

$$+\left\{\sum_{0\le r\le k}\nu_r(y)\cdot T2_{k-r}(y)-\nu_k(y)-stv_k(y)\right\}*T7_{n-k-2}(y)$$

$$-\,(T3_{n-k-2}(y)-u_{n-k-2}(y))*T6_k(y)$$

$$-\,(T4_{n-k-2}(y)-v_{n-k-2}(y))*T8_k(y)\Big\}+T6_{n-2}(y)+T8_{n-2}(y)\Big\}.$$

14. From (4.26), for $n\ge 2$,

$$A5_n(y)=\frac{1}{2}\Big\{\sum_{2\le k\le n}\Big\{(u_{k-2}(y)-q_{k-2}(y))*(T5_{n-k}(y)-T6_{n-k}(y))$$

$$+\,(v_{k-2}(y)-t_{k-2}(y))*(T7_{n-k}(y)-T8_{n-k}(y))\Big\}\Big\}.$$

15. From (4.21), for $n\ge 2$,

$$\hat{B}_n(y)=B_n(y)+A1_n(y)+A2_n(y)+A3_n(y)+A4_n(y)+A5_n(y)$$

$$-\,\frac{1}{2}\Big\{\mu_{n-2}(y)+\nu_{n-2}(y)\Big\}.$$

(4.27)

CHAPTER 5

Counting unlabeled 3–edge–connected blocks

§5.1 Initial derivations

In this section, we first derive $q(x,y)$ and $t(x,y)$. Then using $p(x,y)$ and $s(x,y)$ derived in 3rd chapter, we derive equations to express $\mu(x,y)$ in terms of $p(x,y)$ and $q(x,y)$ and $\nu(x,y)$ in terms of $s(x,y)$ and $t(x,y)$.

§5.1.1 $\tilde{S}^+[x,y,y] = q(x,y)$

Recall from (3.63), $\tilde{S}^+ = M_1^+[a_1, \tilde{D}^+ - \tilde{S}^+, \tilde{D}^- - \tilde{S}^-]$.

Substituting for M_1^+ from (3.41) gives,

$$\tilde{S}^+ = \left\{\frac{a_1b_1^2 - a_1^2b_1^3 + 2a_1b_1^*b_1 - a_1^2b_1^*b_1^2}{(1-a_1b_1)^2}\right\}\left[a_1, \tilde{D}^+ - \tilde{S}^+, \tilde{D}^- - \tilde{S}^-\right].$$

Expanding this and rearranging gives,

$$\begin{aligned}\tilde{S}^+ &= 2a_1\tilde{S}^+(\tilde{D}^+ - \tilde{S}^+) - a_1^2\tilde{S}^+(\tilde{D}^+ - \tilde{S}^+)^2 + a_1(\tilde{D}^+ - \tilde{S}^+)^2 \\ &- a_1^2(\tilde{D}^+ - \tilde{S}^+)^3 + 2a_1b_1(\tilde{D}^+ - \tilde{S}^+) - a_1^2b_1(\tilde{D}^+ - \tilde{S}^+)^2 \\ &= a_1(\tilde{D}^+ - \tilde{S}^+)(2\tilde{S}^+ + \tilde{D}^+ - \tilde{S}^+) - a_1^2(\tilde{D}^+ - \tilde{S}^+)^2(\tilde{S}^+ + \tilde{D}^+ - \tilde{S}^+) \\ &+ 2a_1b_1(\tilde{D}^+ - \tilde{S}^+) - a_1^2b_1(\tilde{D}^+ - \tilde{S}^+)^2 \\ &= a_1(\tilde{D}^+ - \tilde{S}^+)(2b_1 + \tilde{D}^+ + \tilde{S}^+) - a_1^2(\tilde{D}^+ - \tilde{S}^+)^2(\tilde{D}^+ + b_1).\end{aligned}$$

Let $\tilde{S}^+[x,y,y] = q(x,y)$. Since, $\tilde{D}^+[x,y,y] = \delta(x,y) - y$, we have,

$$\begin{aligned}q(x,y) &= x(\delta - y - q)(2y + \delta - y + q) - x^2(\delta - q - y)^2\delta \\ &= x(\delta - y - q)(\delta + y + q) - x^2(\delta - q - y)^2\delta\end{aligned}$$

$$= x(\delta^2 - (y+q)^2) - x^2\delta(\delta - y - q)^2$$

$$= x\delta^2 - x(y^2 + 2yq + q^2) - x^2\delta(y^2 - 2y(\delta - q) + (\delta - q)^2).$$

Then $q_0(y) = 0$.

$$q(x,y) = x\left(\sum_{i\geq 0}\delta_i(y)x^i\right)^2 - x\left\{y^2 + 2y\sum_{i\geq 0} q_i(y)x^i + \left(\sum_{i\geq 0} q_i(y)x^i\right)^2\right\}$$

$$- x^2\left(\sum_{i\geq 0}\delta_i(y)x^i\right)\left\{y^2 - 2y\sum_{i\geq 0}(\delta_i(y) - q_i(y))x^i + \left(\sum_{i\geq 0}(\delta_i(y) - q_i(y))x^i\right)^2\right\}.$$

Then, $q_1(y) = \delta_0(y)^2 - y^2 - 2yq_0(y) - q_0(y)q_0(y) = \delta_0(y)^2 - y^2$

and for $n \geq 2$,

$$q_n(y) = \sum_{1\leq k\leq n}\delta_{k-1}(y)\delta_{n-k}(y) - 2yq_{n-1}(y) - \sum_{1\leq k\leq n} q_{k-1}(y)q_{n-k}(y) - y^2\delta_{n-2}(y)$$

$$+ 2y\sum_{2\leq k\leq n}\delta_{k-2}(y)\Big\{\delta_{n-k}(y) - q_{n-k}(y)\Big\}$$

$$- \sum_{2\leq k\leq n}\delta_{k-2}(y)\sum_{0\leq r\leq n-k}\Big\{\delta_r(y) - q_r(y)\Big\}\Big\{\delta_{n-k-r}(y) - q_{n-k-r}(y)\Big\}$$

$$= -y^2\delta_{n-2}(y) - 2yq_{n-1}(y) + \sum_{1\leq k\leq n}\Big\{\delta_{k-1}(y)\delta_{n-k}(y) - q_{k-1}(y)q_{n-k}(y)\Big\}$$

$$+ \sum_{2\leq k\leq n}\delta_{k-2}(y)\left\{2y\Big\{\delta_{n-k}(y) - q_{n-k}(y)\Big\}\right.$$

$$\left. - \sum_{0\leq r\leq n-k}\Big\{\delta_r(y) - q_r(y)\Big\}\Big\{\delta_{n-k-r}(y) - q_{n-k-r}(y)\Big\}\right\}$$

$$= -y^2\delta_{n-2}(y) - 2yq_{n-1}(y) + \delta_0(y)\delta_{n-1}(y) - q_0(y)q_{n-1}(y)$$

$$+ \sum_{2\leq k\leq n}\left\{\Big\{\delta_{k-1}(y)\delta_{n-k}(y) - q_{k-1}(y)q_{n-k}(y)\Big\}\right.$$

$$+ \delta_{k-2}(y)\left\{2y\Big\{\delta_{n-k}(y) - q_{n-k}(y)\Big\}\right.$$

$$\left.\left. - \sum_{0\leq r\leq n-k}\Big\{\delta_r(y) - q_r(y)\Big\}\Big\{\delta_{n-k-r}(y) - q_{n-k-r}(y)\Big\}\right\}\right\}. \tag{5.1}$$

§5.1.2 $\tilde{S}^-[x,y,y] = t(x,y)$

Also recall (3.64), $\tilde{S}^- = M_1^-[a_1, \tilde{D}^+ - \tilde{S}^+, \tilde{D}^- - \tilde{S}^-]$.

Substituting for M_1^- from (3.42) gives,

$$\tilde{S}^- = \left\{\frac{a_1 b_2 + a_2 b_2 (c_1 + c_1^*)}{1 - a_2 b_2}\right\}\left[a_1, \tilde{D}^+ - \tilde{S}^+, \tilde{D}^- - \tilde{S}^-\right].$$

Expanding,

$$\tilde{S}^-\left\{1 - a_2(\tilde{D}_2^+ - \tilde{S}_2^+)\right\} = a_1(\tilde{D}_2^+ - \tilde{S}_2^+) + a_2(\tilde{D}_2^+ - \tilde{S}_2^+)(\tilde{D}^- - \tilde{S}^- + c_1).$$

i.e., $\tilde{S}^- = a_1(\tilde{D}_2^+ - \tilde{S}_2^+) + a_2(\tilde{D}_2^+ - \tilde{S}_2^+)(\tilde{S}^- + \tilde{D}^- - \tilde{S}^- + c_1)$

$$= a_1(\tilde{D}_2^+ - \tilde{S}_2^+) + a_2(\tilde{D}_2^+ - \tilde{S}_2^+)(\tilde{D}^- + c_1).$$

Let $\tilde{S}^-[x,y,y] = t(x,y)$. Since $\tilde{D}^-[x,y,y] = \epsilon(x,y) - y$,

$t(x,y) = x(\delta_2 - y^2 - q_2) + x^2(\delta_2 - y^2 - q_2)\epsilon(x,y)$. Then, $t_0(y) = 0$.

$$t(x,y) = x\left\{\sum_{i\geq 0}(\delta_i(y^2) - q_i(y^2))x^{2i} - y^2\right\}$$
$$+ x^2\left(\sum_{i\geq 0}\epsilon_i(y)x^i\right)\left\{\sum_{i\geq 0}(\delta_i(y^2) - q_i(y^2))x^{2i} - y^2\right\}.$$

Then, $t_1(y) = \delta_0(y^2) - q_0(y^2) - y^2 = \delta_0(y^2) - y^2$

and for $n \geq 1$,

$$t_{2n}(y) = -y^2\epsilon_{2n-2}(y) + \sum_{1\leq k\leq n}\left\{\delta_{k-1}(y^2) - q_{k-1}(y^2)\right\}\epsilon_{2(n-k)}(y) \text{ and}$$

$$t_{2n+1}(y) = \delta_n(y^2) - q_n(y^2) - y^2\epsilon_{2n-1}(y) + \sum_{1\leq k\leq n}\left\{\delta_{k-1}(y^2) - q_{k-1}(y^2)\right\}\epsilon_{2(n-k)+1}(y).$$
$$(5.2)$$

§5.1.3 $\tilde{D}^+ - \tilde{P}^+$

Recall (3.65), $\tilde{P}^+ = N_1^+[a_1, \tilde{D}^+ - \tilde{P}^+, \tilde{D}^- - \tilde{P}^-]$.

Substituting the expression for N_1^+ from (3.47) gives

$$\tilde{P}^+ = \left\{(1 + b_1^*)\exp\left(\sum_{i\geq 1}\frac{b_i}{i}\right) - (1 + b_1^*) - b_1\right\}\left[a_1, \tilde{D}^+ - \tilde{P}^+, \tilde{D}^- - \tilde{P}^-\right].$$

On expansion, $\tilde{P}^+ = (1+b_1)\exp\left(\sum_{i\geq 1}\frac{b_i}{i}[\tilde{D}^+ - \tilde{P}^+]\right) - (1+b_1) - (\tilde{D}^+ - \tilde{P}^+)$.

Then, $\exp\left(\sum_{i\geq 1}\frac{b_i}{i}[\tilde{D}^+ - \tilde{P}^+]\right) = \dfrac{\tilde{P}^+ + (1+b_1) + \tilde{D}^+ - \tilde{P}^+}{1+b_1} = \dfrac{1+b_1+\tilde{D}^+}{1+b_1}$.

Then by Möbius inversion (3.2), $(\tilde{D}^+ - \tilde{P}^+) = \sum_{i\geq 1}\frac{\mu(i)}{i}a_i\left[\log\left(\dfrac{1+b_1+\tilde{D}^+}{1+b_1}\right)\right]$.

This on composition with $[x, y, y]$, gives

$$(\tilde{D}^+ - \tilde{P}^+)[x,y,y] = \sum_{i\geq 1}\frac{\mu(i)}{i}a_i\left[\log\left(\frac{1+\delta(x,y)}{1+y}\right)\right]. \tag{5.3}$$

§5.1.4 $\tilde{D}^- - \tilde{P}^-$

Recall from (3.66), $\tilde{P}^- = N_1^-[a_1, \tilde{D}^+ - \tilde{P}^+, \tilde{D}^- - \tilde{P}^-]$.

Substituting the expression for N_1^- from (3.48) gives,

$$\tilde{P}^- = \left\{(1+c_1^*)\exp\left(\sum_{i\text{ odd}}\frac{c_i}{i} + \sum_{i\text{ even}}\frac{b_i}{i}\right) - (1+c_1+c_1^*)\right\}\Big[a_1, \tilde{D}^+ - \tilde{P}^+, \tilde{D}^- - \tilde{P}^-\Big].$$

Expanding,

$$\tilde{P}^- = (1+c_1)\left\{\exp\left(\sum_{i\text{ odd}}\frac{a_i}{i}[\tilde{D}^- - \tilde{P}^-]\right) + \exp\left(\sum_{i\text{ even}}\frac{a_i}{i}[\tilde{D}^+ - \tilde{P}^+]\right)\right\}$$
$$- (1+c_1) - (\tilde{D}^- - \tilde{P}^-).$$

Rearranging,

$$\exp\left(\sum_{i\text{ odd}}\frac{a_i}{i}[\tilde{D}^- - \tilde{P}^-]\right) + \exp\left(\sum_{i\text{ even}}\frac{a_i}{i}[\tilde{D}^+ - \tilde{P}^+]\right)$$
$$= \frac{\tilde{P}^- + (1+c_1) + \tilde{D}^- - \tilde{P}^-}{1+c_1} = \frac{1+c_1+\tilde{D}^-}{1+c_1}.$$

Then by Möbius inversion (3.2),

$$\sum_{k\geq 1}\frac{\mu(k)}{k}a_i\left[\log\left(\frac{1+c_1+\tilde{D}^-}{1+c_1}\right)\right] = \sum_{k\text{ odd}}\frac{\mu(k)}{1}\sum_{i\text{ odd}}\frac{a_{ik}}{ik}[\tilde{D}^- - \tilde{P}^-]$$
$$+ \sum_{k\text{ odd}}\frac{\mu(k)}{1}\sum_{i\text{ even}}\frac{a_{ik}}{ik}[\tilde{D}^+ - \tilde{P}^+]$$

$$= \sum_{m \text{ odd}} \frac{a_m}{m} [\tilde{D}^- - \tilde{P}^-] \sum_{i|m} \frac{\mu(i)}{i}$$

$$+ \sum_{m \text{ odd}} \frac{a_m}{m} [\tilde{D}^+ - \tilde{P}^+] \sum_{i|m} \frac{\mu(i)}{i}$$

$$= a_1[\tilde{D}^- - \tilde{P}^-] + \sum_{k \geq 1} \frac{a_{2^k}}{2^k} [\tilde{D}^+ - \tilde{P}^+].$$

i.e., $\tilde{D}^- - \tilde{P}^- = \sum_{k \text{ odd}} \frac{\mu(k)}{k} a_k \left[\log \left(\frac{1 + c_1 + \tilde{D}^-}{1 + c_1} \right) \right] - \sum_{m = 2^{1+e},\, e \geq 0} \frac{a_m}{m} [\tilde{D}^+ - \tilde{P}^+].$

On composing both sides with $[x, y, y]$ and using equation (5.3)

$$(\tilde{D}^- - \tilde{P}^-)[x, y, y] = \sum_{k \text{ odd}} \frac{\mu(k)}{k} a_k \left[\log \left(\frac{1 + \epsilon(x, y)}{1 + y} \right) \right]$$

$$- \sum_{m = 2^{1+e},\, e \geq 0} \frac{1}{m} \sum_{i \geq 1} \frac{\mu(i)}{i} a_{im} \left[\log \left(\frac{1 + \delta(x, y)}{1 + y} \right) \right]. \qquad (5.4)$$

§5.1.5 $\mu(x, y)$

Now, rearranging (3.61), gives

$$T^+[a_1, b_1 + \tilde{D}^+, c_1 + \tilde{D}^-] = \tilde{D}^+ - N_1^+[a_1, \tilde{D}^+ - \tilde{P}^+, \tilde{D}^- - \tilde{P}^-]$$

$$- M_1^+[a_1, \tilde{D}^+ - \tilde{S}^+, \tilde{D}^- - \tilde{S}^-].$$

Applying (3.65) and (3.63), on RHS of this gives

$$T^+[a_1, b_1 + \tilde{D}^+, c_1 + \tilde{D}^-] = \tilde{D}^+ - \tilde{P}^+ - \tilde{S}^+.$$

Composing with $[x, y, y]$, and using (3.75), gives

$$T^+[x, \delta, \epsilon] = (\tilde{D}^+ - \tilde{P}^+)[x, y, y] - \tilde{S}^+[x, y, y]. \qquad (5.5)$$

Applying this with (4.5), gives

$$(D^+ - P^+)[x, \mu, \nu] - \mu(x, y) - S^+[x, \mu, \nu] = (\tilde{D}^+ - \tilde{P}^+)[x, y, y] - \tilde{S}^+[x, y, y]. \qquad (5.6)$$

$$\therefore \tilde{S}^+[x, y, y] - S^+[x, \mu, \nu] = (\tilde{D}^+ - \tilde{P}^+)[x, y, y] - \left\{ (D^+ - P^+)[x, \mu, \nu] - \mu(x, y) \right\}$$

$$= \sum_{i \geq 1} \frac{\mu(i)}{i} a_i \left[\log \left(\frac{1 + \delta(x, y)}{1 + y} \right) \right]$$

$$- \sum_{i\geq 1} \frac{\mu(i)}{i} a_i \left[\log\left(\frac{1+\delta(x,y)}{1+\mu(x,y)}\right)\right] \text{ from (3.85) \& (5.3)}$$

$$= \sum_{i\geq 1} \frac{\mu(i)}{i} a_i \left[\log\left(\frac{(1+\delta)(1+\mu)}{(1+y)(1+\delta)}\right)\right].$$

i.e., $q(x,y) - p(x,y) = \sum_{i\geq 1} \frac{\mu(i)}{i} a_i \left[\log\left(\frac{1+\mu(x,y)}{1+y}\right)\right].$

Then by Möbius inversion (3.2),

$$\frac{1+\mu(x,y)}{1+y} = \exp\left(\sum_{i\geq 1} \frac{a_i}{i}[q(x,y)-p(x,y)]\right).$$

Then, $1+\mu(x,y) = (1+y)\exp\left(\sum_{i\geq 1} \frac{a_i}{i}[q(x,y)-p(x,y)]\right).$ (5.7)

Thus $\mu_n(y)$ can be computed using $E1(1+\mu(x,y), q(x,y)-p(x,y))$,

with $1+\mu_0(y) = 1+y$ and $\mu_1(y) = (1+y)(q_1(y)-p_1(y))$.

§5.1.6 $\nu(x,y)$

Now, rearranging (3.62), gives

$$T^-[a_1, b_1+\tilde{D}^+, c_1+\tilde{D}^-] = \tilde{D}^- - N_1^-[a_1, \tilde{D}^+ - \tilde{P}^+, \tilde{D}^- - \tilde{P}^-]$$
$$- M_1^-[a_1, \tilde{D}^+ - \tilde{S}^+, \tilde{D}^- - \tilde{S}^-].$$

Applying (3.66) and (3.64), on RHS of this gives

$$T^-[a_1, b_1+\tilde{D}^+, c_1+\tilde{D}^-] = \tilde{D}^- - \tilde{P}^- - \tilde{S}^-.$$

Composing with $[x,y,y]$, and using (3.75), gives

$$T^-[x,\delta,\epsilon] = (\tilde{D}^- - \tilde{P}^-)[x,y,y] - \tilde{S}^-[x,y,y]. \quad (5.8)$$

Applying this with (4.9), gives

$$(D^- - P^-)[x,\mu,\nu] - \nu(x,y) - S^-[x,\mu,\nu] = (\tilde{D}^- - \tilde{P}^-)[x,y,y] - \tilde{S}^-[x,y,y]. \quad (5.9)$$

$$\therefore \tilde{S}^-[x,y,y] - S^-[x,\mu,\nu] = (\tilde{D}^- - \tilde{P}^-)[x,y,y] - \left\{(D^- - P^-)[x,\mu,\nu] - \nu(x,y)\right\}$$

$$= \sum_{k \text{ odd}} \frac{\mu(k)}{k} a_k \left[\log\left(\frac{1+\epsilon(x,y)}{1+y}\right)\right] - \sum_{m=2^{1+e},\, e\geq 0} \frac{1}{m} \sum_{i\geq 1} \frac{\mu(i)}{i} a_{im} \left[\log\left(\frac{1+\delta(x,y)}{1+y}\right)\right]$$

$$- \sum_{k \text{ odd}} \frac{\mu(k)}{k} a_k \left[\log\left(\frac{1+\epsilon(x,y)}{1+\nu(x,y)}\right)\right] + \sum_{m=2^{1+e},\, e\geq 0} \frac{1}{m} \sum_{i\geq 1} \frac{\mu(i)}{i} a_{im} \left[\log\left(\frac{1+\delta(x,y)}{1+\mu(x,y)}\right)\right],$$

from (3.86) and (5.4).

i.e., $t(x,y) - s(x,y)$

$$= \sum_{k \text{ odd}} \frac{\mu(k)}{k} a_k \left[\log\left(\frac{1+\nu(x,y)}{1+y}\right)\right] - \sum_{m=2^{1+e},\, e\geq 0} \frac{1}{m} \sum_{i\geq 1} \frac{\mu(i)}{i} a_{im} \left[\log\left(\frac{1+\mu(x,y)}{1+y}\right)\right]$$

$$= \sum_{k \text{ odd}} \frac{\mu(k)}{k} a_k \left[\log\left(\frac{1+\nu(x,y)}{1+y}\right)\right] - \sum_{m=2^{1+e},\, e\geq 0} \frac{a_m}{m}[q(x,y) - p(x,y)].$$

$$\therefore \sum_{k \text{ odd}} \frac{\mu(k)}{k} a_k \left[\log\left(\frac{1+\nu(x,y)}{1+y}\right)\right] \tag{5.10}$$

$$= a_1[t(x,y) - s(x,y)] + \sum_{m=2^{1+e}, e \geq 0} \frac{a_m}{m}[q(x,y) - p(x,y)]$$

$$= \sum_{m \text{ odd}} \frac{a_m}{m}[t(x,y) - s(x,y)] \sum_{i|m} \frac{\mu(i)}{i} + \sum_{m \text{ odd}} \frac{a_m}{m}[q(x,y) - p(x,y)] \sum_{i|m,\, i \text{ odd}} \frac{\mu(i)}{i}$$

$$= \sum_{k \text{ odd}} \frac{\mu(k)}{1} \sum_{i \text{ odd}} \frac{a_{ik}}{ik}[t(x,y) - s(x,y)] + \sum_{k \text{ odd}} \frac{\mu(k)}{1} \sum_{i \text{ even}} \frac{a_{ik}}{ik}[q(x,y) - p(x,y)].$$

Then by Möbius inversion (3.2),

$$1 + \nu(x,y) = (1+y)\exp\left(\sum_{i \text{ odd}} \frac{a_i}{i}[t(x,y) - s(x,y)] + \sum_{i \text{ even}} \frac{a_i}{i}[q(x,y) - p(x,y)]\right). \tag{5.11}$$

Thus $\nu_n(y)$ can be computed using

$E2(1 + \nu(x,y), q(x,y) - p(x,y), t(x,y) - s(x,y))$

with $1 + \nu_0(y) = 1 + y$ and $\nu_1(y) = (1+y)(t_1(y) - s_1(y))$.

§5.2 Main counting equation

Now, using Otter's dissimilarity characteristic equation (3.16) as in the case of 2–connected graphs (3.74), the decomposition of 3–edge–connected blocks can be written as :

$$\begin{aligned}\tilde{B} &= T[a_1, b_1 + \tilde{D}^+, c_1 + \tilde{D}^-] + N_1[a_1, \tilde{D}^+ - \tilde{P}^+, \tilde{D}^- - \tilde{P}^-] \\
&+ M_1[a_1, \tilde{D}^+ - \tilde{S}^+, \tilde{D}^- - \tilde{S}^-] \\
&- \Big\{ \vec{T}[a_1, b_1 + \tilde{D}^+, c_1 + \tilde{D}^-] \diamond \vec{N}_1[a_1, \tilde{D}^+ - \tilde{P}^+, \tilde{D}^- - \tilde{P}^-] \\
&+ \vec{N}_1[a_1, \tilde{D}^+ - \tilde{P}^+, \tilde{D}^- - \tilde{P}^-] \diamond \vec{M}_1[a_1, \tilde{D}^+ - \tilde{S}^+, \tilde{D}^- - \tilde{S}^-] \\
&+ \vec{T}[a_1, b_1 + \tilde{D}^+, c_1 + \tilde{D}^-] \diamond \vec{M}_1[a_1, \tilde{D}^+ - \tilde{S}^+, \tilde{D}^- - \tilde{S}^-] \\
&+ \frac{1}{2}\vec{T}[a_1, b_1 + \tilde{D}^+, c_1 + \tilde{D}^-] \diamond \vec{T}[a_1, b_1 + \tilde{D}^+, c_1 + \tilde{D}^-] \\
&+ \left(\frac{a_1^2 + a_2}{4}\right) a_2 \left[T^+[a_1, b_1 + \tilde{D}^+, c_1 + \tilde{D}^-]\right] \Big\} \\
&+ \left(\frac{a_1^2 + a_2}{2}\right) a_2 \left[T^+[a_1, b_1 + \tilde{D}^+, c_1 + \tilde{D}^-]\right].\end{aligned}$$

Composing on both sides with $[x, y, y]$ and using (3.75), we get,

$$\begin{aligned}\tilde{B}[x, y, y] &= T[x, \delta, \epsilon] + N_1\left[x, (\tilde{D}^+ - \tilde{P}^+)[x, y, y], (\tilde{D}^- - \tilde{P}^-)[x, y, y]\right] \\
&+ M_1\left[x, \delta - \tilde{S}^+[x, y, y] - y, \epsilon - \tilde{S}^-[x, y, y] - y\right] \\
&- \Big\{ \vec{T}[x, \delta, \epsilon] \diamond \vec{N}_1\left[x, (\tilde{D}^+ - \tilde{P}^+)[x, y, y], (\tilde{D}^- - \tilde{P}^-)[x, y, y]\right] \\
&+ \vec{N}_1\left[x, (\tilde{D}^+ - \tilde{P}^+)[x, y, y], (\tilde{D}^- - \tilde{P}^-)[x, y, y]\right] \\
&\diamond \vec{M}_1\left[x, \delta - \tilde{S}^+[x, y, y] - y, \epsilon - \tilde{S}^-[x, y, y] - y\right] \\
&+ \vec{T}[x, \delta, \epsilon] \diamond \vec{M}_1\left[x, \delta - \tilde{S}^+[x, y, y] - y, \epsilon - \tilde{S}^-[x, y, y] - y\right] \\
&+ \frac{1}{2}\vec{T}[x, \delta, \epsilon] \diamond \vec{T}[x, \delta, \epsilon] + \frac{x^2}{2} a_2 \left[T^+[x, \delta, \epsilon]\right] \Big\} + x^2 a_2 \left[T^+[x, \delta, \epsilon]\right].\end{aligned}$$

Now, using (3.76) on RHS of this gives

$$\begin{aligned}\tilde{B}[x, y, y] &= B[x, \mu, \nu] - N_1\left[x, (D^+ - P^+)[x, \mu, \nu] - \mu, (D^- - P^-)[x, \mu, \nu] - \nu\right] \\
&- M\left[x, \delta - S^+[x, \mu, \nu], \epsilon - S^-[x, \mu, \nu]\right] \\
&+ \vec{T}[x, \delta, \epsilon] \diamond \vec{N}_1\left[x, (D^+ - P^+)[x, \mu, \nu] - \mu, (D^- - P^-)[x, \mu, \nu] - \nu\right]\end{aligned}$$

$$
\begin{aligned}
&+ \overrightarrow{N}_1\left[x,(D^+ - P^+)[x,\mu,\nu]-\mu,(D^- - P^-)[x,\mu,\nu]-\nu\right] \\
&\diamond \overrightarrow{M}\left[x,\delta - S^+[x,\mu,\nu],\epsilon - S^-[x,\mu,\nu]\right] \\
&+ \overrightarrow{T}[x,\delta,\epsilon]\diamond \overrightarrow{M}\left[x,\delta - S^+[x,\mu,\nu],\epsilon - S^-[x,\mu,\nu]\right] \\
&+ N_1\left[x,(\tilde{D}^+ - \tilde{P}^+)[x,y,y],(\tilde{D}^- - \tilde{P}^-)[x,y,y]\right] \\
&+ M_1\left[x,\delta - \tilde{S}^+[x,y,y]-y,\epsilon - \tilde{S}^-[x,y,y]-y\right] \\
&- \overrightarrow{T}[x,\delta,\epsilon]\diamond \overrightarrow{N}_1\left[x,(\tilde{D}^+ - \tilde{P}^+)[x,y,y],(\tilde{D}^- - \tilde{P}^-)[x,y,y]\right] \\
&- \overrightarrow{N}_1\left[x,(\tilde{D}^+ - \tilde{P}^+)[x,y,y],(\tilde{D}^- - \tilde{P}^-)[x,y,y]\right] \\
&\diamond \overrightarrow{M}_1\left[x,\delta - \tilde{S}^+[x,y,y]-y,\epsilon - \tilde{S}^-[x,y,y]-y\right] \\
&- \overrightarrow{T}[x,\delta,\epsilon]\diamond \overrightarrow{M}_1\left[x,\delta - \tilde{S}^+[x,y,y]-y,\epsilon - \tilde{S}^-[x,y,y]-y\right] \\
&- \frac{x^2}{2}(\mu(x,y)+\nu(x,y)).
\end{aligned} \tag{5.12}
$$

Rearranging (5.12), gives

$$
\begin{aligned}
\tilde{B}[x,y,y] = {} & B[x,\mu,\nu] + N_1\left[x,(\tilde{D}^+ - \tilde{P}^+)[x,y,y],(\tilde{D}^- - \tilde{P}^-)[x,y,y]\right] \\
&- N_1\left[x,(D^+ - P^+)[x,\mu,\nu]-\mu,(D^- - P^-)[x,\mu,\nu]-\nu\right] \\
&+ M_1\left[x,\delta - \tilde{S}^+[x,y,y]-y,\epsilon - \tilde{S}^-[x,y,y]-y\right] \\
&- M\left[x,\delta - S^+[x,\mu,\nu],\epsilon - S^-[x,\mu,\nu]\right] \\
&+ \overrightarrow{T}[x,\delta,\epsilon]\diamond \overrightarrow{N}_1\left[x,(D^+ - P^+)[x,\mu,\nu]-\mu,(D^- - P^-)[x,\mu,\nu]-\nu\right] \\
&- \overrightarrow{T}[x,\delta,\epsilon]\diamond \overrightarrow{N}_1\left[x,(\tilde{D}^+ - \tilde{P}^+)[x,y,y],(\tilde{D}^- - \tilde{P}^-)[x,y,y]\right] \\
&+ \overrightarrow{N}_1\left[x,(D^+ - P^+)[x,\mu,\nu]-\mu,(D^- - P^-)[x,\mu,\nu]-\nu\right] \\
&\diamond \overrightarrow{M}\left[x,\delta - S^+[x,\mu,\nu],\epsilon - S^-[x,\mu,\nu]\right] \\
&- \overrightarrow{N}_1\left[x,(\tilde{D}^+ - \tilde{P}^+)[x,y,y],(\tilde{D}^- - \tilde{P}^-)[x,y,y]\right] \\
&\diamond \overrightarrow{M}_1\left[x,\delta - \tilde{S}^+[x,y,y]-y,\epsilon - \tilde{S}^-[x,y,y]-y\right] \\
&+ \overrightarrow{T}[x,\delta,\epsilon]\diamond \overrightarrow{M}\left[x,\delta - S^+[x,\mu,\nu],\epsilon - S^-[x,\mu,\nu]\right]
\end{aligned}
$$

$$- \vec{T}[x,\delta,\epsilon] \diamond \vec{M}_1 \left[x, \delta - \tilde{S}^+[x,y,y] - y, \epsilon - \tilde{S}^-[x,y,y] - y\right]$$

$$- \frac{x^2}{2}(\mu(x,y) + \nu(x,y)). \tag{5.13}$$

§5.3 Final counting series

Now, again let $u(x,y) = (\tilde{D}^+ - \tilde{P}^+)[x,y,y]$.

Then (5.5) becomes, $T^+[x,\delta,\epsilon] = u(x,y) - q(x,y)$. (5.14)

Also, by (5.3), $u(x,y) = \sum_{i\geq 1} u_i(y)x^i = \sum_{i\geq 1} \frac{\mu(i)}{i} a_i \left[\log\left(\frac{1+\delta(x,y)}{1+y}\right)\right]$.

Letting $w(x,y) = \log\left(\frac{1+\delta(x,y)}{1+y}\right)$, gives, $u_n(y) = \sum_{i\cdot j = n} \frac{\mu(i)}{i} w_j(y^i)$. (5.15)

Moreover, rearranging (5.6) gives,

$$(D^+ - P^+)[x,\mu,\nu] - \mu(x,y) = p(x,y) - q(x,y) + u(x,y). \tag{5.16}$$

Now, let $v(x,y) = (\tilde{D}^- - \tilde{P}^-)[x,y,y]$.

Then (5.8) becomes, $T^-[x,\delta,\epsilon] = v(x,y) - t(x,y)$. (5.17)

Also by (5.4), $v(x,y) = \sum_{i\geq 1} v_i(y)x^i = \sum_{k \text{ odd}} \frac{\mu(k)}{k} a_k \left[\log\left(\frac{1+\epsilon(x,y)}{1+y}\right)\right]$.

$$- \sum_{m=2^{1+e}, e\geq 0} \frac{a_m}{m}\left[u(x,y)\right].$$

Letting $h(x,y) = \log\left(\frac{1+\epsilon(x,y)}{1+y}\right)$, gives,

$$v_n(y) = \sum_{i\cdot j=n,\ i \text{ odd}} \frac{\mu(i)}{i} h_j(y^i) - \sum_{2|n, m\cdot i = n} \frac{1}{m} u_i(y^m), \text{where, } m = 2^{1+e}, \text{for, } e \geq 0. \tag{5.18}$$

Moreover, rearranging (5.9), gives,

$$(D^- - P^-)[x,\mu,\nu] - \nu(x,y) = s(x,y) - t(x,y) + v(x,y). \tag{5.19}$$

Recall from previous equations,

$$\delta(x,y) - S^+[x,\mu,\nu] = \delta(x,y) - p(x,y),$$

$$\epsilon(x,y) - S^-[x,\mu,\nu] = \epsilon(x,y) - s(x,y),$$

$\delta(x,y) - \hat{S}^{+}[x,y,y] - y = \delta(x,y) - q(x,y) - y$ and

$\epsilon(x,y) - \hat{S}^{-}[x,y,y] - y = \epsilon(x,y) - t(x,y) - y.$

Substituting all of these in equation (5.13) gives,

$$\begin{aligned}
\tilde{B}[x,y,y] &= B[x,\mu,\nu] \text{ (say, } B(x,y) \text{)} \\
&\quad + N_1[x,u,v] - N_1[x,p-q+u,s-t+v] \text{ (say, } A1(x,y) \text{)} \\
&\quad + M_1[x,\delta-q-y,\epsilon-t-y] - M[x,\delta-p,\epsilon-s] \text{ (say, } A2(x,y) \text{)} \\
&\quad + \vec{T}[x,\delta,\epsilon] \diamond \vec{N}_1[x,p-q+u,s-t+v] \\
&\quad - \vec{T}[x,\delta,\epsilon] \diamond \vec{N}_1[x,u,v] \text{ (say, } A3(x,y) \text{)} \\
&\quad + \vec{N}_1[x,p-q+u,s-t+v] \diamond \vec{M}[x,\delta-p,\epsilon-s] \\
&\quad - \vec{N}_1[x,u,v] \diamond \vec{M}_1[x,\delta-q-y,\epsilon-t-y] \text{ (say, } A4(x,y) \text{)} \\
&\quad + \vec{T}[x,\delta,\epsilon] \diamond \vec{M}[x,\delta-p,\epsilon-s] \\
&\quad - \vec{T}[x,\delta,\epsilon] \diamond \vec{M}_1[x,\delta-q-y,\epsilon-t-y] \text{ (say, } A5(x,y) \text{)} \\
&\quad - \frac{x}{2}(\mu(x,y) + \nu(x,y)).
\end{aligned}$$

In terms of nth coefficients, for $n \geq 2$, this is,

$$\begin{aligned}
\tilde{B}_n(y) &= B_n(y) + A1_n(y) + A2_n(y) + A3_n(y) + A4_n(y) + A5_n(y) \\
&\quad - \frac{1}{2}\left\{ \mu_{n-2}(y) + \nu_{n-2}(y) \right\}.
\end{aligned} \tag{5.20}$$

§5.3.1 $A1(x,y)$

$A1(x,y) = N_1[x,u,v] - N_1[x,p-q+u,s-t+v].$

Let pqu to denote $p-q+u$ and stv to denote $s-t+v$.

Then from (3.46), we get

$$\begin{aligned}
A1(x,y) &= \frac{x^2}{2}\left\{ (1+y)\exp\left(\sum_{i\geq 1} \frac{u_i}{i} \right) + (1+y)\exp\left(\sum_{i \text{ even}} \frac{u_i}{i} + \sum_{i \text{ odd}} \frac{v_i}{i} \right) \right. \\
&\quad \left. - \left\{ (1+y)(1+u) + \frac{u^2}{2} + \frac{u_2}{2} + (1+y)(1+v) + \frac{v^2}{2} + \frac{u_2}{2} \right\} \right.
\end{aligned}$$

$$- \left\{ (1+\mu)\exp\left(\sum_{i\geq 1}\frac{pqu_i}{i}\right) + (1+\nu)\exp\left(\sum_{i \text{ even}}\frac{pqu_i}{i} + \sum_{i \text{ odd}}\frac{stv_i}{i}\right)\right.$$

$$- \left\{ (1+\mu)(1+pqu) + \frac{pqu^2}{2} + \frac{pqu_2}{2}\right.$$

$$\left.\left.\left. + (1+\nu)(1+stv) + \frac{stv^2}{2} + \frac{pqu_2}{2}\right\}\right\}\right\}.$$

Applying (3.87) and (3.88), let $T1$ come from $E1(T1, pqu)$; $T2$ come from $E2(T2, pqu, stv)$; $T3$ come from $E1(T3, u)$; $T4$ come from $E2(T4, u, v)$.

Then, $A1(x,y) = \frac{x^2}{2}\left\{(1+y)(T3+T4-u-v) - 2(1+y) - \frac{u^2}{2} - \frac{v^2}{2} - u_2\right.$

$$+ (1+\mu)(pqu - T1) + (1+\nu)(stv - T2)$$

$$\left. + (2+\mu+\nu) + \frac{pqu^2}{2} + \frac{stv^2}{2} + pqu_2\right\}.$$

Since, $T1_0(y) = T2_0(y) = T3_0(y) = T4_0(y) = 1$ and

$pqu_0(y) = stv_0(y) = u_0(y) = v_0(y) = 0$, we have

$$A1_2(y) = \frac{1}{2}\Big\{(1+y)(1+1-0-0) - 2(1+y) - 0 - 0 - 0 + (1+y)(-1)$$

$$+ (1+y)(-1) + (2+2y) + 0 + 0 + 0\Big\}$$

$$= \frac{1}{2}\Big\{(1+y)2 - 2(1+y) - (1+y) - (1+y) + 2(1+y)\Big\}$$

$$= 0.$$

So, for $n \geq 3$,

$$A1_n(y) = \frac{1}{2}\left\{\sum_{2\leq k\leq n}\Big\{(pqu_{n-k} - T1_{n-k})\mu_{k-2} + (stv_{n-k} - T2_{n-k})\nu_{k-2}\Big\}\right.$$

$$+ \frac{1}{2}\sum_{2\leq k\leq n}\Big\{pqu_{n-k}\cdot pqu_{k-2} + stv_{n-k}\cdot stv_{k-2} - u_{n-k}\cdot u_{k-2} - v_{n-k}\cdot v_{k-2}\Big\}$$

$$+ (1+y)(T3_{n-2} + T4_{n-2} - u_{n-2} - v_{n-2}) + \mu_{n-2} + \nu_{n-2} +$$

$$\left. + \begin{cases} pqu_m(y^2) - u_m(y^2) & \text{if } 2|n \text{ for } m = \left(\frac{n-2}{2}\right) \\ 0 & \text{otherwise}\end{cases}\right\}. \qquad (5.21)$$

§5.3.2 $A2(x,y)$

$A2(x,y) = M_1[x, \delta - q - y, \epsilon - t - y] - M[x, \delta - p, \epsilon - s]$.

We let dqy to denote $\delta - q - y$, et to denote $\epsilon - t - y$, dp to denote $\delta - p$ and es to denote $\epsilon - s$.

Then using (3.34), gives

$$
\begin{aligned}
A2(x,y) = & \frac{1}{2}\Bigg\{ \sum_{d\geq 1} \frac{\phi(d)}{d} a_d \left[\log\left(\frac{1 - x\cdot dp}{1 - x\cdot dqy}\right)\right]\Bigg\} \\
& + \frac{x}{2}(dp - dqy) + \frac{x^2}{4}(dp^2 - dqy^2) + \frac{x^2}{4}(dp_2 - dqy_2) \\
& + \frac{1}{4(1 - x^2\cdot dqy_2)}\Big\{2x^3\cdot ety\cdot dqy_2 + x^4\cdot dqy_2^2 + x^4 ety^2\cdot dqy_2\Big\} \\
& - \frac{1}{4(1 - x^2\cdot dp_2)}\Big\{2x^3\cdot es\cdot dp_2 + x^4\cdot dp_2^2 + x^4\cdot es^2\cdot dp_2\Big\} \\
& + M_0[x, \delta - q - y, \epsilon - t - y].
\end{aligned}
$$

Applying (3.96) and (3.97), let $T9$ come from $V(T9, dp, dqy)$; $T10$ come from $W(T10, dqy, ety)$; $T11$ come from $W(T11, dp, es)$;

$T12$ come from $M0(T12, dqy, ety)$.

$$
\begin{aligned}
A2(x,y) = & \frac{1}{2}\Bigg\{ \sum_{n\geq 1}\left(\sum_{i\cdot d = n} \frac{\phi(d)}{d} T9_i(y^d)\right) x^n \Bigg\} \\
& + \frac{x}{2}(dp - dqy) + \frac{x^2}{4}(dp^2 - dqy^2) + \frac{x^2}{4}(dp_2 - dqy_2) + T10 - T11 + T12.
\end{aligned}
$$

Then for $n \geq 2$,

$$
\begin{aligned}
A2_n(y) = & \sum_{i\cdot d = n} \frac{\phi(d)}{2d} T9_i(y^d) + \frac{1}{2}(dp_{n-1} - dqy_{n-1}) \\
& + \frac{1}{4}\sum_{2\leq k\leq n}(dp_{k-2}\cdot dp_{n-k} - dqy_{k-2}\cdot dqy_{n-k}) + T10_n(y) - T11_n(y) + T12_n(y) \\
& + \frac{1}{4}\left\{\begin{array}{ll} dp_m(y^2) - dqy_m(y^2), & \text{if } 2|n \text{ for } m = \left(\frac{n-2}{2}\right) \\ 0, & \text{otherwise} \end{array}\right\}
\end{aligned}
\tag{5.22}
$$

with $A2_0(y) = 0$;

$$
\begin{aligned}
A2_1(y) &= \frac{\phi(1)}{2}T9_1(y) + \frac{1}{2}(dp_0 - dqy_0) \\
&= \frac{\phi(1)}{2}(-dp_0 + dqy_0) + \frac{1}{2}(dp_0 - dqy_0) \\
&= \frac{1}{2}y - \frac{1}{2}y = 0.
\end{aligned}
$$

§5.3.3 $A3(x,y)$

$$
A3(x,y) = \vec{T}[x,\delta,\epsilon] \diamond \vec{N}_1[x,pqu,stv] - \vec{T}[x,\delta,\epsilon] \diamond \vec{N}_1[x,u,v].
$$

Applying the definition of $\diamond$ from (3.73),

$$
\begin{aligned}
A3(x,y) = {} & \frac{x^2}{2}T^+[x,\delta,\epsilon]N_1^+[x,pqu,stv] + \frac{x^2}{2}T^-[x,\delta,\epsilon]N_1^-[x,pqu,stv] \\
& - \frac{x^2}{2}T^+[x,\delta,\epsilon]N_1^+[x,u,v] - \frac{x^2}{2}T^-[x,\delta,\epsilon]N_1^-[x,u,v].
\end{aligned}
$$

Applying (5.14), (5.17), (3.47), (3.48) gives,

$$
\begin{aligned}
A3(x,y) = {} & \frac{x^2}{2}\Bigg\{(u-q)\Bigg\{(1+\mu)\exp\left(\sum_{i\geq 1}\frac{pqu_i}{i}\right) - (1+\mu) - pqu\Bigg\} \\
& + (v-t)\Bigg\{(1+\nu)\exp\left(\sum_{i \text{ even}}\frac{pqu_i}{i} + \sum_{i \text{ odd}}\frac{stv_i}{i}\right) - (1+\nu) - stv\Bigg\} \\
& - (u-q)\Bigg\{(1+y)\exp\left(\sum_{i\geq 1}\frac{u_i}{i}\right) - (1+y) - u\Bigg\} \\
& - (v-t)\Bigg\{(1+y)\exp\left(\sum_{i \text{ even}}\frac{u_i}{i} + \sum_{i \text{ odd}}\frac{v_i}{i}\right) - (1+y) - v\Bigg\}\Bigg\}.
\end{aligned}
$$

As before, let $T1$ come from $E1(T1,pqu)$; $T2$ come from $E2(T2,pqu,stv)$;

$T3$ come from $E1(T3,u)$; $T4$ come from $E2(T4,u,v)$.

$$
\begin{aligned}
A3(x,y) = {} & \frac{x^2}{2}\Bigg\{(u-q)\Bigg\{(1+\mu)T1 - (1+\mu) - pqu\Bigg\} \\
& + (v-t)\Bigg\{(1+\nu)T2 - (1+\nu) - stv\Bigg\} \\
& - (u-q)\Bigg\{(1+y)T3 - (1+y) - u\Bigg\} - (v-t)\Bigg\{(1+y)T4 - (1+y) - v\Bigg\}\Bigg\} \\
= {} & \frac{x^2}{2}\Bigg\{(u-q)\Bigg\{(1+\mu)T1 - (1+\mu) - pqu - (1+y)T3 + (1+y) + u\Bigg\}
\end{aligned}
$$

$$+ (v-t)\Big\{(1+\nu)T2 - (1+\nu) - stv - (1+y)T4 + (1+y) + v\Big\}\Big\}.$$

Then, for $n \geq 2$,

$$A3_n(y) = \frac{1}{2}\Bigg\{\sum_{0\leq k\leq n-2}\Bigg\{\Big(u_{n-k-2}(y) - q_{n-k-2}(y)\Big)$$
$$* \Bigg\{\Big(\sum_{0\leq r\leq k}\mu_r(y)T1_{k-r}(y)\Big) - \mu_k(y) - pqu_k(y) - (1+y)T3_k(y) + u_k(y)\Bigg\}$$
$$+ \Big(v_{n-k-2}(y) - t_{n-k-2}(y)\Big)$$
$$* \Bigg\{\Big(\sum_{0\leq r\leq k}\nu_r(y)T2_{k-r}(y)\Big) - \nu_k(y) - stv_k(y) - (1+y)T4_k(y) + v_k(y)\Bigg\}\Bigg\}$$
$$+ (1+y)(u_{n-2}(y) - q_{n-2}(y) + v_{n-2}(y) - t_{n-2}(y))\Bigg\}. \tag{5.23}$$

§5.3.4 $A4(x,y)$

$$A4(x,y) = \vec{N}_1[x,pqu,stv] \diamond \vec{M}[x,dp,es] - \vec{N}_1[x,u,v] \diamond \vec{M}_1[x,dqy,ety].$$

Applying the definition of $\diamond$ from (3.73),

$$A4(x,y) = \frac{x^2}{2}N_1^+[x,pqu,stv]M^+[x,dp,es] + \frac{x^2}{2}N_1^-[x,pqu,stv]M^-[x,dp,es]$$
$$- \frac{x^2}{2}N_1^+[x,u,v]M_1^+[x,dqy,ety] - \frac{x^2}{2}N_1^-[x,u,v]M_1^-[x,dqy,ety].$$

Applying (3.47), (3.48) gives,

$$A4(x,y) = \frac{x^2}{2}\Bigg\{\Bigg\{(1+\mu)\exp\Big(\sum_{i\geq 1}\frac{pqu_i}{i}\Big) - (1+\mu) - pqu\Bigg\}M^+[x,dp,es]$$
$$+ \Bigg\{(1+\nu)\exp\Big(\sum_{i\text{ even}}\frac{pqu_i}{i} + \sum_{i\text{ odd}}\frac{stv_i}{i}\Big) - (1+\nu) - stv\Bigg\}M^-[x,dp,es]$$
$$- \Bigg\{(1+y)\exp\Big(\sum_{i\geq 1}\frac{u_i}{i}\Big) - (1+y) - u\Bigg\}M_1^+[x,dqy,ety]$$
$$- \Bigg\{(1+y)\exp\Big(\sum_{i\text{ even}}\frac{u_i}{i} + \sum_{i\text{ odd}}\frac{v_i}{i}\Big) - (1+y) - v\Bigg\}M_1^-[x,dqy,ety]\Bigg\}.$$

As before, let $T1$ come from $E1(T1,pqu)$; $T2$ come from $E2(T2,pqu,stv)$;

$T3$ come from $E1(T3, u)$; $T4$ come from $E2(T4, u, v)$.

Also, using (3.89) and (3.90), $T5$ come from $MP(T5, dp)$; $T6$ come from $M1P(T6, dqy)$; $T7$ come from $MN(T7, dp, es)$; $T8$ come from $M1N(T8, dqy, ety)$.

$$A4(x,y) = \frac{x^2}{2}\Bigg\{\Big\{(1+\mu)T1 - (1+\mu) - pqu\Big\}T5 + \Big\{(1+\nu)T2 - (1+\nu) - stv\Big\}T7 \\ - \Big\{(1+y)T3 - (1+y) - u\Big\}T6 - \Big\{(1+y)T4 - (1+y) - v\Big\}T8\Bigg\}.$$

Then, for $n \geq 2$,

$$A4_n(y) = \frac{1}{2}\Bigg\{\sum_{0\leq k\leq n-2}\Bigg\{\Big\{\sum_{0\leq r\leq k}\mu_r(y)\cdot T1_{k-r}(y) - \mu_k(y) - pqu_k(y)\Big\} * T5_{n-k-2}(y) \\ + \Big\{\sum_{0\leq r\leq k}\nu_r(y)\cdot T2_{k-r}(y) - \nu_k(y) - stv_k(y)\Big\} * T7_{n-k-2}(y) \\ - \Big\{(1+y)T3_{n-k-2}(y) - u_{n-k-2}(y)\Big\} * T6_k(y) \\ - \Big\{(1+y)T4_{n-k-2}(y) - v_{n-k-2}(y)\Big\} * T8_k(y)\Bigg\} \\ + (1+y)(T6_{n-2}(y) + T8_{n-2}(y))\Bigg\}. \quad (5.24)$$

§5.3.5 $A5(x, y)$

$$A5(x,y) = \vec{T}[x,\delta,\epsilon] \diamond \vec{M}[x,dp,es] - \vec{T}[x,\delta,\epsilon] \diamond \vec{M}_1[x,dqy,ety].$$

Applying the definition of $\diamond$ from (3.73),

$$A5(x,y) = \frac{x^2}{2}T^+[x,\delta,\epsilon]M^+[x,dp,es] + \frac{x^2}{2}T^-[x,\delta,\epsilon]M^-[x,dp,es] \\ - \frac{x^2}{2}T^+[x,\delta,\epsilon]M_1^+[x,dqy,ety] - \frac{x^2}{2}T^-[x,\delta,\epsilon]M_1^-[x,dqy,ety].$$

Again, using (5.14) and (5.17), gives

$$A5(x,y) = \frac{x^2}{2}\Bigg\{(u-q)\Big\{M^+[x,dp,es] - M_1^+[x,dqy,ety]\Big\} \\ + (v-t)\Big\{M^-[x,dp,es] - M_1^-[x,dqy,ety]\Big\}\Bigg\}.$$

As before let $T5$ come from $MP(T5, dp)$; $T6$ come from $M1P(T6, dqy)$;

$T7$ come from $MN(T7, dp, es)$; $T8$ come from $M1N(T8, dqy, ety)$.

$$A5(x,y) = \frac{x^2}{2}\Big\{(u-q)(T5-T6) + (v-t)(T7-T8)\Big\}.$$

$$\therefore \text{ for } n \geq 2,\ A5_n(y) = \frac{1}{2}\Bigg\{\sum_{2\leq k\leq n}\Big\{(u_{k-2}(y) - q_{k-2}(y)) * (T5_{n-k}(y) - T6_{n-k}(y)) + (v_{k-2}(y) - t_{k-2}(y)) * (T7_{n-k}(y) - T8_{n-k}(y))\Big\}\Bigg\}. \tag{5.25}$$

§5.4 Summary of the counting algorithm

In the computations below, power series are computed term by term. At any given time nth term of each of the power series (except in the case of $\eta(x,y)$, for which $(n+1)$th term is computed) is computed one after the other in the order given here, utilizing all its previous terms and all the available terms of the power series computed before it. Steps are numbered for easy perusal.

1. From (3.83), compute, $p_0(0) = 0$; $p_1(y) = \delta_0(y)^2$

and for $n \geq 2$, $p_n(y) = \sum_{1\leq k\leq n} \delta_{k-1}(y)\Big\{\delta_{n-k}(y) - p_{n-k}(y)\Big\}.$

Then from (5.1), compute, $q_0(y) = 0$; $q_1(y) = \delta_0(y)^2 - y^2$

and for $n \geq 2$,

$$q_n(y) = -y^2\delta_{n-2}(y) - 2yq_{n-1}(y) + \delta_0(y)\delta_{n-1}(y) - q_0(y)q_{n-1}(y) + \sum_{2\leq k\leq n}\Bigg\{\Big\{\delta_{k-1}(y)\delta_{n-k}(y) - q_{k-1}(y)q_{n-k}(y)\Big\} + \delta_{k-2}(y)\Big\{2y\Big\{\delta_{n-k}(y) - q_{n-k}(y)\Big\} - \sum_{0\leq r\leq n-k}\Big\{\delta_r(y) - q_r(y)\Big\}\Big\{\delta_{n-k-r}(y) - q_{n-k-r}(y)\Big\}\Big\}\Bigg\}.$$

Then from (5.7), compute, for $n \geq 2$, nth term of μ using these in

$E1(1+\mu, q-p)$ and $1 + \mu_0(y) = 1 + y$ and $\mu_1(y) = (1+y)(q_1(y) - p_1(y))$.

2. From (3.84), compute, $s_0(y)=0$; $s_1(y)=\delta_0(y^2)-p_0(y^2)=\delta_0(y^2)$

and for $n\geq 2$, $s_n(y)$ using,

$$s_{2n}(y)=\sum_{1\leq k\leq n}\left\{\delta_{k-1}(y^2)-p_{k-1}(y^2)\right\}\epsilon_{2n-2k}(y) \text{ and}$$

$$s_{2n+1}(y)=\delta_n(y^2)-p_n(y^2)+\sum_{1\leq k\leq n}\left\{\delta_{k-1}(y^2)-p_{k-1}(y^2)\right\}\epsilon_{2n+1-2k}(y).$$

From (5.2) compute, $t_0(y)=0$; $t_1(y)=\delta_0(y^2)-y^2$

and for $n\geq 1$, $t_n(y)$ using

$$t_{2n}(y)=-y^2\epsilon_{2n-2}(y)+\sum_{1\leq k\leq n}\left\{\delta_{k-1}(y^2)-q_{k-1}(y^2)\right\}\epsilon_{2(n-k)}(y) \text{ and}$$

$$t_{2n+1}(y)=\delta_n(y^2)-q_n(y^2)-y^2\epsilon_{2n-1}(y)+\sum_{1\leq k\leq n}\left\{\delta_{k-1}(y^2)-q_{k-1}(y^2)\right\}\epsilon_{2(n-k)+1}(y).$$

Then from (5.11), compute, $1+\nu_0(y)=1+y$ and $\nu_1(y)=(1+y)(t_1(y)-s_1(y))$

for $n\geq 2$, nth term of ν using these in $E2(1+\nu,q-p,t-s)$.

3. From (3.80), $\eta(x,y)-x=xF(x,y)-a_1\left(\dfrac{\partial \boldsymbol{K}}{\partial a_1}-1\right)[\eta,1+\mu,1+\nu]$

where $F(x,y)=(\boldsymbol{K}-1)[\eta,1+\mu,1+\nu]$. We compute the nth term $F(x,y)$ by doing the composition on the terms generated for $\boldsymbol{K}$.

Let $\boldsymbol{K}_a$ denote $a_1\left(\dfrac{\partial \boldsymbol{K}}{\partial a_1}-1\right)$. Compute the power series

$R(x,y)=\boldsymbol{K}_a[\eta,1+\mu,1+\nu]$ by doing the composition on the terms

generated for $\boldsymbol{K}_a$. Compute, for $n\geq 2$, the $(n+1)$th term

of $\eta(x,y)$, using $\eta_{n+1}(y)=F_n(y)-R_{n+1}(y)$.

4. From (3.78),

$$\delta(x,y)+1=\left(\frac{\eta(x,y)}{x}\right)^2\frac{2}{a_1^2}\left(b_1\frac{\partial \boldsymbol{K}}{\partial b_1}\right)[\eta,1+\mu,1+\nu]-(1+\delta(x,y))F(x,y).$$

Let $\boldsymbol{K}_b$ denote $\dfrac{2}{a_1^2}\left(b_1\dfrac{\partial \boldsymbol{K}}{\partial b_1}\right)$. Compute the power series

$S(x,y)=\boldsymbol{K}_b[\eta,1+\mu,1+\nu]$ by composing on the terms generated for $\boldsymbol{K}_b$.

Then, $\delta(x,y)+1 = \left(\frac{\eta(x,y)}{x}\right)^2 S(x,y) - (1+\delta(x,y))F(x,y)$.

Let $E(x,y) = \sum_{i\geq 0} E_i(y)x^i = \left(\frac{\eta(x,y)}{x}\right)$. Then,

$$\delta_n(y) = \sum_{0\leq k\leq n} S_{n-k}(y)\left(\sum_{0\leq r\leq k} E_r(y)E_{k-r}(y)\right)$$
$$- F_n(y) - \sum_{0\leq k\leq n} \delta_k(y)F_{n-k}(y).$$

This can be written as

$$\delta_n(y) = \sum_{0\leq k\leq n}\left\{S_{n-k}(y)\left(\sum_{0\leq r\leq k} E_r(y)E_{k-r}(y)\right) - \delta_k(y)F_{n-k}(y)\right\} - F_n(y).$$

5. From (3.79),

$$\epsilon(x,y)+1 = \left(\frac{\eta(x^2,y^2)}{x^2}\right)\frac{2}{a_2}\left(c_1\frac{\partial \boldsymbol{K}}{\partial c_1}\right)[\eta, 1+\mu, 1+\nu] - (1+\epsilon(x,y))F(x,y).$$

Let $\boldsymbol{K}_c$ denote $\frac{2}{a_2}\left(c_1\frac{\partial \boldsymbol{K}}{\partial c_1}\right)$. Compute the power series

$T(x,y) = \boldsymbol{K}_c[\eta, 1+\mu, 1+\nu]$ by composing on the terms generated for $\boldsymbol{K}_c$.

Then, $\epsilon(x,y)+1 = \left(\frac{\eta(x^2,y^2)}{x^2}\right)T(x,y) - (1+\epsilon(x,y))F(x,y)$. Then,

$$\epsilon_n(y) = \sum_{1\leq k\leq \lfloor \frac{n}{2}\rfloor} E_k(y^2)T_{n-2k}(y) - \sum_{0\leq k\leq n-1} \epsilon_k(y)F_{n-k} + T_n(y) - F_n(y).$$

6. Let $B(x,y) = B[x,\mu,\nu]$. Then, from (3.82), for $n\geq 2$,

$$B_n(y) = \sum_{k\cdot i=n} \frac{\mu(k)}{k} f_i(y^k) - \sum_{k\cdot i=n-1} \frac{\mu(k)}{k} e_i(y^k).$$

7. From (5.3), $u(x,y) = \sum_{i\geq 1} u_i(y)x^i = \sum_{i\geq 1} \frac{\mu(i)}{i} a_i \left[\log\left(\frac{1+\delta(x,y)}{1+y}\right)\right]$.

Let $w(x,y) = \log\left(\frac{1+\delta(x,y)}{1+y}\right)$, so that, from (5.15)$u_n(y) = \sum_{i\cdot j=n} \frac{\mu(i)}{i} w_j(y^i)$.

8. From (5.4),

$$v(x,y) = \sum_{i\geq 1} v_i(y)x^i = \sum_{k \text{ odd}} \frac{\mu(k)}{k} a_k \left[\log\left(\frac{1+\epsilon(x,y)}{1+y}\right)\right] - \sum_{m=2^{1+e},\, e\geq 0} \frac{a_m}{m}\left[u(x,y)\right].$$

Let $h(x,y) = \log\left(\dfrac{1+\epsilon(x,y)}{1+y}\right)$, so that, from (5.18)

$$v_n(y) = \sum_{i\cdot j=n,\ i\ \text{odd}} \frac{\mu(i)}{i} h_j(y^i) - \sum_{2|n, m\cdot i\cdot j=n} \frac{1}{m} u_i(y^m), \text{where, } m = 2^{1+e}, \text{for, } e \geq 0.$$

9. Having computed the essential power series, we now compute some auxiliary power series that are used in the remaining computation. For all $n \geq 2$,

$pqu_n(y) = p_n(y) - q_n(y) + u_n(y)$,

$stv_n(y) = s_n(y) - t_n(y) + v_n(y)$,

$dp_n(y) = \delta_n(y) - p_n(y)$,

$es_n(y) = \epsilon_n(y) - s_n(y)$,

$dqy_n(y) = \delta_n(y) - q_n(y)$ and

$ety_n(y) = \epsilon_n(y) - t_n(y)$.

Now, we pass on these into routines $E1$, $E2$, MP, MN, $M0$, $M1P$, $M1N$, V, W to compute the nth term of power series $T1(x,y)$ through $T12(x,y)$.

$E1(T1, pqu)$, $E2(T2, pqu, stv)$, $E1(T3, u)$, $E2(T4, u, v)$,

$MP(T5, dp)$, $M1P(T6, dqy)$, $MN(T7, dp, es)$, $M1N(T8, dqy, ety)$,

$V(T9, dp, dqy)$, $W(T10, dqy, ety)$, $W(T11, dp, es)$, and $M0(T12, dqy, ety)$.

10. From (5.21), for $n \geq 2$,

$$A1_n(y) = \frac{1}{2}\Bigg\{ \sum_{2\leq k\leq n} \Big\{ (pqu_{n-k} - T1_{n-k})\mu_{k-2} + (stv_{n-k} - T2_{n-k})\nu_{k-2} \Big\}$$
$$+ \frac{1}{2} \sum_{2\leq k\leq n} \Big\{ pqu_{n-k} \cdot pqu_{k-2} + stv_{n-k} \cdot stv_{k-2} - u_{n-k} \cdot u_{k-2} - v_{n-k} \cdot v_{k-2} \Big\}$$
$$+ (1+y)(T3_{n-2} + T4_{n-2} - u_{n-2} - v_{n-2}) + \mu_{n-2} + \nu_{n-2} +$$
$$+ \begin{Bmatrix} pqu_m(y^2) - u_m(y^2) & \text{if } 2|n \text{ for } m = \left(\frac{n-2}{2}\right) \\ 0 & \text{otherwise} \end{Bmatrix} \Bigg\}.$$

11. From (5.22), for $n \geq 2$,

$$A2_n(y) = \sum_{i\cdot d=n} \frac{\phi(d)}{2d} T9_i(y^d) + \frac{1}{2}(dp_{n-1} - dqy_{n-1})$$
$$+ \frac{1}{4} \sum_{2\le k\le n} (dp_{k-2}\cdot dp_{n-k} - dqy_{k-2}\cdot dqy_{n-k}) + T10_n(y) - T11_n(y) + T12_n(y)$$
$$+ \frac{1}{4} \begin{Bmatrix} dp_m(y^2) - dqy_m(y^2), & \text{if } 2|n \text{ for } m = (\frac{n-2}{2}) \\ 0, & \text{otherwise} \end{Bmatrix}.$$

12. From (5.23), for $n \ge 2$,

$$A3_n(y) = \frac{1}{2}\Bigg\{ \sum_{0\le k\le n-2} \Bigg\{ \Big(u_{n-k-2}(y) - q_{n-k-2}(y) \Big)$$
$$* \Bigg\{ \Big(\sum_{0\le r\le k} \mu_r(y) T1_{k-r}(y) \Big) - \mu_k(y) - pqu_k(y) - (1+y)T3_k(y) + u_k(y) \Bigg\}$$
$$+ \Big(v_{n-k-2}(y) - t_{n-k-2}(y) \Big)$$
$$* \Bigg\{ \Big(\sum_{0\le r\le k} \nu_r(y) T2_{k-r}(y) \Big) - \nu_k(y) - stv_k(y) - (1+y)T4_k(y) + v_k(y) \Bigg\} \Bigg\}$$
$$+ (1+y)(u_{n-2}(y) - q_{n-2}(y) + v_{n-2}(y) - t_{n-2}(y)) \Bigg\}.$$

13. From (5.24), for $n \ge 2$,

$$A4_n(y) = \frac{1}{2}\Bigg\{ \sum_{0\le k\le n-2} \Bigg\{ \Bigg\{ \sum_{0\le r\le k} \mu_r(y)\cdot T1_{k-r}(y) - \mu_k(y) - pqu_k(y) \Bigg\} * T5_{n-k-2}(y)$$
$$+ \Bigg\{ \sum_{0\le r\le k} \nu_r(y)\cdot T2_{k-r}(y) - \nu_k(y) - stv_k(y) \Bigg\} * T7_{n-k-2}(y)$$
$$- \Big\{ (1+y)T3_{n-k-2}(y) - u_{n-k-2}(y) \Big\} * T6_k(y)$$
$$- \Big\{ (1+y)T4_{n-k-2}(y) - v_{n-k-2}(y) \Big\} * T8_k(y) \Bigg\}$$
$$+ (1+y)(T6_{n-2}(y) + T8_{n-2}(y)) \Bigg\}.$$

14. From (5.25), for $n \ge 2$,

$$A5_n(y) = \frac{1}{2}\Bigg\{ \sum_{2\le k\le n} \Big\{ (u_{k-2}(y) - q_{k-2}(y)) * (T5_{n-k}(y) - T6_{n-k}(y))$$

$$+ (v_{k-2}(y) - t_{k-2}(y)) * (T7_{n-k}(y) - T8_{n-k}(y))\Big\}\Bigg\}.$$

15. From (5.20), for $n \geq 2$,

$$\tilde{B}_n(y) = B_n(y) + A1_n(y) + A2_n(y) + A3_n(y) + A4_n(y) + A5_n(y) - \frac{1}{2}\Big\{\mu_{n-2}(y) + \nu_{n-2}(y)\Big\}. \tag{5.26}$$

CHAPTER 6

Counting unlableled minimally 2–edge–connected blocks

§6.1 Initial derivations

In this section, we first derive $q(x,y)$ and $t(x,y)$. Then using $p(x,y)$ and $s(x,y)$ derived in 3rd chapter, we derive equations to express $\mu(x,y)$ in terms of $p(x,y)$ and $q(x,y)$ and $\nu(x,y)$ in terms of $s(x,y)$ and $t(x,y)$.

§6.1.1 $\check{S}^{+}[x,y,y] = q(x,y)$

Recall from (3.69),

$$\check{S}^{+} = M^{+}[a_1, b_1 + \check{D}^{+} - \check{S}^{+}, c_1 + \check{D}^{-} - \check{S}^{-}] - M_0^{+}[a_1, \check{D}^{+} - \check{S}^{+}, \check{D}^{-} - \check{S}^{-}].$$

Substituting for M^{+} from (3.35) and M_0^{+} from (3.38),

$$\check{S}^{+} = \frac{a_1 b_1^2}{1 - a_1 b_1}[a_1, b_1 + \check{D}^{+} - \check{S}^{+}, c_1 + \check{D}^{-} - \check{S}^{-}]$$
$$- \frac{b_1^{*}(2a_1 b_1 - a_1^2 b_1^2)}{(1 - a_1 b_1)^2}[a_1, \check{D}^{+} - \check{S}^{+}, \check{D}^{+} - \check{S}^{-}].$$

Expanding, $\check{S}^{+} = \dfrac{a_1(b_1 + \check{D}^{+} - \check{S}^{+})^2}{1 - a_1(b_1 + \check{D}^{+} - \check{S}^{+})} - \dfrac{b_1(2a_1(\check{D}^{+} - \check{S}^{+}) - a_1^2(\check{D}^{+} - \check{S}^{+})^2)}{(1 - a_1(\check{D}^{+} - \check{S}^{+}))^2}.$

$$\check{S}^{+}(1 - a_1(b_1 + \check{D}^{+} - \check{S}^{+}))(1 - a_1(\check{D}^{+} - \check{S}^{+}))^2$$
$$= a_1(b_1 + \check{D}^{+} - \check{S}^{+})^2(1 - a_1(\check{D}^{+} - \check{S}^{+}))^2$$
$$- b_1(1 - a_1(b_1 + \check{D}^{+} - \check{S}^{+}))\Big\{2a_1(\check{D}^{+} - \check{S}^{+}) - a_1^2(\check{D}^{+} - \check{S}^{+})^2\Big\}.$$

$$\check{S}^{+} = \check{S}^{+}\Big\{(1 - a_1(b_1 + \check{D}^{+} - \check{S}^{+})) * (2a_1(\check{D}^{+} - \check{S}^{+}) - a_1^2(\check{D}^{+} - \check{S}^{+})^2)\Big\}$$
$$+ a_1\check{S}^{+}(b_1 + \check{D}^{+} - \check{S}^{+})$$
$$+ a_1(b_1 + \check{D}^{+} - \check{S}^{+})^2\Big\{1 - 2a_1(\check{D}^{+} - \check{S}^{+}) + a_1^2(\check{D}^{+} - \check{S}^{+})^2\Big\}$$

$$+ (1 - a_1(b_1 + \check{D}^+ - \check{S}^+)) * \Big\{ a_1^2 b_1 (\check{D}^+ - \check{S}^+)^2 - 2a_1 b_1 (\check{D}^+ - \check{S}^+) \Big\}$$

$$= 2a_1^2(\check{D}^+ - \check{S}^+)(b_1 + \check{D}^+ - \check{S}^+)\Big\{ -\check{S}^+ - b_1 - \check{D}^+ + \check{S}^+ \Big\}$$

$$+ a_1^3(\check{D}^+ - \check{S}^+)^2(b_1 + \check{D}^+ - \check{S}^+)\Big\{ \check{S}^+ + b_1 + \check{D}^+ - \check{S}^+ \Big\}$$

$$+ 2a_1\check{S}^+(\check{D}^+ - \check{S}^+) - a_1^2\check{S}^+(\check{D}^+ - \check{S}^+)^2 + a_1(b_1 + \check{D}^+ - \check{S}^+)^2$$

$$+ a_1\check{S}^+(b_1 + \check{D}^+ - \check{S}^+)$$

$$+ (1 - a_1(b_1 + \check{D}^+ - \check{S}^+))\Big\{ a_1^2 b_1 (\check{D}^+ - \check{S}^+)^2 - 2a_1 b_1 (\check{D}^+ - \check{S}^+) \Big\}$$

$$= 2a_1^2(\check{D}^+ - \check{S}^+)(b_1 + \check{D}^+ - \check{S}^+)\Big\{ -b_1 - \check{D}^+ + b_1 \Big\}$$

$$+ a_1^3(\check{D}^+ - \check{S}^+)^2(b_1 + \check{D}^+ - \check{S}^+)\Big\{ b_1 + \check{D}^+ - b_1 \Big\}$$

$$+ 2a_1\check{S}^+(\check{D}^+ - \check{S}^+) - a_1^2\check{S}^+(\check{D}^+ - \check{S}^+)^2$$

$$+ a_1(b_1 + \check{D}^+ - \check{S}^+) * (b_1 + \check{D}^+ - \check{S}^+ + \check{S}^+)$$

$$+ a_1^2 b_1(\check{D}^+ - \check{S}^+)^2 - 2a_1 b_1(\check{D}^+ - \check{S}^+)$$

$$= a_1^3(\check{D}^+ - \check{S}^+)^2(b_1 + \check{D}^+ - \check{S}^+)\check{D}^+ - 2a_1^2(\check{D}^+ - \check{S}^+)(b_1 + \check{D}^+ - \check{S}^+)\check{D}^+$$

$$- 2a_1(\check{D}^+ - \check{S}^+)(b_1 - \check{S}^+) + a_1^2(\check{D}^+ - \check{S}^+)^2(b_1 - \check{S}^+)$$

$$+ a_1(\check{D}^+ - \check{S}^+ + b_1)(b_1 + \check{D}^+)$$

$$= a_1^3(\check{D}^+ - \check{S}^+)^2(b_1 + \check{D}^+ - \check{S}^+)\check{D}^+$$

$$+ a_1^2(\check{D}^+ - \check{S}^+)(b_1 - \check{S}^+)\Big\{ \check{D}^+ - \check{S}^+ - 2\check{D}^+ \Big\}$$

$$- 2a_1^2\check{D}^{+2}(\check{D}^+ - \check{S}^+) - 2a_1(\check{D}^+ - \check{S}^+)(b_1 - \check{S}^+)$$

$$+ a_1(\check{D}^+ - \check{S}^+)(b_1 + \check{D}^+) + a_1 b_1(b_1 + \check{D}^+)$$

$$= a_1^3(\check{D}^+ - \check{S}^+)^2(b_1 + \check{D}^+ - \check{S}^+)\check{D}^+ - a_1^2(b_1 - \check{S}^+)(\check{D}^{+2} - \check{S}^{+2})$$

$$- 2a_1^2\check{D}^{+2}(\check{D}^+ - \check{S}^+) + a_1(\check{D}^+ - \check{S}^+)\Big\{ -2b_1 + 2\check{S}^+ + b_1 + \check{D}^+ \Big\} + a_1 b_1(b_1 + \check{D}^+)$$

$$= a_1^3(\check{D}^+ - \check{S}^+)^2(b_1 + \check{D}^+ - \check{S}^+)\check{D}^+ - a_1^2\check{D}^{+2}(b_1 - \check{S}^+ + 2\check{D}^+ - 2\check{S}^+)$$

$$+ a_1^2\check{S}^{+2}(b_1 - \check{S}^+) + a_1(\check{D}^+ - \check{S}^+)(-b_1) + a_1 b_1(b_1 + \check{D}^+)$$

$$+ a_1(\check{D}^+ - \check{S}^+)(2\check{S}^+ + \check{D}^+)$$

$$= a_1^3(\check{D}^+ - \check{S}^+)^2(b_1 + \check{D}^+ - \check{S}^+)\check{D}^+ - a_1^2\check{D}^{+2}(b_1 + 2\check{D}^+ - 3\check{S}^+)$$

$$+ a_1^2\check{S}^{+2}(b_1 - \check{S}^+)$$

$$- a_1b_1\check{D}^+ + a_1b_1\check{D}^+ + a_1b_1\check{S}^+ + a_1b_1^2 + a_1(\check{D}^+ - \check{S}^+)(2\check{S}^+ + \check{D}^+)$$

$$= a_1^3(\check{D}^+ - \check{S}^+)^2(b_1 + \check{D}^+ - \check{S}^+)\check{D}^+ - a_1^2\check{D}^{+2}(b_1 + 2\check{D}^+ - 3\check{S}^+)$$

$$+ a_1^2\check{S}^{+2}(b_1 - \check{S}^+)$$

$$+ a_1(\check{D}^+ - \check{S}^+)(2\check{S}^+ + \check{D}^+) + a_1b_1\check{S}^+ + a_1b_1^2.$$

Composing with $[x, y, y]$ on both sides, and letting $\check{S}^+[x, y, y] = q(x, y)$,

$$q(x, y) = x^3(\delta - q)^2(y + \delta - q)\delta - x^2\delta^2(y + 2\delta - 3q)$$

$$+ x^2q^2(y - q) + x(\delta - q)(2q + \delta) + xyq(x, y) + xy^2.$$

Expanding and collecting the coefficients, gives

$$q(x, y) = x\Big\{\delta^2 + \delta q - 2q^2 + yq + y^2\Big\} + x^2\Big\{3\delta^2 q - 2\delta^3 - q^3 - y\delta^2 + yq^2\Big\}$$

$$+ x^3\Big\{\delta^4 - 3\delta^3 q + 3\delta^2 q^2 - \delta q^3 + y\delta^3 - 2y\delta^2 q + y\delta q^2\Big\}$$

$$= x\Big\{\delta(\delta + q) - 2q^2\Big\} + xyq(x, y) + xy^2 + x^2\Big\{\delta^2(3q - 2\delta) - q^3 + y(q^2 - \delta^2)\Big\}$$

$$+ x^3\Big\{\delta^3(\delta - 3q) + \delta q^2(3\delta - q) + y\delta(\delta^2 - 2\delta q + q^2)\Big\}.$$

Then, $q_0(y) = 0$,

$$q_1(y) = \delta_0(y)^2 + \delta_0(y)q_0(y) - 2q_0(y)^2 + yq_0(y) + y^2 = \delta_0(y)^2 + y^2$$

and for $n > 1$,

$$q_n(y) = yq_{n-1}(y) + \sum_{1 \le k \le n} \Big\{\delta_{k-1}(y)\Big\{\delta_{n-k}(y) + q_{n-k}(y)\Big\} - 2q_{k-1}(y)q_{n-k}(y)\Big\}$$

$$+ \sum_{2 \le k \le n} \Big\{\Big\{3q_{n-k}(y) - 2\delta_{n-k}(y)\Big\} * \sum_{0 \le r \le k-2} \delta_r(y)\delta_{k-2-r}(y)$$

$$- q_{n-k}(y) * \sum_{0 \le r \le k-2} q_r(y)q_{k-2-r}(y) + y\Big\{q_{n-k}(y)q_{k-2}(y) - \delta_{n-k}(y)\delta_{k-2}(y)\Big\}\Big\}$$

$$+ \sum_{3 \le k \le n} \Big\{\Big\{\delta_{n-k}(y) - 3q_{n-k}(y)\Big\} * \sum_{0 \le r \le k-3} \delta_{k-3-r}(y) \sum_{0 \le i \le r} \delta_i(y)\delta_{r-i}(y)$$

$$+\Big\{3\delta_{n-k}(y)-q_{n-k}(y)\Big\}*\sum_{0\le r\le k-3}\delta_{k-3-r}(y)\sum_{0\le i\le r}q_i(y)q_{r-i}(y)$$

$$+y\delta_{n-k}(y)\Big\{\sum_{0\le r\le k-3}\delta_{k-3-r}(y)\delta_r(y)-2\delta_{k-3-r}(y)q_r(y)+q_{k-3-r}(y)q_r(y)\Big\}\Big\}. \tag{6.1}$$

§6.1.2 $\check{S}^-[x,y,y]=t(x,y)$

On the other hand, recall from (3.70),

$$\check{S}^- = M^-[a_1, b_1+\check{D}^+-\check{S}^+, c_1+\check{D}^--\check{S}^-] - M_0^-[a_1, \check{D}^+-\check{S}^+, \check{D}^--\check{S}^-].$$

Substituting for M^- from (3.36) and M_0^- from (3.39) gives,

$$\check{S}^- = \left(\frac{a_1b_2+a_2b_2c_1}{1-a_2b_2}\right)[a_1, b_1+\check{D}^+-\check{S}^+, c_1+\check{D}^--\check{S}^-]$$
$$-\left(\frac{c_1^*a_2b_2}{1-a_2b_2}\right)[a_1, \check{D}^+-\check{S}^+, \check{D}^--\check{S}^-].$$

Expanding,

$$\check{S}^- = \frac{a_1(b_2+\check{D}_2^+-\check{S}_2^+)+a_2(b_2+\check{D}_2^+-\check{S}_2^+)(c_1+\check{D}^--\check{S}^-)}{1-a_2(b_2+\check{D}_2^+-\check{S}_2^+)}$$
$$-\frac{c_1a_2(\check{D}_2^+-\check{S}_2^+)}{1-a_2(\check{D}_2^+-\check{S}_2^+)}.$$

$$\check{S}^-(1-a_2(b_2+\check{D}_2^+-\check{S}_2^+))*(1-a_2(\check{D}_2^+-\check{S}_2^+))$$
$$=(1-a_2(\check{D}_2^+-\check{S}_2^+))*\Big\{a_1(b_2+\check{D}_2^+-\check{S}_2^+)+a_2(b_2+\check{D}_2^+-\check{S}_2^+)(c_1+\check{D}^--\check{S}^-)\Big\}$$
$$-\Big\{1-a_2(b_2+\check{D}_2^+-\check{S}_2^+)\Big\}*\Big\{c_1a_2(\check{D}_2^+-\check{S}_2^+)\Big\}.$$

$$\check{S}^- = \check{S}^-\Big\{a_2(b_2+\check{D}_2^+-\check{S}_2^+)+a_2(\check{D}_2^+-\check{S}_2^+)-a_2^2(\check{D}_2^+-\check{S}_2^+)(b_2+\check{D}_2^+-\check{S}_2^+)\Big\}$$
$$+a_2(b_2+\check{D}_2^+-\check{S}_2^+)*(c_1+\check{D}^--\check{S}^-)$$
$$-a_2^2(\check{D}_2^+-\check{S}_2^+)*(b_2+\check{D}_2^+-\check{S}_2^+)*(c_1+\check{D}^--\check{S}^-)$$
$$+(1-a_2(\check{D}_2^+-\check{S}_2^+))a_1(b_2+\check{D}_2^+-\check{S}_2^+)$$
$$-(1-a_2(b_2+\check{D}_2^+-\check{S}_2^+))c_1a_2(\check{D}_2^+-\check{S}_2^+)$$
$$=a_2(b_2+\check{D}_2^+-\check{S}_2^+)(c_1+\check{D}^--\check{S}^-+\check{S}^-)$$

$$- a_2^2(\check{D}_2^+ - \check{S}_2^+)(b_2 + \check{D}_2^+ - \check{S}_2^+)(c_1 + \check{D}^- - \check{S}^- + \check{S}^-)$$

$$+ a_2(\check{D}_2^+ - \check{S}_2^+)\check{S}^- + a_1(b_2 + \check{D}_2^+ - \check{S}_2^+) - a_1 a_2(\check{D}_2^+ - \check{S}_2^+)(b_2 + \check{D}_2^+ - \check{S}_2^+)$$

$$- a_2 c_1(\check{D}_2^+ - \check{S}_2^+) + a_2^2 c_1(\check{D}_2^+ - \check{S}_2^+)(b_2 + \check{D}_2^+ - \check{S}_2^+)$$

$$= a_2^2(\check{D}_2^+ - \check{S}_2^+)(b_2 + \check{D}_2^+ - \check{S}_2^+)\Big\{-c_1 - \check{D}^- + c_1\Big\}$$

$$- a_1 a_2(\check{D}_2^+ - \check{S}_2^+)(b_2 + \check{D}_2^+ - \check{S}_2^+) + a_2 b_2(c_1 + \check{D}^-)$$

$$+ a_2(\check{D}_2^+ - \check{S}_2^+)c_1 + a_2(\check{D}_2^+ - \check{S}_2^+)\check{D}^- - a_2 c_1(\check{D}_2^+ - \check{S}_2^+)$$

$$+ a_2\check{S}^-(\check{D}_2^+ - \check{S}_2^+) + a_1(b_2 + \check{D}_2^+ - \check{S}_2^+)$$

$$= -a_2^2(\check{D}_2^+ - \check{S}_2^+)(b_2 + \check{D}_2^+ - \check{S}_2^+)\check{D}^- - a_1 a_2(\check{D}_2^+ - \check{S}_2^+)(b_2 + \check{D}_2^+ - \check{S}_2^+)$$

$$+ a_2\Big\{b_2(c_1 + \check{D}^-) + (\check{D}^- + \check{S}^-)(\check{D}_2^+ - \check{S}_2^+)\Big\} + a_1(b_2 + \check{D}_2^+ - \check{S}_2^+).$$

Composing with $[x,y,y]$ and letting $\check{S}^-[x,y,y] = t(x,y)$,

$$t(x,y) = -x^4(\delta_2 - q_2)(y^2 + \delta_2 - q_2)\epsilon(x,y) - x^3(\delta_2 - q_2)(y^2 + \delta_2 - q_2)$$

$$+ x^2\Big\{y^2(y + \epsilon(x,y)) + (\epsilon + t)(\delta_2 - q_2)\Big\} + x(y^2 + \delta_2 - q_2)$$

$$= xy^2 + x(\delta_2 - q_2) + x^2\Big\{(\epsilon + t)(\delta_2 - q_2) + y^2\epsilon + y^3\Big\}$$

$$+ x^3\Big\{y^2(q_2 - \delta_2) - (q_2 - \delta_2)^2\Big\} + x^4\Big\{y^2\epsilon(q_2 - \delta_2) - \epsilon(q_2 - \epsilon_2)^2\Big\}.$$

Then, $t_0(y) = 0$,

$$t_1(y) = y^2 + \delta_0(y^2) - q_0(y^2) = y^2 + \delta_0(y^2).$$

$$t_2(y) = \Big\{\epsilon_0(y) + t_0(y)\Big\} * \Big\{\delta_0(y^2) - q_0(y^2)\Big\} + y^2\epsilon_0(y) + y^3$$

$$= \epsilon_0(y)\delta_0(y^2) + y^2\epsilon_0(y) + y^3.$$

Let $m = \begin{cases} \frac{n}{2}, & \text{n is even} \\ \frac{n-1}{2}, & \text{n is odd.} \end{cases}$

For $n > 3$,

$$t_n(y) = \sum_{1 \le k \le m}\Big\{\Big\{\epsilon_{n-2k}(y) + t_{n-2k}(y)\Big\} * \Big\{\delta_{k-1}(y^2) - q_{k-1}(y^2)\Big\}\Big\} + y^2\epsilon_{n-2}(y)$$

$$+\sum_{2\le k\le m}\Big\{y^2\epsilon_{n-2k}(y)*\Big\{(q_{k-2}(y^2)-\delta_{k-2}(y^2)\Big\}$$
$$-\,\epsilon_{n-2k}(y)*\sum_{0\le r\le k-2}\Big\{q_r(y^2)-\delta_r(y^2)\Big\}*\Big\{q_{k-2-r}(y^2)-\delta_{k-2-r}(y^2)\Big\}\Big\}$$
$$+\left\{\begin{array}{cr} 0, & \text{if n is even} \\ \delta_m(y^2)-q_m(y^2)+y^2\Big\{q_{m-1}(y^2)-\delta_{m-1}(y^2)\Big\} & \\ -\sum_{1\le k\le m}\Big\{q_{k-1}(y^2)-\delta_{k-1}(y^2)\Big\}*\Big\{q_{m-k}(y^2-\delta_{m-k}(y^2)\Big\}, & \text{Otherwise}\end{array}\right\}. \tag{6.2}$$

§6.1.3 $\check{D}^+ - \check{P}^+$

Now, $\check{P}^+ = N^+[a_1, \check{D}^+ - \check{P}^+, \check{D}^- - \check{P}^-]$.

Proceeding just as in the case of minimally 2–connected graphs gives,

$$(\check{D}^+ - \check{P}^+)[x,y,y] = \sum_{i\ge 1}\frac{\mu(i)}{i}a_i[\log(1+\delta(x,y))]. \tag{6.3}$$

§6.1.4 $\check{D}^- - \check{P}^-$

Also, $\check{P}^- = N^-[a_1, \check{D}^+ - \check{P}^+, \check{D}^- - \check{P}^-]$.

Proceeding just as in the case of minimally 2–connected graphs gives,

$$(\check{D}^- - \check{P}^-)[x,y,y] = \sum_{k\text{ odd}}\frac{\mu(k)}{k}a_k[\log(1+\epsilon(x,y))]$$
$$-\sum_{m=2^{1+e},\ e\ge 0}\frac{1}{m}\sum_{i\ge 1}\frac{\mu(i)}{i}a_{im}[\log(1+\delta(x,y))]. \tag{6.4}$$

§6.1.5 $\mu(x,y)$

Now, rearranging (3.67) gives

$$T^+[a_1,\check{D}^+,\check{D}^-] = \check{D}^+ - N^+[a_1,\check{D}^+ - \check{P}^+,\check{D}^- - \check{P}^-]$$
$$-\,M^+[a_1, b_1+\check{D}^+ - \check{S}^+, c_1+\check{D}^- - \check{S}^-]$$
$$+\,M_0^+[a_1,\check{D}^+ - \check{S}^+,\check{D}^- - \check{S}^-].$$

Applying (3.71) and (3.69), on RHS of this gives

$T^+[a_1, \check{D}^+, \check{D}^-] = \check{D}^+ - \check{P}^+ - \check{S}^+.$

Composing this with $[x, y, y]$, and applying (3.75), gives

$$T^+[x, \delta, \epsilon] = (\check{D}^+ - \check{P}^+)[x, y, y] - \check{S}^+[x, y, y]. \tag{6.5}$$

Applying this with (4.5), gives

$$(D^+ - P^+)[x, \mu, \nu] - \mu(x, y) - S^+[x, \mu, \nu] = (\check{D}^+ - \check{P}^+)[x, y, y] - \check{S}^+[x, y, y]. \tag{6.6}$$

$$\therefore \check{S}^+[x, y, y] - S^+[x, \mu, \nu] = (\check{D}^+ - \check{P}^+)[x, y, y] - \left\{(D^+ - P^+)[x, \mu, \nu] - \mu(x, y)\right\}.$$

Since the expression for $(\check{D}^+ - \check{P}^+)[x, y, y]$ looks the same as $(\hat{D}^+ - \hat{P}^+)[x, y, y]$, proceeding just as in the case of minimally 2–connected graphs gives

$$1 + \mu(x, y) = \exp\left(\sum_{i \geq 1} \frac{a_i}{i}[q(x, y) - p(x, y)]\right). \tag{6.7}$$

Thus $\mu_n(y)$ can be computed applying $E1(1 + \mu(x, y), q(x, y) - p(x, y))$ with $\mu_0(y) = 0$ and $\mu_1(y) = q_1(y) - p_1(y)$.

§6.1.6 $\nu(x, y)$

Also, rearranging (3.68) gives

$$\begin{aligned} T^-[a_1, \check{D}^+, \check{D}^-] = \check{D}^- &- N^-[a_1, \check{D}^+ - \check{P}^+, \check{D}^- - \check{P}^-] \\ &- M^-[a_1, b_1 + \check{D}^+ - \check{S}^+, c_1\check{D}^- - \check{S}^-] \\ &+ M_0^-[a_1, \check{D}^+ - \check{S}^+, \check{D}^- - \check{S}^-]. \end{aligned}$$

Applying (3.72) and (3.70), on RHS of this gives

$T^-[a_1, \check{D}^+, \check{D}^-] = \check{D}^- - \check{P}^- - \check{S}^-.$

Composing this with $[x, y, y]$, and applying (3.75) gives

$$T^-[x, \delta, \epsilon] = (\check{D}^- - \check{P}^-)[x, y, y] - \check{S}^-[x, y, y]. \tag{6.8}$$

Applying this with (4.9), gives

$$(D^- - P^-)[x,\mu,\nu] - \nu(x,y) - S^-[x,\mu,\nu] = (\check{D}^- - \check{P}^-)[x,y,y] - \check{S}^-[x,y,y]. \tag{6.9}$$

$$\therefore\ \check{S}^-[x,y,y] - S^-[x,\mu,\nu] = (\check{D}^- - \check{P}^-)[x,y,y] - \Big\{(D^- - P^-)[x,\mu,\nu] - \nu(x,y)\Big\}.$$

Since the expression for $(\check{D}^- - \check{P}^-)[x,y,y]$ looks the same as $(\hat{D}^- - \hat{P}^-)[x,y,y]$, proceeding just as in the case of Minimally 2–connected graphs, gives

$$1 + \nu(x,y) = \exp\left(\sum_{i \text{ odd}} \frac{a_i}{i}[t(x,y) - s(x,y)] + \sum_{i \text{ even}} \frac{a_i}{i}[q(x,y) - p(x,y)]\right). \tag{6.10}$$

Thus $\nu_n(y)$ can be computed applying

$E2(1 + \nu(x,y), q(x,y) - p(x,y), t(x,y) - s(x,y))$

with $\nu_0(y) = 0$ and $\nu_1(y) = t_1(y) - s_1(y)$.

§6.2 Main counting equation

Now, applying Otter's dissimilarity characteristic equation (3.16) as in the case of 2–connected graphs (3.74), the decomposition of minimally 2–edge–connected blocks can be written as :

$$\begin{aligned}
\check{B} &= T[a_1, \check{D}^+, \check{D}^-] + N[a_1, \check{D}^+ - \check{P}^+, \check{D}^- - \check{P}^-] \\
&\quad + M[a_1, b_1 + \check{D}^+ - \check{S}^+, c_1 + \check{D}^- - \check{S}^-] - M_0[a_1, \check{D}^+ - \check{S}^+, \check{D}^- - \check{S}^-] \\
&\quad - \Big\{\vec{T}[a_1, \check{D}^+, \check{D}^-] \diamond \vec{N}[a_1, \check{D}^+ - \check{P}^+, \check{D}^- - \check{P}^-] \\
&\quad + \vec{N}[a_1, \check{D}^+ - \check{P}^+, \check{D}^- - \check{P}^-] \diamond \vec{M}[a_1, b_1 + \check{D}^+ - \check{S}^+, c_1 + \check{D}^- - \check{S}^-] \\
&\quad - \vec{N}[a_1, \check{D}^+ - \check{P}^+, \check{D}^- - \check{P}^-] \diamond \vec{M}_0[a_1, \check{D}^+ - \check{S}^+, \check{D}^- - \check{S}^-] \\
&\quad + \vec{T}[a_1, \check{D}^+, \check{D}^-] \diamond \vec{M}[a_1, b_1 + \check{D}^+ - \check{S}^+, c_1\check{D}^- - \check{S}^-] \\
&\quad - \vec{T}[a_1, \check{D}^+, \check{D}^-] \diamond \vec{M}_0[a_1, \check{D}^+ - \check{S}^+, \check{D}^- - \check{S}^-] \\
&\quad + \frac{1}{2}\vec{T}[a_1, \check{D}^+, \check{D}^-] \diamond \vec{T}[a_1, \check{D}^+, \check{D}^-] \\
&\quad + \left(\frac{a_1^2 + a_2}{4}\right) a_2 \left[T^+[a_1, \check{D}^+, \check{D}^-]\right]\Big\} + \left(\frac{a_1^2 + a_2}{2}\right) a_2 \left[T^+[a_1, \check{D}^+, \check{D}^-]\right].
\end{aligned}$$

Composing on both sides with $[x, y, y]$ and applying (3.75), we get,

$$
\begin{aligned}
\check{B}[x,y,y] = {} & T[x,\delta,\epsilon] + N\left[x, (\check{D}^{+} - \check{P}^{+})[x,y,y], (\check{D}^{-} - \check{P}^{-})[x,y,y]\right] \\
& + M\left[x, y+\delta - \check{S}^{+}[x,y,y], y+\epsilon - \check{S}^{-}[x,y,y]\right] \\
& - M_0\left[x, \delta - \check{S}^{+}[x,y,y], \epsilon - \check{S}^{-}[x,y,y]\right] \\
& - \Big\{ \vec{T}[x,\delta,\epsilon] \diamond \vec{N}\left[x, (\check{D}^{+} - \check{P}^{+})[x,y,y], (\check{D}^{-} - \check{P}^{-})[x,y,y]\right] \\
& + \vec{N}\left[x, (\check{D}^{+} - \check{P}^{+})[x,y,y], (\check{D}^{-} - \check{P}^{-})[x,y,y]\right] \\
& \diamond \vec{M}\left[x, y+\delta - \check{S}^{+}[x,y,y], y+\epsilon - \check{S}^{-}[x,y,y]\right] \\
& - \vec{N}\left[x, (\check{D}^{+} - \check{P}^{+})[x,y,y], (\check{D}^{-} - \check{P}^{-})[x,y,y]\right] \\
& \diamond \vec{M}_0\left[x, \delta - \check{S}^{+}[x,y,y], \epsilon - \check{S}^{-}[x,y,y]\right] \\
& + \vec{T}[x,\delta,\epsilon] \diamond \vec{M}\left[x, y+\delta - \check{S}^{+}[x,y,y], y+\epsilon - \check{S}^{-}[x,y,y]\right] \\
& - \vec{T}[x,\delta,\epsilon] \diamond \vec{M}_0\left[x, \delta - \check{S}^{+}[x,y,y], \epsilon - \check{S}^{-}[x,y,y]\right] \\
& + \frac{1}{2}\vec{T}[x,\delta,\epsilon] \diamond \vec{T}[x,\delta,\epsilon] + \frac{x^2}{2} a_2\left[T^{+}[x,\delta,\epsilon]\right] \Big\} + x^2 a_2\left[T^{+}[x,\delta,\epsilon]\right].
\end{aligned}
$$

Now, applying (3.76) on RHS of this gives

$$
\begin{aligned}
\check{B}[x,y,y] = {} & B[x,\mu,\nu] - N_1\left[x, (D^{+} - P^{+})[x,\mu,\nu] - \mu, (D^{-} - P^{-})[x,\mu,\nu] - \nu\right] \\
& - M\left[x, \delta - S^{+}[x,\mu,\nu], \epsilon - S^{-}[x,\mu,\nu]\right] \\
& + \vec{T}[x,\delta,\epsilon] \diamond \vec{N}_1\left[x, (D^{+} - P^{+})[x,\mu,\nu] - \mu, (D^{-} - P^{-})[x,\mu,\nu] - \nu\right] \\
& + \vec{N}_1\left[x, (D^{+} - P^{+})[x,\mu,\nu] - \mu, (D^{-} - P^{-})[x,\mu,\nu] - \nu\right] \\
& \diamond \vec{M}\left[x, \delta - S^{+}[x,\mu,\nu], \epsilon - S^{-}[x,\mu,\nu]\right] \\
& + \vec{T}[x,\delta,\epsilon] \diamond \vec{M}\left[x, \delta - S^{+}[x,\mu,\nu], \epsilon - S^{-}[x,\mu,\nu]\right] \\
& + N\left[x, (\check{D}^{+} - \check{P}^{+})[x,y,y], (\check{D}^{-} - \check{P}^{-})[x,y,y]\right] \\
& + M\left[x, y+\delta - \check{S}^{+}[x,y,y], y+\epsilon - \check{S}^{-}[x,y,y]\right] \\
& - M_0\left[x, \delta - \check{S}^{+}[x,y,y], \epsilon - \check{S}^{-}[x,y,y]\right] \\
& - \vec{T}[x,\delta,\epsilon] \diamond \vec{N}\left[x, (\check{D}^{+} - \check{P}^{+})[x,y,y], (\check{D}^{-} - \check{P}^{-})[x,y,y]\right]
\end{aligned}
$$

$$
\begin{aligned}
&- \overrightarrow{N}\left[x, (\check{D}^+ - \check{P}^+)[x,y,y], (\check{D}^- - \check{P}^-)[x,y,y]\right] \\
&\diamond \overrightarrow{M}\left[x, y+\delta - \check{S}^+[x,y,y], y+\epsilon - \check{S}^-[x,y,y]\right] \\
&+ \overrightarrow{N}\left[x, (\check{D}^+ - \check{P}^+)[x,y,y], (\check{D}^- - \check{P}^-)[x,y,y]\right] \\
&\diamond \overrightarrow{M}_0\left[x, \delta - \check{S}^+[x,y,y], \epsilon - \check{S}^-[x,y,y]\right] \\
&- \overrightarrow{T}[x,\delta,\epsilon] \diamond \overrightarrow{M}\left[x, y+\delta - \check{S}^+[x,y,y], y+\epsilon - \check{S}^-[x,y,y]\right] \\
&+ \overrightarrow{T}[x,\delta,\epsilon] \diamond \overrightarrow{M}_0\left[x, \delta - \check{S}^+[x,y,y], \epsilon - \check{S}^-[x,y,y]\right] \\
&- \frac{x^2}{2}(\mu(x,y) + \nu(x,y)). \qquad (6.11)
\end{aligned}
$$

Rearranging (6.11), gives

$$
\begin{aligned}
\check{B}[x,y,y] = {} & B[x,\mu,\nu] + N\left[x, (\check{D}^+ - \check{P}^+)[x,y,y], (\check{D}^- - \check{P}^-)[x,y,y]\right] \\
&- N_1\left[x, (D^+ - P^+)[x,\mu,\nu] - \mu, (D^- - P^-)[x,\mu,\nu] - \nu\right] \\
&+ M\left[x, y+\delta - \check{S}^+[x,y,y], y+\epsilon - \check{S}^-[x,y,y]\right] \\
&- M_0\left[x, \delta - \check{S}^+[x,y,y], \epsilon - \check{S}^-[x,y,y]\right] \\
&- M\left[x, \delta - S^+[x,\mu,\nu], \epsilon - S^-[x,\mu,\nu]\right] \\
&+ \overrightarrow{T}[x,\delta,\epsilon] \diamond \overrightarrow{N}_1\left[x, (D^+ - P^+)[x,\mu,\nu] - \mu, (D^- - P^-)[x,\mu,\nu] - \nu\right] \\
&- \overrightarrow{T}[x,\delta,\epsilon] \diamond \overrightarrow{N}\left[x, (\check{D}^+ - \check{P}^+)[x,y,y], (\check{D}^- - \check{P}^-)[x,y,y]\right] \\
&+ \overrightarrow{N}_1\left[x, (D^+ - P^+)[x,\mu,\nu] - \mu, (D^- - P^-)[x,\mu,\nu] - \nu\right] \\
&\diamond \overrightarrow{M}\left[x, \delta - S^+[x,\mu,\nu], \epsilon - S^-[x,\mu,\nu]\right] \\
&- \overrightarrow{N}\left[x, (\check{D}^+ - \check{P}^+)[x,y,y], (\check{D}^- - \check{P}^-)[x,y,y]\right] \\
&\diamond \overrightarrow{M}\left[x, y+\delta - \check{S}^+[x,y,y], y+\epsilon - \check{S}^-[x,y,y]\right] \\
&+ \overrightarrow{N}\left[x, (\check{D}^+ - \check{P}^+)[x,y,y], (\check{D}^- - \check{P}^-)[x,y,y]\right] \\
&\diamond \overrightarrow{M}_0\left[x, \delta - \check{S}^+[x,y,y], \epsilon - \check{S}^-[x,y,y]\right] \\
&+ \overrightarrow{T}[x,\delta,\epsilon] \diamond \overrightarrow{M}\left[x, \delta - S^+[x,\mu,\nu], \epsilon - S^-[x,\mu,\nu]\right] \\
&- \overrightarrow{T}[x,\delta,\epsilon] \diamond \overrightarrow{M}\left[x, y+\delta - \check{S}^+[x,y,y], y+\epsilon - \check{S}^-[x,y,y]\right]
\end{aligned}
$$

$$+\overrightarrow{T}[x,\delta,\epsilon]\diamond \overrightarrow{M}_0\left[x,\delta-\check{S}^{+}[x,y,y],\epsilon-\check{S}^{-}[x,y,y]\right]$$
$$-\frac{x^2}{2}(\mu(x,y)+\nu(x,y)). \qquad (6.12)$$

§6.3 Final counting series

Now, again let $u(x,y)=(\check{D}^{+}-\check{P}^{+})[x,y,y]$.

Then (6.5) becomes, $T^{+}[x,\delta,\epsilon]=u(x,y)-q(x,y)$. (6.13)

Also, by (6.3), $u(x,y)=\sum_{i\geq 1}u_i(y)x^i=\sum_{i\geq 1}\frac{\mu(i)}{i}a_i[\log(1+\delta(x,y))]$.

Letting $w(x,y)=\log(1+\delta(x,y))$, gives, $u_n(y)=\sum_{i\cdot j=n}\frac{\mu(i)}{i}w_j(y^i)$. (6.14)

Moreover, rearranging (6.6) gives,

$$(D^{+}-P^{+})[x,\mu,\nu]-\mu(x,y)=p(x,y)-q(x,y)+u(x,y) \qquad (6.15)$$

Now, let $v(x,y)=(\check{D}^{-}-\check{P}^{-})[x,y,y]$.

Then (6.8) becomes, $T^{-}[x,\delta,\epsilon]=v(x,y)-t(x,y)$. (6.16)

Also by (6.4), $v(x,y)=\sum_{i\geq 1}v_i(y)x^i=\sum_{k\text{ odd}}\frac{\mu(k)}{k}a_k[\log(1+\epsilon(x,y))]$

$$-\sum_{m=2^{1+e},\, e\geq 0}\frac{a_m}{m}\left[u(x,y)\right].$$

Letting $h(x,y)=\log(1+\epsilon(x,y))$ gives,

$$v_n(y)=\sum_{i\cdot j=n,\ i\ odd}\frac{\mu(i)}{i}h_j(y^i)-\sum_{2|n,m\cdot i=n}\frac{1}{m}u_i(y^m),\text{where, } m=2^{1+e},\text{for } e\geq 0.$$

(6.17)

Moreover, rearranging (6.9), gives,

$$(D^{-}-P^{-})[x,\mu,\nu]-\nu(x,y)=s(x,y)-t(x,y)+v(x,y). \qquad (6.18)$$

Recall from previous equations,

$\delta(x,y)-S^{+}[x,\mu,\nu]=\delta(x,y)-p(x,y)$;

$\epsilon(x,y)-S^{-}[x,\mu,\nu]=\epsilon(x,y)-s(x,y)$;

$y + \delta(x, y) - \check{S}^{+}[x, y, y] = y + \delta(x, y) - q(x, y);$

$y + \epsilon(x, y) - \check{S}^{-}[x, y, y] = y + \epsilon(x, y) - t(x, y).$

Substituting all of these in equation (6.12), we get

$$
\begin{aligned}
\check{B}[x, y, y] = {} & B[x, \mu, \nu](\text{ say, } B(x, y)\ ) \\
& + N[x, u, v] - N_1[x, p - q + u, s - t + v](\text{ say, } A1(x, y)\ ) \\
& + M[x, y + \delta - q, y + \epsilon - t] - M_0[x, \delta - q, \epsilon - t] \\
& - M[x, \delta - p, \epsilon - s](\text{ say, } A2(x, y)\ ) \\
& + \overrightarrow{T}[x, \delta, \epsilon] \diamond \overrightarrow{N}_1[x, p - q + u, s - t + v] \\
& - \overrightarrow{T}[x, \delta, \epsilon] \diamond \overrightarrow{N}[x, u, v](\text{ say, } A3(x, y)\ ) \\
& + \overrightarrow{N}_1[x, p - q + u, s - t + v] \diamond \overrightarrow{M}[x, \delta - p, \epsilon - s] \\
& - \overrightarrow{N}[x, u, v] \diamond \overrightarrow{M}[x, y + \delta - q, y + \epsilon - t] \\
& + \overrightarrow{N}[x, u, v] \diamond \overrightarrow{M}_0[x, \delta - q, \epsilon - t](\text{ say, } A4(x, y)\ ) \\
& + \overrightarrow{T}[x, \delta, \epsilon] \diamond \overrightarrow{M}[x, \delta - p, \epsilon - s] - \overrightarrow{T}[x, \delta, \epsilon] \diamond \overrightarrow{M}[x, y + \delta - q, y + \epsilon - t] \\
& + \overrightarrow{T}[x, \delta, \epsilon] \diamond \overrightarrow{M}_0[x, \delta - q, \epsilon - t](\text{ say, A5(x,y) }) \\
& - \frac{x}{2}(\mu(x, y) + \nu(x, y)).
\end{aligned}
$$

In terms of nth coefficients, for $n \geq 2$, this is,

$$
\begin{aligned}
\check{B}_n(y) = {} & B_n(y) + A1_n(y) + A2_n(y) + A3_n(y) + A4_n(y) + A5_n(y) \\
& - \frac{1}{2}\left\{\mu_{n-2}(y) + \nu_{n-2}(y)\right\}.
\end{aligned}
\tag{6.19}
$$

§6.3.1 $A1(x, y)$

$A1(x, y) = N[x, u, v] - N_1[x, p - q + u, s - t + v].$

Let pqu to denote $p - q + u$ and stv to denote $s - t + v$.

Then from (3.43) and (3.46) gives,

$$A1(x,y) = \frac{x^2}{2}\Bigg\{\exp\left(\sum_{i\geq 1}\frac{u_i}{i}\right) + \exp\left(\sum_{i \text{ even}}\frac{u_i}{i} + \sum_{i \text{ odd}}\frac{v_i}{i}\right)$$
$$- \left(1 + u + \frac{u^2}{2} + \frac{u_2}{2}\right) - \left(1 + v + \frac{v^2}{2} + \frac{u_2}{2}\right)\Bigg\}$$
$$- \frac{x^2}{2}\Bigg\{(1+\mu)\exp\left(\sum_{i\geq 1}\frac{pqu_i}{i}\right) + (1+\nu)\exp\left(\sum_{i \text{ even}}\frac{pqu_i}{i} + \sum_{i \text{ odd}}\frac{stv_i}{i}\right)$$
$$- \left\{(1+\mu)(1+pqu) + \frac{pqu^2}{2} + \frac{pqu_2}{2} + (1+\nu)(1+stv) + \frac{stv^2}{2} + \frac{pqu_2}{2}\right\}\Bigg\}.$$

Applying (3.87) and (3.88), let $T1$ come from $E1(T1, pqu)$, $T2$ come from $E2(T2, pqu, stv)$, $T3$ come from $E1(T3, u)$ and $T4$ come from $E2(T4, u, v)$. Then,

$$A1(x,y) = \frac{x^2}{2}\Bigg\{T3 + T4 - \left(1 + u + \frac{u^2}{2}\right) - \left(1 + v + \frac{v^2}{2}\right) - u_2 - (1+\mu)T1$$
$$- (1+\nu)T2 + (1+\mu)(1+pqu) + (1+\nu)(1+stv) + \frac{pqu^2}{2} + \frac{stv^2}{2} + pqu_2\Bigg\}$$
$$= \frac{x^2}{2}\Bigg\{(pqu - T1)(1+\mu) + (stv - T2)(1+\nu) + \frac{1}{2}(pqu^2 + stv^2 - u^2 - v^2)$$
$$- 2 + 2 + (T3 + T4 + \mu + \nu - u - v) + pqu_2 - u_2\Bigg\}.$$

Since, $pqu_0(y) = stv_0(y) = u_0(y) = v_0(y) = 0$

and $T1_0(y) = T2_0(y) = T3_0(y) = T4_0(y) = 1$

we have, $\frac{1}{2}\left\{(-1) + (-1) + \frac{1}{2}0 + 2 - 2 + (1+1) + 0 - 0\right\} = 0.$

Thus, for $n \geq 2$,

$$A1_n(y) = \frac{1}{2}\Bigg\{\sum_{2\leq k\leq n}\left\{(pqu_{n-k} - T1_{n-k})\mu_{k-2} + (stv_{n-k} - T2_{n-k})\nu_{k-2}\right\}$$
$$+ \frac{1}{2}\sum_{2\leq k\leq n}\left\{pqu_{n-k}pqu_{k-2} + stv_{n-k}stv_{k-2} - u_{n-k}u_{k-2} - v_{n-k}v_{k-2}\right\}$$
$$+ T3_{n-2} + T4_{n-2} + \mu_{n-2} + \nu_{n-2} - u_{n-2} - v_{n-2}$$
$$+ \begin{Bmatrix} pqu_m(y^2) - u_m(y^2) & \text{if } 2|n \text{ for } m = \left(\frac{n-2}{2}\right) \\ 0 & \text{otherwise} \end{Bmatrix}\Bigg\}. \tag{6.20}$$

§6.3.2 $A2(x,y)$

$$A2(x,y) = M[x, y+\delta-q, y+\epsilon-t] - M_0[x, \delta-q, \epsilon-t] - M[x, \delta-p, \epsilon-s].$$

Let dq denote $\delta - q$, et denote $\epsilon - t$, ydq to denote $y + \delta - q$,

yet to denote $y + \epsilon - t$, dp to denote $\delta - p$ and es to denote $\epsilon - s$.

Then applying (3.34) gives

$$\begin{aligned} A2(x,y) = & \frac{1}{2}\left\{\sum_{d\geq 1}\frac{\phi(d)}{d}a_d\left[\log\left(\frac{1-x\cdot dp}{1-x\cdot ydq}\right)\right]\right\} \\ & + \frac{x}{2}(dp - ydq) + \frac{x^2}{4}(dp^2 - ydq^2) + \frac{x^2}{4}(dp_2 - ydq_2) \\ & + \frac{1}{4(1-x^2\cdot ydq_2)}\left\{2x^3\cdot yet\cdot ydq_2 + x^4\cdot ydq_2^2 + x^4 yet^2\cdot ydq_2\right\} \\ & - \frac{1}{4(1-x^2\cdot dp_2)}\left\{2x^3\cdot es\cdot dp_2 + x^4\cdot dp_2^2 + x^4\cdot es^2\cdot dp_2\right\} \\ & - M_0[x, \delta-q, \epsilon-t]. \end{aligned}$$

Applying (3.96) and (3.97), let $T9$ come from $V(T9, dp, ydq)$, $T10$ come from $W(T10, ydq, yet)$, $T11$ come from $W(T11, dp, es)$ and $T12$ come from $M0(T12, dq, et)$.

$$\begin{aligned} A2(x,y) = & \frac{1}{2}\left\{\sum_{n\geq 1}\left(\sum_{i\cdot d=n}\frac{\phi(d)}{d}T9_i(y^d)\right)x^n\right\} + \frac{x}{2}(dp - ydq) + \frac{x^2}{4}(dp^2 - ydq^2) \\ & + \frac{x^2}{4}(dp_2 - ydq_2) + T10(x,y) - T11(x,y) - T12(x,y). \end{aligned}$$

Then for $n \geq 2$,

$$\begin{aligned} A2_n(y) = & \sum_{i\cdot d=n}\frac{\phi(d)}{2d}T9_i(y^d) + \frac{1}{2}(dp_{n-1} - ydq_{n-1}) \\ & + \frac{1}{4}\sum_{2\leq k\leq n}(dp_{k-2}\cdot dp_{n-k} - ydq_{k-2}\cdot ydq_{n-k}) \\ & + T10_n(y) - T11_n(y) - T12_n(y) \\ & + \frac{1}{4}\left\{\begin{matrix} dp_m(y^2) - ydq_m(y^2) & \text{if } 2|n \text{ for } m = \left(\frac{n-2}{2}\right) \\ 0 & \text{otherwise}\end{matrix}\right\} \end{aligned} \tag{6.21}$$

with $A2_0(y) = 0; A2_1(y) = \frac{\phi(1)}{2}T9_1(y) + \frac{1}{2}(dp_0 - dq_0)$
$= \frac{\phi(1)}{2}(-dp_0 + dq_0) + \frac{1}{2}(dp_0 - dq_0) = \frac{1}{2}y - \frac{1}{2}y = 0.$

§6.3.3 $A3(x, y)$

$$A3(x,y) = \overrightarrow{T}[x,\delta,\epsilon] \diamond \overrightarrow{N}_1[x, pqu, stv] - \overrightarrow{T}[x,\delta,\epsilon] \diamond \overrightarrow{N}[x,u,v].$$

Applying the definition of $\diamond$ from (3.73),

$$A3(x,y) = \frac{x^2}{2}T^+[x,\delta,\epsilon]N_1^+[x,pqu,stv] + \frac{x^2}{2}T^-[x,\delta,\epsilon]N_1^-[x,pqu,stv] - \frac{x^2}{2}T^+[x,\delta,\epsilon]N^+[x,u,v] - \frac{x^2}{2}T^-[x,\delta,\epsilon]N^-[x,u,v].$$

Applying (6.13), (6.16), (3.47), (3.48), (3.44) and (3.45) gives

$$A3(x,y) = \frac{x^2}{2}\Bigg\{(u-q)\Bigg\{(1+\mu)\exp\left(\sum_{i\geq 1}\frac{pqu_i}{i}\right) - (1+\mu) - pqu\Bigg\} + (v-t)\Bigg\{(1+\nu)\exp\left(\sum_{i \text{ even}}\frac{pqu_i}{i} + \sum_{i \text{ odd}}\frac{stv_i}{i}\right) - (1+\nu) - stv\Bigg\} - (u-q)\Bigg\{\exp\left(\sum_{i\geq 1}\frac{u_i}{i}\right) - (1+u)\Bigg\} - (v-t)\Bigg\{\exp\left(\sum_{i \text{ even}}\frac{u_i}{i} + \sum_{i \text{ odd}}\frac{v_i}{i}\right) - (1+v)\Bigg\}\Bigg\}.$$

As before, let $T1$ come from $E1(T1, pqu)$, $T2$ come from $E2(T2, pqu, stv)$, $T3$ come from $E1(T3, u)$ and $T4$ come from $E2(T4, u, v)$.

$$A3(x,y) = \frac{x^2}{2}\Big\{(u-q)\Big\{(1+\mu)T1 - (1+\mu) - pqu\Big\} + (v-t)\Big\{(1+\nu)T2 - (1+\nu) - stv\Big\} - (u-q)\Big\{T3 - (1+u)\Big\} - (v-t)\Big\{T4 - (1+v)\Big\}\Big\}$$
$$= \frac{x^2}{2}\Big\{(u-q)\Big\{(1+\mu)T1 - (1+\mu) - pqu - T3 + 1 + u\Big\} + (v-t)\Big\{(1+\nu)T2 - (1+\nu) - stv - T4 + 1 + v\Big\}\Big\}.$$

Thus for $n \geq 2$,

$$A3_n(y) = \frac{1}{2}\Bigg\{ \sum_{0\le k\le n-2} \Bigg\{ \Big\{ u_{n-k-2}(y) - q_{n-k-2}(y) \Big\}$$

$$* \Bigg\{ \Bigg(\sum_{0\le r\le k} \mu_r(y) T1_{k-r}(y) \Bigg) - \mu_k(y) - pqu_k(y) - T3_k(y) + u_k(y) \Bigg\}$$

$$+ \Big\{ v_{n-k-2}(y) - t_{n-k-2}(y) \Big\}$$

$$* \Bigg\{ \Bigg(\sum_{0\le r\le k} \nu_r(y) T2_{k-r}(y) \Bigg) - \nu_k(y) - stv_k(y) - T4_k(y) + v_k(y) \Bigg\} \Bigg\}$$

$$+ u_{n-2}(y) - q_{n-2}(y) + v_{n-2}(y) - t_{n-2}(y) \Bigg\}. \tag{6.22}$$

§6.3.4 $A4(x, y)$

$$A4(x,y) = \overrightarrow{N}_1[x, pqu, stv] \diamond \overrightarrow{M}[x, dp, es] - \overrightarrow{N}[x, u, v] \diamond \overrightarrow{M}[x, ydq, yet]$$
$$+ \overrightarrow{N}[x, u, v] \diamond \overrightarrow{M}_0[x, dq, et].$$

Applying the definition of $\diamond$ from (3.73),

$$A4(x,y) = \frac{x^2}{2} N_1^+[x, pqu, stv] M^+[x, dp, es] + \frac{x^2}{2} N_1^-[x, pqu, stv] M^-[x, dp, es]$$
$$- \frac{x^2}{2} N^+[x, u, v] M^+[x, ydq, yet] - \frac{x^2}{2} N^-[x, u, v] M^-[x, ydq, yet]$$
$$+ \frac{x^2}{2} N^+[x, u, v] M_0^+[x, dq, et] + \frac{x^2}{2} N^-[x, u, v] M_0^-[x, dq, et].$$

Applying (3.47), (3.48) gives,

$$A4(x,y) = \frac{x^2}{2}\Bigg\{ \Bigg\{ (1+\mu) \exp\Bigg(\sum_{i\ge 1} \frac{pqu_i}{i} \Bigg) - (1+\mu) - pqu \Bigg\} M^+[x, dp, es]$$

$$+ \Bigg\{ (1+\nu) \exp\Bigg(\sum_{i \text{ even}} \frac{pqu_i}{i} + \sum_{i \text{ odd}} \frac{stv_i}{i} \Bigg) - (1+\nu) - stv \Bigg\} M^-[x, dp, es]$$

$$- \Bigg\{ \exp\Bigg(\sum_{i\ge 1} \frac{u_i}{i} \Bigg) - (1+u) \Bigg\} M^+[x, ydq, yet]$$

$$- \Bigg\{ \exp\Bigg(\sum_{i \text{ even}} \frac{u_i}{i} + \sum_{i \text{ odd}} \frac{v_i}{i} \Bigg) - (1+v) \Bigg\} M^-[x, ydq, yet]$$

$$+ \Bigg\{ \exp\Bigg(\sum_{i\ge 1} \frac{u_i}{i} \Bigg) - (1+u) \Bigg\} M_0^+[x, dq, et]$$

$$+\left\{\exp\left(\sum_{i\text{ even}}\frac{u_i}{i}+\sum_{i\text{ odd}}\frac{v_i}{i}\right)-(1+v)\right\}M_0^-[x,dq,et]\Bigg\}.$$

As before, let $T1$ come from $E1(T1,pqu)$, $T2$ come from $E2(T2,pqu,stv)$,

$T3$ come from $E1(T3,u)$, $T4$ come from $E2(T4,u,v)$.

Also, applying (3.89) and (3.90), $T5$ come from $MP(T5,dp)$,

$T6$ come from $MP(T6,ydq)$, $T7$ come from $MN(T7,dp,es)$,

$T8$ come from $MN(T8,ydq,yet)$, $T13$ come from $M0P(T13,dq)$,

$T14$ come from $M0N(T14,dq)$.

$$\begin{aligned}A4(x,y)=\frac{x^2}{2}\Bigg\{&\Big\{(1+\mu)T1-(1+\mu)-pqu\Big\}T5\\&+\Big\{(1+\nu)T2-(1+\nu)-stv\Big\}T7\\&-\Big\{T3-(1+u)\Big\}(T6-T13)\\&-\Big\{T4-(1+v)\Big\}(T8-T14)\Bigg\}.\end{aligned}$$

Thus for $n\geq 2$,

$$\begin{aligned}A4_n(y)=\frac{1}{2}\Bigg\{\sum_{0\leq k\leq n-2}\Bigg\{&\Bigg\{\sum_{0\leq r\leq k}\mu_r(y)\cdot T1_{k-r}(y)-\mu_k(y)-pqu_k(y)\Bigg\}*T5_{n-k-2}(y)\\&+\Bigg\{\sum_{0\leq r\leq k}\nu_r(y)\cdot T2_{k-r}(y)-\nu_k(y)-stv_k(y)\Bigg\}*T7_{n-k-2}(y)\\&-\Big\{T3_{n-k-2}(y)-u_{n-k-2}(y)\Big\}*\Big\{T6_k(y)-T13_k(y)\Big\}\\&-\Big\{T4_{n-k-2}(y)-v_{n-k-2}(y)\Big\}*\Big\{T8_k(y)-T14_k(y)\Big\}\\&+(T6_{n-2}(y)-T13_{n-2}(y)+T8_{n-2}(y)-T14_{n-2}(y))\Bigg\}.\end{aligned}\tag{6.23}$$

§6.3.5 $A5(x,y)$

$$\begin{aligned}A5(x,y)=&\overrightarrow{T}[x,\delta,\epsilon]\diamond\overrightarrow{M}[x,dp,es]-\overrightarrow{T}[x,\delta,\epsilon]\diamond\overrightarrow{M}[x,ydq,yet]\\&+\overrightarrow{T}[x,\delta,\epsilon]\diamond\overrightarrow{M}_0[x,dq,et].\end{aligned}$$

Applying the definition of $\diamond$ from (3.73),

$$\begin{aligned}A5(x,y) &= \frac{x^2}{2}T^+[x,\delta,\epsilon]M^+[x,dp,es] + \frac{x^2}{2}T^-[x,\delta,\epsilon]M^-[x,dp,es]\\ &- \frac{x^2}{2}T^+[x,\delta,\epsilon]M^+[x,dq,et] - \frac{x^2}{2}T^-[x,\delta,\epsilon]M^-[x,ydq,yet]\\ &+ \frac{x^2}{2}T^+[x,\delta,\epsilon]M_0^+[x,dq,et] + \frac{x^2}{2}T^-[x,\delta,\epsilon]M_0^-[x,dq,et].\end{aligned}$$

Again, applying (6.13) and (6.16), gives

$$\begin{aligned}A5(x,y) = \frac{x^2}{2}\Bigg\{&(u-q)\Big\{M^+[x,dp,es] - M^+[x,ydq,yet] + M_0^+[x,dq,et]\Big\}\\ &+ (v-t)\Big\{M^-[x,dp,es] - M^-[x,ydq,yet] + M_0^-[x,dq,et]\Big\}\Bigg\}.\end{aligned}$$

As before let $T5$ come from $MP(T5,dp)$, $T6$ come from $MP(T6,ydq)$,

$T7$ come from $MN(T7,dp,es)$, $T8$ come from $MN(T8,ydq,yet)$,

$T13$ come from $M0P(T13,dq)$ and $T14$ come from $M0N(T14,dq)$.

Then, $A5(x,y) = \dfrac{x^2}{2}\Big\{(u-q)(T5 - T6 + T13) + (v-t)(T7 - T8 + T14)\Big\}.$

Thus for $n \geq 2$,

$$\begin{aligned}A5_n(y) = \frac{1}{2}\Bigg\{\sum_{2\leq k\leq n} &\Big\{(u_{k-2}(y) - q_{k-2}(y)) * (T5_{n-k}(y) - T6_{n-k}(y) + T13_{n-k}(y))\\ &+ (v_{k-2}(y) - t_{k-2}(y)) * (T7_{n-k}(y) - T8_{n-k}(y) + T14_{n-k}(y))\Big\}\Bigg\}.\end{aligned} \tag{6.24}$$

§6.4 Summary of the counting algorithm

In the computations below, power series are computed term by term. At any given time nth term of each of the power series (except in the case of $\eta(x,y)$, for which $(n+1)$th term is computed) is computed one after the other in the order given here, utilizing all its previous terms and all the available terms of the power series computed before it. Steps are numbered for easy perusal.

1. From (3.83), compute, $p_0(0) = 0$; $p_1(y) = \delta_0(y)^2$

and for $n \geq 2$, $p_n(y) = \displaystyle\sum_{1\leq k\leq n} \delta_{k-1}(y)\Big\{\delta_{n-k}(y) - p_{n-k}(y)\Big\}$

then from (6.1), compute, $q_0(y) = 0$; $q_1(y) = \delta_0(y)^2 - y^2$

and for $n > 1$,

$$q_n(y) = yq_{n-1}(y) + \sum_{1\le k\le n}\Big\{\delta_{k-1}(y)\Big\{\delta_{n-k}(y) + q_{n-k}(y)\Big\} - 2q_{k-1}(y)q_{n-k}(y)\Big\}$$

$$+ \sum_{2\le k\le n}\Big\{\Big\{3q_{n-k}(y) - 2\delta_{n-k}(y)\Big\} * \sum_{0\le r\le k-2}\delta_r(y)\delta_{k-2-r}(y)$$

$$- q_{n-k}(y) * \sum_{0\le r\le k-2} q_r(y)q_{k-2-r}(y) + y\Big\{q_{n-k}(y)q_{k-2}(y) - \delta_{n-k}(y)\delta_{k-2}(y)\Big\}\Big\}$$

$$+ \sum_{3\le k\le n}\Big\{\Big\{\delta_{n-k}(y) - 3q_{n-k}(y)\Big\} * \sum_{0\le r\le k-3}\delta_{k-3-r}(y)\sum_{0\le i\le r}\delta_i(y)\delta_{r-i}(y)$$

$$+ \Big\{3\delta_{n-k}(y) - q_{n-k}(y)\Big\} * \sum_{0\le r\le k-3}\delta_{k-3-r}(y)\sum_{0\le i\le r} q_i(y)q_{r-i}(y)$$

$$+ y\delta_{n-k}(y)\Big\{\sum_{0\le r\le k-3}\delta_{k-3-r}(y)\delta_r(y) - 2\delta_{k-3-r}(y)q_r(y) + q_{k-3-r}(y)q_r(y)\Big\}\Big\}.$$

Then from (6.7), compute, for $n \ge 2$, nth term of $\mu(x, y)$ applying

these in $E1(1 + \mu, q - p)$ and $1 + \mu_0(y) = 1 + y$ and $\mu_1(y) = q_1(y) - p_1(y)$.

2. From (3.84), compute, $s_0(y) = 0$; $s_1(y) = \delta_0(y^2) - p_0(y^2) = \delta_0(y^2)$

and for $n \ge 2$, $s_n(y)$ applying,

$$s_{2n}(y) = \sum_{1\le k\le n}\Big\{\delta_{k-1}(y^2) - p_{k-1}(y^2)\Big\}\epsilon_{2n-2k}(y) \text{ and}$$

$$s_{2n+1}(y) = \delta_n(y^2) - p_n(y^2) + \sum_{1\le k\le n}\Big\{\delta_{k-1}(y^2) - p_{k-1}(y^2)\Big\}\epsilon_{2n+1-2k}(y).$$

From (6.2) compute, $t_0(y) = 0$; $t_1(y) = \delta_0(y^2) - y^2$

$$t_2(y) = \Big(\epsilon_0(y) + t_0(y)\Big) * \Big(\delta_0(y^2) - q_0(y^2)\Big) + y^2\epsilon_0(y) + y^3$$

$$= \epsilon_0(y)\delta_0(y^2) + y^2\epsilon_0(y) + y^3.$$

$$\text{Let } m = \begin{cases} \frac{n}{2}, & \text{n is even} \\ \frac{n-1}{2}, & \text{n is odd.} \end{cases}$$

For $n > 3$,

$$t_n(y) = \sum_{1\le k\le m} \Big\{ \Big\{ \epsilon_{n-2k}(y) + t_{n-2k}(y) \Big\} * \Big\{ \delta_{k-1}(y^2) - q_{k-1}(y^2) \Big\} \Big\} + y^2 \epsilon_{n-2}(y)$$

$$+ \sum_{2\le k\le m} \Big\{ y^2 \epsilon_{n-2k}(y) * \Big\{ (q_{k-2}(y^2) - \delta_{k-2}(y^2) \Big\}$$

$$- \epsilon_{n-2k}(y) * \sum_{0\le r\le k-2} \Big\{ q_r(y^2) - \delta_r(y^2) \Big\} * \Big\{ q_{k-2-r}(y^2) - \delta_{k-2-r}(y^2) \Big\} \Big\}$$

$$+ \left\{ \begin{array}{cr} 0, & \text{if n is even} \\ \delta_m(y^2) - q_m(y^2) + y^2 \Big\{ q_{m-1}(y^2) - \delta_{m-1}(y^2) \Big\} & \\ - \sum_{1\le k\le m} \Big\{ q_{k-1}(y^2) - \delta_{k-1}(y^2) \Big\} * \Big\{ q_{m-k}(y^2 - \delta_{m-k}(y^2) \Big\}, & \text{Otherwise} \end{array} \right\}.$$

Then from (6.10), compute, $1 + \nu_0(y) = 1 + y$ and $\nu_1(y) = t_1(y) - s_1(y)$

for $n \ge 2$, nth term of $\nu(x, y)$ applying these in $E2(1 + \nu, q - p, t - s)$.

3. From (3.80), $\eta(x, y) - x = xF(x, y) - a_1 \left(\dfrac{\partial \boldsymbol{K}}{\partial a_1} - 1 \right) [\eta, 1 + \mu, 1 + \nu]$

where $F(x, y) = (\boldsymbol{K} - 1)[\eta, 1 + \mu, 1 + \nu]$. We compute the nth term $F(x, y)$ by doing the composition on the terms generated for $\boldsymbol{K}$.

Let $\boldsymbol{K}_a$ denote $a_1 \left(\dfrac{\partial \boldsymbol{K}}{\partial a_1} - 1 \right)$. Compute the power series

$R(x, y) = \boldsymbol{K}_a[\eta, 1 + \mu, 1 + \nu]$ by composing on the terms generated for $\boldsymbol{K}_a$.

Compute, for $n \ge 2$, the $(n + 1)$th term

of $\eta(x, y)$, applying $\eta_{n+1}(y) = F_n(y) - R_{n+1}(y)$.

4. From (3.78),

$$\delta(x, y) + 1 = \left(\frac{\eta(x, y)}{x} \right)^2 \frac{2}{a_1^2} \left(b_1 \frac{\partial \boldsymbol{K}}{\partial b_1} \right) [\eta, 1 + \mu, 1 + \nu] - (1 + \delta(x, y))F(x, y).$$

Let $\boldsymbol{K}_b$ denote $\dfrac{2}{a_1^2} \left(b_1 \dfrac{\partial \boldsymbol{K}}{\partial b_1} \right)$. Compute the power series

$S(x, y) = \boldsymbol{K}_b[\eta, 1 + \mu, 1 + \nu]$ by composing on the terms generated for $\boldsymbol{K}_b$.

Then, $\delta(x, y) + 1 = \left(\dfrac{\eta(x, y)}{x} \right)^2 S(x, y) - (1 + \delta(x, y))F(x, y)$.

Let $E(x,y) = \sum_{i\geq 0} E_i(y)x^i = \left(\frac{\eta(x,y)}{x}\right)$. Then,

$$\delta_n(y) = \sum_{0\leq k\leq n} S_{n-k}(y)\left(\sum_{0\leq r\leq k} E_r(y)E_{k-r}(y)\right) - F_n(y) - \sum_{0\leq k\leq n} \delta_k(y)F_{n-k}(y)$$

$$= \sum_{0\leq k\leq n}\left\{S_{n-k}(y)\left(\sum_{0\leq r\leq k} E_r(y)E_{k-r}(y)\right) - \delta_k(y)F_{n-k}(y)\right\} - F_n(y).$$

5. From (3.79),

$$\epsilon(x,y)+1 = \left(\frac{\eta(x^2,y^2)}{x^2}\right)\frac{2}{a_2}\left(c_1\frac{\partial \boldsymbol{K}}{\partial c_1}\right)[\eta, 1+\mu, 1+\nu] - (1+\epsilon(x,y))F(x,y).$$

Let $\boldsymbol{K}_c$ denote $\frac{2}{a_2}\left(c_1\frac{\partial \boldsymbol{K}}{\partial c_1}\right)$. Compute the power series

$T(x,y) = \boldsymbol{K}_c[\eta, 1+\mu, 1+\nu]$ by composing on the terms generated for $\boldsymbol{K}_c$.

Then, $\epsilon(x,y)+1 = \left(\frac{\eta(x^2,y^2)}{x^2}\right)T(x,y) - (1+\epsilon(x,y))F(x,y)$. Then,

$$\epsilon_n(y) = \sum_{1\leq k\leq \lfloor \frac{n}{2}\rfloor} E_k(y^2)T_{n-2k}(y) - \sum_{0\leq k\leq n-1} \epsilon_k(y)F_{n-k} + T_n(y) - F_n(y).$$

6. Let $B(x,y) = \boldsymbol{B}[x,\mu,\nu]$. Then, from (3.82), for $n\geq 2$,

$$B_n(y) = \sum_{k\cdot i=n}\frac{\mu(k)}{k}f_i(y^k) - \sum_{k\cdot i=n-1}\frac{\mu(k)}{k}e_i(y^k).$$

7. From (6.3), $u(x,y) = \sum_{i\geq 1} u_i(y)x^i = \sum_{i\geq 1}\frac{\mu(i)}{i}a_i\left[\log\left(\frac{1+\delta(x,y)}{1+y}\right)\right]$.

Let $w(x,y) = \log\left(\frac{1+\delta(x,y)}{1+y}\right)$, so that, from (6.14), $u_n(y) = \sum_{i\cdot j=n}\frac{\mu(i)}{i}w_j(y^i)$.

8. From (6.4),

$$v(x,y) = \sum_{i\geq 1} v_i(y)x^i = \sum_{k \text{ odd}}\frac{\mu(k)}{k}a_k\left[\log\left(\frac{1+\epsilon(x,y)}{1+y}\right)\right] - \sum_{m=2^{1+e},\ e\geq 0}\frac{a_m}{m}[u(x,y)].$$

Let $h(x,y) = \log\left(\frac{1+\epsilon(x,y)}{1+y}\right)$. So that, from (6.17),

$$v_n(y) = \sum_{i\cdot j=n,\ i \text{ odd}}\frac{\mu(i)}{i}h_j(y^i) - \sum_{2|n,\ m\cdot i\cdot j=n}\frac{1}{m}u_i(y^m), \text{where, } m = 2^{1+e}, \text{for, } e\geq 0.$$

9. Having computed the essential power series, we now compute some auxiliary power series that are used in the remaining computation. For all $n \geq 2$,

$pqu_n(y) = p_n(y) - q_n(y) + u_n(y)$,

$stv_n(y) = s_n(y) - t_n(y) + v_n(y)$,

$dp_n(y) = \delta_n(y) - p_n(y)$,

$es_n(y) = \epsilon_n(y) - s_n(y)$,

$dq_n(y) = \delta_n(y) - q_n(y)$,

$ydq_n(y) = \delta_n(y) - q_n(y)$,

$et_n(y) = \epsilon_n(y) - t_n(y)$ and

$yet_n(y) = \epsilon_n(y) - t_n(y)$.

Now, we pass on these into routines $E1$, $E2$, MP, MN, $M0$, $M0P$, $M0N$, V, W to compute the nth term of power series $T1(x, y)$ through $T12(x, y)$.

$E1(T1, pqu)$, $E2(T2, pqu, stv)$, $E1(T3, u)$ $E2(T4, u, v)$,

$MP(T5, dp)$, $MP(T6, ydq)$, $MN(T7, dp, es)$, $MN(T8, ydq, yet)$,

$V(T9, dp, ydq)$, $W(T10, ydq, yet)$, $W(T11, dp, es)$, $M0(T12, dq, et)$,

$M0P(T13, dq)$ and $M0N(T14, dq)$.

10. From (6.20), for $n \geq 3$,

$$
\begin{aligned}
A1_n(y) = \frac{1}{2}\Bigg\{ & \sum_{2\leq k\leq n} \Big\{ (pqu_{n-k} - T1_{n-k})\mu_{k-2} + (stv_{n-k} - T2_{n-k})\nu_{k-2} \Big\} \\
& + \frac{1}{2} \sum_{2\leq k\leq n} \Big\{ pqu_{n-k}pqu_{k-2} + stv_{n-k}stv_{k-2} - u_{n-k}u_{k-2} - v_{n-k}v_{k-2} \Big\} \\
& + T3_{n-2} + T4_{n-2} + \mu_{n-2} + \nu_{n-2} - u_{n-2} - v_{n-2} \\
& + \begin{Bmatrix} pqu_m(y^2) - u_m(y^2) & \text{if } 2|n \text{ for } m = \left(\frac{n-2}{2}\right) \\ 0 & \text{otherwise} \end{Bmatrix} \Bigg\}.
\end{aligned}
$$

11. From (6.21), for $n \geq 2$,

$$A2_n(y) = \sum_{i\cdot d=n} \frac{\phi(d)}{2d} T9_i(y^d) + \frac{1}{2}(dp_{n-1} - ydq_{n-1})$$

$$+ \frac{1}{4} \sum_{2\le k\le n} (dp_{k-2} \cdot dp_{n-k} - ydq_{k-2} \cdot ydq_{n-k})$$

$$+ T10_n(y) - T11_n(y) - T12_n(y)$$

$$+ \frac{1}{4} \left\{ \begin{matrix} dp_m(y^2) - ydq_m(y^2) & \text{if } 2|n \text{ for } m = (\frac{n-2}{2}) \\ 0 & \text{otherwise} \end{matrix} \right\}$$

with $A2_0(y) = 0$; $A2_1(y) = \frac{\phi(1)}{2} T9_1(y) + \frac{1}{2}(dp_0 - dq_0)$

$$= \frac{\phi(1)}{2}(-dp_0 + dq_0) + \frac{1}{2}(dp_0 - dq_0) = \frac{1}{2}y - \frac{1}{2}y = 0.$$

12. From (6.22), for $n \ge 2$,

$$A3_n(y) = \frac{1}{2}\Bigg\{ \sum_{0\le k\le n-2} \Bigg\{ \Big\{ u_{n-k-2}(y) - q_{n-k-2}(y) \Big\}$$

$$* \Bigg\{ \Bigg(\sum_{0\le r\le k} \mu_r(y) T1_{k-r}(y) \Bigg) - \mu_k(y) - pqu_k(y) - T3_k(y) + u_k(y) \Bigg\}$$

$$+ \Big\{ v_{n-k-2}(y) - t_{n-k-2}(y) \Big\}$$

$$* \Bigg\{ \Bigg(\sum_{0\le r\le k} \nu_r(y) T2_{k-r}(y) \Bigg) - \nu_k(y) - stv_k(y) - T4_k(y) + v_k(y) \Bigg\} \Bigg\}$$

$$+ u_{n-2}(y) - q_{n-2}(y) + v_{n-2}(y) - t_{n-2}(y) \Bigg\}.$$

13. From (6.23), for $n \ge 2$,

$$A4_n(y) = \frac{1}{2}\Bigg\{ \sum_{0\le k\le n-2} \Bigg\{ \Bigg\{ \sum_{0\le r\le k} \mu_r(y) \cdot T1_{k-r}(y) - \mu_k(y) - pqu_k(y) \Bigg\} * T5_{n-k-2}(y)$$

$$+ \Bigg\{ \sum_{0\le r\le k} \nu_r(y) \cdot T2_{k-r}(y) - \nu_k(y) - stv_k(y) \Bigg\} * T7_{n-k-2}(y)$$

$$- \Big\{ T3_{n-k-2}(y) - u_{n-k-2}(y) \Big\} * \Big\{ T6_k(y) - T13_k(y) \Big\}$$

$$- \Big\{ T4_{n-k-2}(y) - v_{n-k-2}(y) \Big\} * \Big\{ T8_k(y) - T14_k(y) \Big\}$$

$$+ (T6_{n-2}(y) - T13_{n-2}(y) + T8_{n-2}(y) - T14_{n-2}(y)) \Bigg\}.$$

14. From (6.24), for $n \geq 2$,

$$A5_n(y) = \frac{1}{2}\Bigg\{ \sum_{2\leq k\leq n} \Big\{(u_{k-2}(y) - q_{k-2}(y)) * (T5_{n-k}(y) - T6_{n-k}(y) + T13_{n-k}(y)) + (v_{k-2}(y) - t_{k-2}(y)) * (T7_{n-k}(y) - T8_{n-k}(y) + T14_{n-k}(y))\Big\}\Bigg\}.$$

15. From (6.19), for $n \geq 2$,

$$\check{B}_n(y) = B_n(y) + A1_n(y) + A2_n(y) + A3_n(y) + A4_n(y) + A5_n(y) - \frac{1}{2}\Big\{\mu_{n-2}(y) + \nu_{n-2}(y)\Big\}. \tag{6.25}$$

CHAPTER 7

Conclusions

§7.1 Numerical Results

Most of the sequences computed in this dissertation are new, according to N. J. Sloane's online encyclopedia of Integer sequences ⟨*http://www.research.att.com/ ~njas/sequences/Seis.html*⟩. In case of unlabeled minimally 2–connected graphs, numbers up to 12 and 16 nodes are given in [47] and [29] respectively. They agree with our numbers except that the values for 15 and 16 nodes in [29] are a little less than our values. Our calculations have provided numbers up to 32 nodes for this class. In Sloane's online encyclopedia, the numbers of unlabeled minimally 2–connected graphs is sequence number $A003317$.

The numbers computed for the other two classes are not found elsewhere. The numbers of unlabeled 2–connected 3-edge–connected graphs is sequence number $A054316$ and we have them computed up to 25 nodes. The numbers of unlabeled 2–connected minimally 2–edge–connected graphs is sequence number $A054317$ and we have them computed up to 34 nodes.

In all the three computations for this dissertations, we were using fixed precision. Each number was stored as a sequence of unsigned short (16 bit) integers respectively the residues modulo primes such as 65323, 65287, etc,. So each multiplication is an elementary operation, as in [51]. For the computations of minimally 2–connected graphs and minimally 2–edge–connected graphs, we only had to use four primes. But for the computation of 3–edge–connected graphs, we had to use 14 primes. After the computations we applied the Chinese remainder Theorem to calculate these rather long numbers. All our power series are series in x with coefficients that are polynomials in y. Here the power of x denotes the number of nodes and the power of y denotes edges. The number of power series we used in the computations of minimally 2–connected, 3–edge–connected and minimally 2–edge–connected graphs were 42, 45 and 47 respectively.

As noted by Dirac [8, pp. 214], there are at most $2n-4$ edges in any minimally 2–connected graph on n nodes. These contain the 2–connected minimally 2–edge–connected graphs, so these also have at most $2n-4$ edges. In both the computations, all our polynomials were of fixed size $2N$, where N is the maximum number of nodes that we wanted to compute for. For 3–edge–connected graphs the maximum number of edges is $\binom{n}{2}$ and we stored N^2+N coefficients for each of our polynomials. The extra terms were needed in some polynomials that were intermediate results. But for programming convenience, we had to hold all the polynomials with N^2+N coefficients. Thus each of the three computations runs in polynomial space as a function of n.

Programs for the computations were written in the popular object–oriented language C++. About 4500 lines of code was common to the three graph counting algorithms. Each algorithm then had about 1000 lines of code specific to the problem. Thus the client routines and the base routines consist of some 7500 lines of C++ code. But for each graph counting problem there were bugs that were extremely hard to locate in C++ code, as the computations were done modulo primes. So, for all three of the counting problems, to compute the first few terms for debugging purposes, it was found necessary to write independent programs in Maple, amounting in all to some 3000 lines of Maple code.

The computational cost for any of the three classes is dominated by compositions such as $\boldsymbol{K}[\eta, 1+\mu, 1+\nu]$. Up to a constant factor this is exactly the same as for counting 3–connected graphs in [51]. There it is argued that the time complexity is $O(n^{11}p'(n))$, where $p'(n)$ is the number of partitions of all numbers up to n. This can be related to $p(n)$ by noting that $p'(n) = O(\sqrt{n}p(n))$. The analysis assumes that arithmetic operations are all $O(1)$ in time. The bit complexity can be obtained by multiplying by the number of primes needed as a function of n, which is $O(n^2)$ for 3–edge–connected blocks. The data seems to indicate that for minimally 2–connected or 2–edge–connected blocks the number of primes needed is closer to $O(n)$, but it has not even been proved that the number is $O(n^2)$.

A complete analysis of the bit complexity would need to take into consideration the availability of prime numbers, which will force the use of primes too long for a single word. If m is the number of bits used to store a prime number, then multiplications will take $O(m^2)$ time using the most straightforward algorithm. This can be replaced by $O(m \log m \log\log m)$ using Schönhage and Strassen's algorithm [52], but assume $O(m^2)$ is used.

The $\log_2$ of the product of the prime numbers expressible with m bits is approximately 2^m, so $m = O(\log n)$ will suffice for applying the Chinese remainder Theorem to calculate the unlabeled numbers for graphs on up to n nodes. For the latter are bounded above by $2^{n^2/2}$. Hence an additional factor of $O(\log^2 n)$ will cover the bit complexity of the arithmetic operations required by any of our counting algorithms.

The program for counting minimally 2–connected graphs was run in background under lowest priority for seventeen days on a Sun Solaris Workstation (330 MHz) named MoonStone that belongs to a student laboratory of Computer Science Department at UGA. Since it was running when the classes were not in session, this computation was able to use an average of 99% of the CPU time.

The program for counting 3–edge–connected graphs was run in background under lowest priority for eight days on a personal computer (550 MHz) named Aditya, running Red Hat Linux, that belongs to Lawrence Berkeley National Laboratory, Berkeley, CA and was assigned to author's wife, Mrs. Vijaya L. Natarajan. Since there was no other use for the PC at that time, this computation was also able to use an average of 99% of the CPU time.

The program for counting minimally 2–edge–connected graphs was run in background under lowest priority for seven weeks on a Sun Solaris Workstation (248 MHz) named Kepler that belongs to Mathematics Department at UGA and was assigned to Dr. William Graham. The average CPU time available for this computation was approximately 50%.

§7.2 Related results and problems

In this dissertation we have counted 2–connected 3–edge–connected graphs and 2–connected minimally 2–edge–connected graphs rather than all 3–edge-connected graphs or all minimally 2–edge–connected graphs. Counting the latter classes would not be very different from the ones we have counted. In fact, one could apply (3.22) to solve these problems.

We hope that the Tutte decomposition approach developed in this dissertation can be extended to obtain unique characterizations and counting algorithms for many additional classes of 2–connected graphs. We contend that it is only a matter of applying our methods to suitable classes of 2–connected graphs. We are sure that this might also inspire many improvements to the counting methods used in this dissertation and also inspire many new methods of general counting.

We are confident that the decomposition characterizations developed here can be applied to efficiently enumerate and catalogue the graphs of classes such as minimally 2–connected. However, we leave this fertile ground untouched.

Before launching this dissertation project, we were trying to find a winning strategy for an achievement graph game called *Do disconnect-it*. The computations we did for that produced the number of minimally 2–edge–connected graphs up to 10. Starting for $n = 3$, they are 1, 1, 3, 4, 11, 23, 63, 159. These numbers are sequence number A001072 in Sloane's online encyclopedia.

Computing the asymptotic estimates for minimally 2–connected graphs and for minimally 2–connected blocks is still unsolved. In the case of 3–edge–connected blocks, almost all graphs are 3–edge–connected blocks so the asymptotics are the same as for all graphs. The latter are well known [19, pp. 196].

Finally, in [42], inspired by this dissertation work, we have counted the labeled versions of the three classes of graphs counted in this dissertation.

APPENDIX A

Numbers of unlabeled minimally 2–connected graphs

Table A.1: Numbers of unlabeled minimally 2–connected graphs by the number n of nodes.

Number	n
1	3
1	4
2	5
3	6
6	7
12	8
28	9
68	10
184	11
526	12
1602	13
5075	14
16711	15
56428	16
195003	17
685649	18
2447882	19
8850157	20
32359428	21
119492766	22
445236635	23
1672636369	24
6331624545	25
24138404479	26
92640942148	27
357805122286	28
1390318899884	29
5433781135206	30
21356209420251	31
84393903909663	32

Table A.2: Numbers of unlabeled minimally 2–connected graphs by number of edges m and nodes n.

Number	m	n	Number	m	n
1	3	3	62	14	12
1	4	4	156	15	12
1	5	5	178	16	12
1	6	5	88	17	12
1	6	6	28	18	12
1	7	6	4	19	12
1	8	6	1	20	12
1	7	7	1	13	13
2	8	7	10	14	13
2	9	7	98	15	13
1	10	7	360	16	13
1	8	8	544	17	13
3	9	8	398	18	13
5	10	8	149	19	13
2	11	8	36	20	13
1	12	8	5	21	13
1	9	9	1	22	13
4	10	9	1	14	14
10	11	9	12	15	14
9	12	9	155	16	14
3	13	9	749	17	14
1	14	9	1576	18	14
1	10	10	1510	19	14
5	11	10	793	20	14
21	12	10	227	21	14
23	13	10	46	22	14
14	14	10	5	23	14
3	15	10	1	24	14
1	16	10	1	15	15
1	11	11	14	16	15
7	12	11	229	17	15
35	13	11	1514	18	15
66	14	11	4132	19	15
50	15	11	5362	20	15
20	16	11	3629	21	15
4	17	11	1429	22	15
1	18	11	337	23	15
1	12	12	57	24	15
8	13	12	6	25	15

Table A.2: Numbers of unlabeled minimally 2–connected graphs by number of edges m and nodes n (continued).

Number	m	n	Number	m	n
1	26	15	97	30	18
1	16	16	7	31	18
16	17	16	1	32	18
340	18	16	1	19	19
2852	19	16	24	20	19
10242	20	16	910	21	19
17121	21	16	15459	22	19
15236	22	16	109593	23	19
7669	23	16	367132	24	19
2406	24	16	650492	25	19
469	25	16	664974	26	19
69	26	16	416474	27	19
6	27	16	168058	28	19
1	28	16	45129	29	19
1	17	17	8419	30	19
19	18	17	1096	31	19
477	19	17	112	32	19
5216	20	17	8	33	19
23709	21	17	1	34	19
50965	22	17	1	20	20
57664	23	17	27	21	20
37556	24	17	1227	22	20
14852	25	17	25363	23	20
3811	26	17	220775	24	20
643	27	17	904505	25	20
82	28	17	1967359	26	20
7	29	17	2469275	27	20
1	30	17	1908372	28	20
1	18	18	950021	29	20
21	19	18	317113	30	20
671	20	18	72706	31	20
9106	21	18	11894	32	20
52274	22	18	1381	33	20
140853	23	18	129	34	20
201379	24	18	8	35	20
164973	25	18	1	36	20
83009	26	18	1	21	21
26636	27	18	30	22	21
5777	28	18	1612	23	21
844	29	18	40657	24	21

Table A.2: Numbers of unlabeled minimally 2–connected graphs by number of edges m and nodes n (continued).

Number	**m**	**n**	**Number**	**m**	**n**
427634	25	21	84836617	30	23
2124732	26	21	114787516	31	23
5599535	27	21	101008571	32	23
8543435	28	21	59997824	33	23
8036612	29	21	24784870	34	23
4892330	30	21	7287165	35	23
1997605	31	21	1556379	36	23
564382	32	21	244738	37	23
112634	33	21	28920	38	23
16346	34	21	2562	39	23
1726	35	21	186	40	23
147	36	21	10	41	23
9	37	21	1	42	23
1	38	21	1	24	24
1	22	22	40	25	24
33	23	22	3445	26	24
2111	24	22	145322	27	24
63436	25	22	2573372	28	24
801566	26	22	21499576	29	24
4774144	27	22	95514635	30	24
15109331	28	22	246487547	31	24
27707680	29	22	394640425	32	24
31418133	30	22	411730307	33	24
23102095	31	22	290776506	34	24
11440078	32	22	143180369	35	24
3924677	33	22	50360575	36	24
956812	34	22	12893303	37	24
168416	35	22	2443751	38	24
21968	36	22	346488	39	24
2109	37	22	37430	40	24
166	38	22	3058	41	24
9	39	22	208	42	24
1	40	22	10	43	24
1	23	23	1	44	24
37	24	23	1	25	25
2701	25	23	44	26	25
97045	26	23	4321	27	25
1455995	27	23	213935	28	25
10315987	28	23	4432169	29	25
38829510	29	23	43374839	30	25

Table A.2: Numbers of unlabeled minimally 2–connected graphs by number of edges m and nodes n (continued).

Number	m	n	Number	m	n
225717631	31	25	52	28	27
682986281	32	25	6641	29	27
1283676861	33	25	441182	30	27
1575284541	34	25	12286017	31	27
1311400842	35	25	161747360	32	27
763310455	36	25	1133095396	33	27
318189093	37	25	4620457048	34	27
96889509	38	25	11726168438	35	27
21891385	39	25	19486003077	36	27
3721062	40	25	22048970645	37	27
480011	41	25	17520755376	38	27
47690	42	25	10021498631	39	27
3633	43	25	4209846518	40	27
230	44	25	1320358866	41	27
11	45	25	313438958	42	27
1	46	25	56931054	43	27
1	26	26	7986988	44	27
48	27	26	870210	45	27
5393	28	26	74429	46	27
309460	29	26	4969	47	27
7459783	30	26	279	48	27
84916135	31	26	12	49	27
514295822	32	26	1	50	27
1812062253	33	26	1	28	28
3970372644	34	26	56	29	28
5688538296	35	26	8148	30	28
5540336832	36	26	619720	31	28
3781377066	37	26	19840963	32	28
1853404235	38	26	300355260	33	28
665377394	39	26	2420481934	34	28
177857189	40	26	11358027356	35	28
35859231	41	26	33194905139	36	28
5516407	42	26	63591257772	37	28
651818	43	26	83075102573	38	28
59949	44	26	76353338507	39	28
4257	45	26	50621176228	40	28
254	46	26	24706336467	41	28
11	47	26	9026893726	42	28
1	48	26	2503293604	43	28
1	27	27	532998940	44	28

Table A.2: Numbers of unlabeled minimally 2–connected graphs by number of edges m and nodes n (continued).

Number	m	n	Number	m	n
87920340	45	28	10157819623	36	30
11324550	46	28	62173942440	37	30
1143532	47	28	237252464495	38	30
91417	48	28	594402630873	39	30
5735	49	28	1017896442763	40	30
305	50	28	1229995568862	41	30
12	51	28	1075943494282	42	30
1	52	28	695680835321	43	30
1	29	29	338247018327	44	30
61	30	29	125450344387	45	30
9877	31	29	35918899718	46	30
859635	32	29	8018457398	47	30
31454609	33	29	1407152170	48	30
544819639	34	29	195336227	49	30
5024688877	35	29	21564457	50	30
26993241006	36	29	1897014	51	30
90364786361	37	29	133950	52	30
198470302612	38	29	7522	53	30
297632317761	39	29	361	54	30
314516981658	40	29	13	55	30
240197467221	41	29	1	56	30
135331795619	42	29	1	31	31
57214093727	43	29	70	32	31
18407924859	44	29	14280	33	31
4560063913	45	29	1595366	34	31
878238872	46	29	75210204	35	31
132494221	47	29	1681549335	36	31
15758948	48	29	20033956931	37	31
1482313	49	29	139099683838	38	31
111149	50	29	602319896981	39	31
6599	51	29	1713400277682	40	31
332	52	29	3334508119330	41	31
13	53	29	4584606087011	42	31
1	54	29	4569765183940	43	31
1	30	30	3372586916193	44	31
65	31	30	1875284905013	45	31
11933	32	30	797095484258	46	31
1177541	33	30	262166360979	47	31
49028125	34	30	67407384135	48	31
966907337	35	30	13663910694	49	31

Table A.2: Numbers of unlabeled minimally 2–connected graphs by number of edges m and nodes n (continued).

Number	m	n	Number	m	n
2198872568	50	31	10495296792320	42	32
282377934	51	31	16343482297317	43	32
29063831	52	31	18474667030034	44	32
2400631	53	31	15486219617759	45	32
160089	54	31	9797128516826	46	32
8552	55	31	4747009877278	47	32
390	56	31	1783532167911	48	32
14	57	31	525061808968	49	32
1	58	31	122186329482	50	32
1	32	32	22639189551	51	32
75	33	32	3359618424	52	32
17024	34	32	401007345	53	32
2138160	35	32	38635627	54	32
113684089	36	32	3005991	55	32
2869422677	37	32	189908	56	32
38612897637	38	32	9647	57	32
302873088330	39	32	421	58	32
1482020482312	40	32	14	59	32
4766386084534	41	32	1	60	32

APPENDIX B

Numbers of unlabeled 3–edge–connected blocks

Table B.1: Numbers of unlabeled 3–edge–connected blocks by number of nodes n.

Number	**n**
1	4
3	5
19	6
149	7
2578	8
84127	9
5201474	10
577043450	11
113371887160	12
39618007594177	13
24916462201698733	14
28563626901315427552	15
60366734333061918085970	16
237406975833857522512566844	17
1750330441805919888169149998870	18
24333391253852871050342717789839845	19
640811881791468124412863736794079226908	20
32084542287292947547944886850696505721433875	21
3063386300429730850488173013366487457836083134305	22
559167842804102595767704835583730549059387248495497588	23
195549903344652082788231213269977915834450437312095461140244	24
131272508535413162857992403425071268772960060318310234038396921320	25

Table B.2: Numbers of unlabeled 3–edge–connected blocks by number of edges m and nodes n.

Number	**m,n**
1	6,4
1	8,5
1	9,5
1	10,5
2	9,6
4	10,6
5	11,6
4	12,6
2	13,6
1	14,6
1	15,6
4	11,7
17	12,7
30	13,7
34	14,7
29	15,7
17	16,7
9	17,7
5	18,7
2	19,7
1	20,7
1	21,7
4	12,8
32	13,8
132	14,8
307	15,8
464	16,8
505	17,8
438	18,8
310	19,8
188	20,8
103	21,8
52	22,8
23	23,8
11	24,8
5	25,8
2	26,8
1	27,8
1	28,8

Table B.2: Numbers of unlabeled 3–edge–connected blocks by number of edges m and nodes n (continued).

Number	m,n
22	14,9
269	15,9
1344	16,9
3949	17,9
7938	18,9
11897	19,9
14131	20,9
13827	21,9
11465	22,9
8235	23,9
5226	24,9
2966	25,9
1537	26,9
737	27,9
333	28,9
144	29,9
62	30,9
25	31,9
11	32,9
5	33,9
2	34,9
1	35,9
1	36,9
14	15,10
306	16,10
3302	17,10
18326	18,10
64633	19,10
163495	20,10
318925	21,10
503147	22,10
663882	23,10
750334	24,10
739320	25,10
643802	26,10
500700	27,10
350608	28,10
222644	29,10
129030	30,10
68623	31,10

Table B.2: Numbers of unlabeled 3–edge–connected blocks by number of edges m and nodes n (continued).

Number	m,n
33736	32,10
15464	33,10
6657	34,10
2735	35,10
1091	36,10
424	37,10
164	38,10
66	39,10
26	40,10
11	41,10
5	42,10
2	43,10
1	44,10
1	45,10
159	17,11
4489	18,11
48818	19,11
299028	20,11
1234058	21,11
3788667	22,11
9202115	23,11
18425652	24,11
31314682	25,11
46149871	26,11
59938786	27,11
69460512	28,11
72516698	29,11
68718646	30,11
59457124	31,11
47188372	32,11
34478942	33,11
23260219	34,11
14522369	35,11
8408495	36,11
4523777	37,11
2266257	38,11
1060080	39,11
464664	40,11
191795	41,11
75099	42,11

Table B.2: Numbers of unlabeled 3–edge–connected blocks by number of edges m and nodes n (continued).

Number	m,n
28156	43,11
10216	44,11
3652	45,11
1301	46,11
466	47,11
172	48,11
67	49,11
26	50,11
11	51,11
5	52,11
2	53,11
1	54,11
1	55,11
57	18,12
3600	19,12
79990	20,12
856620	21,12
5682799	22,12
26736078	23,12
96597828	24,12
282157954	25,12
690385379	26,12
1451919830	27,12
2675662077	28,12
4385359470	29,12
6467108849	30,12
8660535465	31,12
10609740093	32,12
11960701231	33,12
12467075127	34,12
12061064775	35,12
10862830624	36,12
9130263691	37,12
7175024938	38,12
5279454413	39,12
3641236401	40,12
2355839991	41,12
1430631045	42,12
815792804	43,12
436981807	44,12

Table B.2: Numbers of unlabeled 3–edge–connected blocks by number of edges m and nodes n (continued).

Number	**m,n**
219967703	45,12
104117726	46,12
46384251	47,12
19478407	48,12
7728519	49,12
2907759	50,12
1042758	51,12
358989	52,12
119787	53,12
39209	54,12
12751	55,12
4187	56,12
1404	57,12
485	58,12
175	59,12
68	60,12
26	61,12
11	62,12
5	63,12
2	64,12
1	65,12
1	66,12
1483	20,13
80652	21,13
1584694	22,13
17053755	23,13
121880202	24,13
643120107	25,13
2674261919	26,13
9153309987	27,13
26596489536	28,13
67122726016	29,13
149725747512	30,13
299248802681	31,13
541741902002	32,13
896142388631	33,13
1364206434811	34,13
1922376481380	35,13
2519660300835	36,13
3084017372636	37,13

Table B.2: Numbers of unlabeled 3–edge–connected blocks by number of edges m and nodes n (continued).

Number	m,n
3536623701638	38,13
3810075504836	39,13
3864696015298	40,13
3697591088759	41,13
3341758405204	42,13
2856156073413	43,13
2310588596232	44,13
1770436320119	45,13
1285439037164	46,13
884622166514	47,13
577108170145	48,13
356906697757	49,13
209225006560	50,13
116243299258	51,13
61197938253	52,13
30524037079	53,13
14422160333	54,13
6455096425	55,13
2737493426	56,13
1100574143	57,13
419904672	58,13
152299258	59,13
52653237	60,13
17419588	61,13
5545595	62,13
1711589	63,13
517018	64,13
154624	65,13
46403	66,13
14136	67,13
4434	68,13
1446	69,13
492	70,13
176	71,13
68	72,13
26	73,13
11	74,13
5	75,13
2	76,13
1	77,13

Table B.2: Numbers of unlabeled 3–edge–connected blocks by number of edges m and nodes n (continued).

Number	**m,n**
1	78,13
341	21,14
49885	22,14
1937072	23,14
34739369	24,14
379124019	25,14
2903529142	26,14
16965265803	27,14
79814504124	28,14
313797963056	29,14
1059277053085	30,14
3133438751472	31,14
8252041741014	32,14
19592143399443	33,14
42362118598427	34,14
84108161229888	35,14
154392323134554	36,14
263514852982140	37,14
420184445842969	38,14
628443424587665	39,14
884603171010062	40,14
1175235664317119	41,14
1477210295605181	42,14
1760290635737501	43,14
1992026376773437	44,14
2143865501808684	45,14
2196896778493476	46,14
2145642134558098	47,14
1998873297525771	48,14
1777324453721065	49,14
1509075844494511	50,14
1223970418163301	51,14
948523355501837	52,14
702422710569755	53,14
497093083860893	54,14
336158568543539	55,14
217202069001957	56,14
134064673658068	57,14
79029272594608	58,14
44479206703795	59,14

Table B.2: Numbers of unlabeled 3–edge–connected blocks by number of edges m and nodes n (continued).

Number	**m,n**
23893410506483	60,14
12246112437802	61,14
5986345417173	62,14
2790101477262	63,14
1239490431612	64,14
524730990798	65,14
211671673174	66,14
81370868134	67,14
29821328781	68,14
10427575732	69,14
3483575905	70,14
1114203919	71,14
342248478	72,14
101400526	73,14
29147822	74,14
8191288	75,14
2271757	76,14
628601	77,14
175640	78,14
50155	79,14
14774	80,14
4541	81,14
1464	82,14
495	83,14
177	84,14
68	85,14
26	86,14
11	87,14
5	88,14
2	89,14
1	90,14
1	91,14
16976	23,15
1549230	24,15
48924608	25,15
831742419	26,15
9285087005	27,15
75917811181	28,15
486369095534	29,15
2554396907343	30,15

Table B.2: Numbers of unlabeled 3–edge–connected blocks by number of edges m and nodes n (continued).

Number	m,n
11357092251432	31,15
43778937016885	32,15
149016622472673	33,15
454395076424079	34,15
1255695889625261	35,15
3174547857484640	36,15
7399567022250986	37,15
16005803896344948	38,15
32304816868143370	39,15
61120167394297336	40,15
108827865184323172	41,15
182977419092510085	42,15
291348493169416815	43,15
440419527626157683	44,15
633417223044538349	45,15
868330706398958253	46,15
1136432883867690049	47,15
1421871812880927030	48,15
1702727785659757023	49,15
1953591960401413750	50,15
2149296664640731428	51,15
2269051771620753190	52,15
2300051829469322328	53,15
2239696010620576521	54,15
2095897032535661901	55,15
1885443782468554385	56,15
1630869112758883377	57,15
1356608122065395085	58,15
1085322786956761254	59,15
835114294268116508	60,15
618022982536375203	61,15
439847417368532492	62,15
301012545262330546	63,15
198052167018485254	64,15
125255462794760113	65,15
76125313063238525	66,15
44448295908864292	67,15
24925247739821443	68,15
13419460591304722	69,15
6934015590627559	70,15

Table B.2: Numbers of unlabeled 3–edge–connected blocks by number of edges m and nodes n (continued).

Number	**m,n**
3437354278984202	71,15
1634123213232008	72,15
744727693516854	73,15
325236259307747	74,15
136062476565831	75,15
54511402849225	76,15
20909946627057	77,15
7678824958741	78,15
2699903455429	79,15
909189857098	80,15
293427305342	81,15
90859264535	82,15
27039872281	83,15
7753364082	84,15
2149511697	85,15
578888289	86,15
152373438	87,15
39499850	88,15
10176557	89,15
2632386	90,15
690989	91,15
185983	92,15
51813	93,15
15036	94,15
4583	95,15
1471	96,15
496	97,15
177	98,15
68	99,15
26	100,15
11	101,15
5	102,15
2	103,15
1	104,15
1	105,15
2828	24,16
785564	25,16
47742218	26,16
1310239034	27,16
21624840345	28,16

Table B.2: Numbers of unlabeled 3–edge–connected blocks by number of edges m and nodes n (continued).

Number	**m,n**
248258232232	29,16
2159705797392	30,16
15049208799722	31,16
87288590627953	32,16
433404721521187	33,16
1881612810192398	34,16
7261736940247177	35,16
25242954834061840	36,16
79885252728676127	37,16
232186733790389820	38,16
624357388495921769	39,16
1562898274193667864	40,16
3661009746768471508	41,16
8060915897368977479	42,16
16747469402354245720	43,16
32940938482944013622	44,16
61516856122347560926	45,16
109347994561026950302	46,16
185410550474968187258	47,16
300465070520245407204	48,16
466138585880238884329	49,16
693319080565510436984	50,16
989931085944898084831	51,16
1358369134960019291890	52,16
1793071985699083175099	53,16
2278853499868900467035	54,16
2790596968007940091828	55,16
3294730175789077860268	56,16
3752543977641624299137	57,16
4124973576519276428874	58,16
4378049711956547989865	59,16
4487975274244116058928	60,16
4444783389464481929065	61,16
4253804759700305987374	62,16
3934649401644258739498	63,16
3517959643638702790338	64,16
3040663644911162805954	65,16
2540726792385293617105	66,16
2052405079672889348226	67,16
1602774217654097703928	68,16

Table B.2: Numbers of unlabeled 3–edge–connected blocks by number of edges m and nodes n (continued).

Number	**m,n**
1209930539668632186099	69,16
882853902709977590978	70,16
622596283715769133785	71,16
424278002517748421130	72,16
279348539717269853543	73,16
177667827106268177090	74,16
109129379058620164528	75,16
64720058804962930372	76,16
37049681000895634991	77,16
20466947054737212306	78,16
10907130849675958491	79,16
5605501880145849191	80,16
2777252189839554804	81,16
1326041287755565278	82,16
609928613076670687	83,16
270157943818830322	84,16
115188644452303711	85,16
47260098697696063	86,16
18651793939202057	87,16
7078643854491827	88,16
2582665755417853	89,16
905720157132715	90,16
305277501328157	91,16
98901969013308	92,16
30807093039918	93,16
9231707217534	94,16
2663868661576	95,16
741261969825	96,16
199325781356	97,16
51944565858	98,16
13169750704	99,16
3264728256	100,16
796276223	101,16
192524702	102,16
46540921	103,16
11353423	104,16
2821104	105,16
720294	106,16
190442	107,16
52496	108,16

Table B.2: Numbers of unlabeled 3–edge–connected blocks by number of edges m and nodes n (continued).

Number	**m,n**
15144	109,16
4601	110,16
1474	111,16
497	112,16
177	113,16
68	114,16
26	115,16
11	116,16
5	117,16
2	118,16
1	119,16
1	120,16
227642	26,17
31677060	27,17
1488985320	28,17
37271409318	29,17
607259104468	30,17
7193835659919	31,17
66380164248122	32,17
499747352507621	33,17
3172977632844284	34,17
17413612006031254	35,17
84188163107834958	36,17
363961427394595318	37,17
1424101858756803592	38,17
5093310318792697235	39,17
16787912611317168584	40,17
51348053586973224950	41,17
146595236310213948275	42,17
392602927894036831128	43,17
990585522004913295217	44,17
2363477873970216606691	45,17
5349814741332519831712	46,17
11520805305288304072695	47,17
23662583366358710664001	48,17
46454268316828467466933	49,17
87339455094355869886432	50,17
157527729045134495983344	51,17
272973768162607909567993	52,17
455075735239514270668157	53,17

Table B.2: Numbers of unlabeled 3–edge–connected blocks by number of edges m and nodes n (continued).

Number	**m,n**
7307375767410760124327500	54,17
11313904811904405691804210	55,17
16906118441131118199248120	56,17
24401518095157958682011980	57,17
34044870464251058246885610	58,17
45944348809284830715719810	59,17
60007947636786176285267550	60,17
75893159815384215582998730	61,17
92983888761494392827832010	62,17
110406235758223317245562890	63,17
127088942055847126332440150	64,17
141865700059004046857789690	65,17
153607090661004475635993340	66,17
161361953225089233033746310	67,17
164483856957625503035950820	68,17
162719489434313923478478620	69,17
156242305873751497194923340	70,17
145625284523721348191065690	71,17
131758560884851275125414490	72,17
115728119909154467530180260	73,17
98678139261282099970232390	74,17
81680704494313951969865500	75,17
65632643343701016676960760	76,17
51191636789192134776644070	77,17
38754787785552328426056930	78,17
28474721664350274094825210	79,17
20302719047512923621995980	80,17
14046068867859858523403720	81,17
9427556689833950631543920	82,17
6137873375482591858971180	83,17
3875561116822465335816870	84,17
2372824076330302847160670	85,17
1408380940718330191834400	86,17
810215930665099085419290	87,17
451650962027249590962300	88,17
243901954450682220704510	89,17
127561539187134734966840	90,17
64593939255715952808850	91,17
31659240133917414322130	92,17
15014440228393327007130	93,17

Table B.2: Numbers of unlabeled 3–edge–connected blocks by number of edges m and nodes n (continued).

Number	**m,n**
688772211504861683291	94,17
305529488244754499045	95,17
131006030661955888704	96,17
54279696917608551911	97,17
21723930826471532615	98,17
8395434971499986169	99,17
3131886216890559234	100,17
1127432279400850236	101,17
391539480210411695	102,17
131148292896286953	103,17
42362846690341489	104,17
13195349684541114	105,17
3963768188271187	106,17
1148593787386434	107,17
321234371472252	108,17
86784355715478	109,17
22676298047750	110,17
5741050447394	111,17
1411765781454	112,17
338296035201	113,17
79325090712	114,17
18296522341	115,17
4177278638	116,17
950868541	117,17
217511848	118,17
50411379	119,17
11932162	120,17
2905508	121,17
732471	122,17
192218	123,17
52763	124,17
15186	125,17
4608	126,17
1475	127,17
497	128,17
177	129,17
68	130,17
26	131,17
11	132,17
5	133,17

Table B.2: Numbers of unlabeled 3–edge–connected blocks by number of edges m and nodes n (continued).

Number	m,n
2	134,17
1	135,17
1	136,17
30468	27,18
13669120	28,18
1210873170	29,18
47841442385	30,18
1127136025603	31,18
18334461233106	32,18
224568380060117	33,18
2191608814091101	34,18
17725742263943658	35,18
122284536418181669	36,18
735443035711132345	37,18
3922337060370798650	38,18
18805243620686282812	39,18
81953951455741433947	40,18
327642293604137086638	41,18
1210882104600056022170	42,18
4163844669982617165315	43,18
13396270537957616687256	44,18
40517212836127754672174	45,18
115678754661997417540200	46,18
312886633249911475581824	47,18
804278289961599382635059	48,18
1970213558350503108493538	49,18
4610719451141330153744465	50,18
10330281467295952802662247	51,18
22201300308761345855695181	52,18
45846998661188325375436478	53,18
91111507129516927491600536	54,18
174485266287360213941376907	55,18
322402894290482533099765766	56,18
575400551295931702430330593	57,18
992888509191990097566999110	58,18
1657958622309123602996425789	59,18
2681235723758783962040260247	60,18
4202372108532034866470171218	61,18
6387497990691971856216229667	62,18
9420933094454651929284464808	63,18

Table B.2: Numbers of unlabeled 3–edge–connected blocks by number of edges m and nodes n (continued).

Number	**m,n**
13489886621902436054143079272	64,18
18761818842804088091606960258	65,18
25355585505391458082187970016	66,18
33309253926666483067307660482	67,18
42549218121748870395402244758	68,18
52866483091592813680210586727	69,18
63906257374359861785102591066	70,18
75175942832933235709606113208	71,18
86074170620939467494478934213	72,18
95939995118343125583731500379	73,18
104117372069909621664905854796	74,18
110026495688305671444868662166	75,18
113231351864430414048875448132	76,18
113492624568320431097113022296	77,18
110797086318020892340282503336	78,18
105358484046076532182823150799	79,18
97589887421317988432799289854	80,18
88052406397822882308952111589	81,18
77389026672404697345043656927	82,18
66254268680265882493763684913	83,18
55250148562912914284763558192	84,18
44876743451502554397573261735	85,18
35502202821053575856856304134	86,18
27353183095502020135134661059	87,18
20523263852988587859449692306	88,18
14994545122095972306503500768	89,18
10666595192485858195378599129	90,18
7387140872496153139995772768	91,18
4980035950800667624438882991	92,18
3267658563829964261997809263	93,18
2086539468185025289793188349	94,18
1296387916131780615968412584	95,18
783591104941337199953367559	96,18
460694328656968769343989319	97,18
263404203965772767074226812	98,18
146430488278036974350518691	99,18
79130996656083417729476743	100,18
41559491847017540749644583	101,18
21207976438916891535720066	102,18
10512969173511023672927825	103,18

Table B.2: Numbers of unlabeled 3–edge–connected blocks by number of edges m and nodes n (continued).

Number	m,n
5060987912734632192601392	104,18
2365432202419169276153297	105,18
1073069052269954823005142	106,18
472344474151200545557701	107,18
201684172286637021938775	108,18
83508504530000433653345	109,18
33519431650416824653335	110,18
13038535475687289081357	111,18
4913441120002496164033	112,18
1793193532103427440271	113,18
633601477154712285017	114,18
216681799910068618558	115,18
71701123296693282621	116,18
22952092380411139798	117,18
7106061760865574416	118,18
2127620103942714203	119,18
616037162567006123	120,18
172509015864489864	121,18
46733207470892001	122,18
12253485307598559	123,18
3112054194732070	124,18
766437438104117	125,18
183329411358621	126,18
42681361461580	127,18
9698621506676	128,18
2158737329341	129,18
472752589612	130,18
102405803387	131,18
22077089148	132,18
4769059905	133,18
1039651192	134,18
230341682	135,18
52211400	136,18
12180204	137,18
2939611	138,18
737248	139,18
192917	140,18
52872	141,18
15204	142,18
4611	143,18

Table B.2: Numbers of unlabeled 3–edge–connected blocks by number of edges m and nodes n (continued).

Number	**m,n**
1476	144,18
497	145,18
177	146,18
68	147,18
26	148,18
11	149,18
5	150,18
2	151,18
1	152,18
1	153,18
3454125	29,19
687215281	30,19
45590910469	31,19
1598343801247	32,19
36212257732357	33,19
592677952730020	34,19
7513500111890517	35,19
77345398976291906	36,19
668782568126919011	37,19
4981567667409602982	38,19
32594392881939476954	39,19
190245827223724423150	40,19
1002986230869423264536	41,19
4825347013273988055411	42,19
21365824470264567468491	43,19
87698061661020735602575	44,19
335735325413550040151576	45,19
1205098440845544656619366	46,19
4074179505314196090522011	47,19
13024716949848724826234888	48,19
39510403798632156646690280	49,19
114075825009436326773348618	50,19
314329334105441571926985251	51,19
828559428992455146086057950	52,19
2093796117136615141264305292	53,19
5082095494141433910140840354	54,19
11868298768421888842907396077	55,19
26707622843270844640235053219	56,19
57993643395043079467562997285	57,19
121664057770532602998570778457	58,19

Table B.2: Numbers of unlabeled 3–edge–connected blocks by number of edges m and nodes n (continued).

Number	m,n
246869230244529273971258656319	59,19
484991107534416036012344632219	60,19
923337401901733267830956274605	61,19
1704932639667275542070862054938	62,19
3055630610703621485279156427106	63,19
5319094202325011975531663884026	64,19
8998843341362280175564587941294	65,19
14804441925919760642553493236002	66,19
23696017477302462683014105207187	67,19
36917968347807475750785263518045	68,19
56009513362040586219508777754468	69,19
82777592818805056258495572873027	70,19
119217659660958607299468465996729	71,19
167371199197833639657869014863218	72,19
229116106766986974568361506432338	73,19
305897298520221214969450553864460	74,19
398419022066792959768264869297336	75,19
506334984728532052003062880039391	76,19
627984385178826341667053090334199	77,19
760227611343299756597191417205104	78,19
898431669859563808256080256145735	79,19
1036640771979798363031229983112348	80,19
1167942670422027514488360883089610	81,19
1285009714899251378459325769463161	82,19
1380760832354002018633786770196519	83,19
1449063584873797751321238756835701	84,19
1485380549440595232477610149796231	85,19
1487265836635581144819646288958465	86,19
1454636559596513162539677040792291	87,19
1389777530865367621674394043942923	88,19
1297079088015628788508672447198451	89,19
1182549484922775729818730329620406	90,19
1053176532044904634258449318348170	91,19
916231920447540049670876099817342	92,19
778613105810343903903397241307583	93,19
646302805874291168145951665199085	94,19
523999450488627127995173682016844	95,19
414939700576512581764434467221125	96,19
320903139009705460729439935104097	97,19
242364936026324429078590866626304	98,19

Table B.2: Numbers of unlabeled 3–edge–connected blocks by number of edges m and nodes n (continued).

Number	**m,n**
178747958260163796348861061942994	99,19
128722098088868693715202781341825	100,19
90503984747699286996751750411253	101,19
62121697309535800977199780932087	102,19
41623152776341164799923784297655	103,19
27220384298562258790592948048696	104,19
17372776314627298565174256106029	105,19
10819431335415975030099759687800	106,19
6574168228850041730329154082433	107,19
3896867561748753914594575930438	108,19
2253006774933706592187640580187	109,19
1270314172353116084063635144287	110,19
698373097043152722397848291217	111,19
374294599021133895328895416767	112,19
195527260502689330375807447087	113,19
99536576884730201406690843218	114,19
49368455177219678650445328464	115,19
23851378746262026290573592457	116,19
11222149140340924297201214754	117,19
5140844622256533277296075696	118,19
2292355455981962856249189723	119,19
994735355829206704548658657	120,19
419948834447718036251126408	121,19
172436744666936387664382835	122,19
68847121487848097227983663	123,19
26720168099932402377324520	124,19
10077723373825822777940474	125,19
3692548744222607289243455	126,19
1314015831075018776754445	127,19
454000278979922831471452	128,19
152252951717333220339927	129,19
49545799159046304052004	130,19
15640990884664100544422	131,19
4788878091280899817016	132,19
1421761719822775849680	133,19
409238373256706059234	134,19
114194303294988894881	135,19
30890984720159107504	136,19
8101950566836871327	137,19
2060795852081343281	138,19

Table B.2: Numbers of unlabeled 3–edge–connected blocks by number of edges m and nodes n (continued).

Number	m,n
508591904958081802	139,19
121871515964695983	140,19
28384299488607543	141,19
6434443979614378	142,19
1422396921546447	143,19
307378985990161	144,19
65136315649478	145,19
13587381672925	146,19
2802900889488	147,19
574836378916	148,19
117896318057	149,19
24331970472	150,19
5084971965	151,19
1082481155	152,19
236007961	153,19
52951697	154,19
12277229	155,19
2952621	156,19
739070	157,19
193186	158,19
52914	159,19
15211	160,19
4612	161,19
1476	162,19
497	163,19
177	164,19
68	165,19
26	166,19
11	167,19
5	168,19
2	169,19
1	170,19
1	171,19
396150	30,20
258690891	31,20
31809846020	32,20
1731259169371	33,20
55808813896800	34,20
1234207512751078	35,20
20434751888269012	36,20

Table B.2: Numbers of unlabeled 3–edge–connected blocks by number of edges m and nodes n (continued).

Number	**m,n**
268232579371457436	37,20
2905416434573810815	38,20
26743758128731955780	39,20
213926509011604335618	40,20
1513342073620541267364	41,20
9601260364097970424575	42,20
55259256081899936858450	43,20
291260933176347223213832	44,20
1417139760293936298591000	45,20
6408044470472211178413565	46,20
27085008840334811206349991	47,20
107544540906088772672398346	48,20
402889709343816232619103087	49,20
1429450817699996391829006137	50,20
4819323452430380015209336865	51,20
15485212290870431106593339764	52,20
47544378721184075944130708229	53,20
139812222431357216128749542522	54,20
394603562801045641310372193044	55,20
1070926065424256104338402692550	56,20
2799443682078517319172484438281	57,20
7059226783028209516078191139404	58,20
17195377954561714226284503689288	59,20
40511225128517823027867901545186	60,20
92414041795222613879924801305133	61,20
204335924985130111857873988217226	62,20
438329721812728755673794611749355	63,20
913010702591147912294617029878397	64,20
1848019456123922539316408286885644	65,20
3637480448730754315038446018490435	66,20
6966899217917365740722362248398779	67,20
12992130851183850340654156253814727	68,20
23602536505457488171772864868955303	69,20
41791692199881757406798723019110663	70,20
72155863322152057736962925344039127	71,20
121530282052246298178576480640264993	72,20
199753417459214001150298299616891058	73,20
320518025963200213513936628498105130	74,20
502224772019000251689096945567395025	75,20
768702626124484006057897898039543554	76,20

Table B.2: Numbers of unlabeled 3–edge–connected blocks by number of edges m and nodes n (continued).

Number	**m,n**
114960715639040249878058001335922668 6	77,20
168026473596192762344622494001301277 0	78,20
240071274440911252367351569090604625 1	79,20
335370956124976518771121355143042805 8	80,20
458156758425527026501816373860691894 2	81,20
612180434723934492289358835524430119 5	82,20
800180548543288384438949053036740680 2	83,20
1023292752990628983129877023721517792 2	84,20
1280470102565233390945990268755235292 4	85,20
1567997600023909925136084152909330170 5	86,20
1879192976001143501294844841326750613 0	87,20
2204378756590864825099455426914871287 3	88,20
2531186808872625886023025081250437787 2	89,20
2845216834736239928524506506855177669 0	90,20
3131019475628859352880989381706555140 4	91,20
3373320832478347583356343665158600184 3	92,20
3558358260161288600419260117054173025 8	93,20
3675167110417077750304444820042454183 2	94,20
3716652109957997423645028272301259524 2	95,20
3680298358392788016892424268001907933 0	96,20
3568423200541830298467814013323459694 5	97,20
3387933955695284322404320160948270242 0	98,20
3149626422591194295044061024366990593 8	99,20
2867122639029967810314467528755337130 6	100,20
2555592377709381756580626270317875329 1	101,20
2230423904278624327163711006051745536 9	102,20
1906003416177505112588412622730000056 0	103,20
1594732492482919456534082683635333525 4	104,20
1306366274127335039852800090125239684 0	105,20
1047701752662755629281569401074709883 0	106,20
822595326207919200672801543932178533 4	107,20
632249583085014922203650625050019599 5	108,20
475685739629593517254480807129816092 2	109,20
350311281559783233507112186260898684 8	110,20
252499943783298580952837949438307659 5	111,20
178118888073776495196991466859209493 7	112,20
122960642929977339972001813929737956 0	113,20
83060246700161846036672825842511147 3	114,20
54897459365849844514964879886258012 5	115,20
35497820370160249198223140095181051 1	116,20

Table B.2: Numbers of unlabeled 3–edge–connected blocks by number of edges m and nodes n (continued).

Number	m,n
22454193005628357181888073210246602	117,20
13892897676461702706846322256580310	118,20
84069503545908087806711552322468185	119,20
49748765787351868946856554532498073	120,20
28785214493335304590108224034229810	121,20
16283294373790287328311460872843969	122,20
9004096473986090967912577652698410	123,20
4866307159146317323129139033867070	124,20
2570122539147463713338054309160937	125,20
1326274436853807627974932906331080	126,20
668598473556260870148077992593571	127,20
329210517673293588544437395102008	128,20
158299300366202866530399156251497	129,20
74318775832062197174455080899135	130,20
34060125762762958818056733415028	131,20
15234652498049483201389970919937	132,20
6649124552467094405504389658143	133,20
2831037016943169886900313679425	134,20
1175644238694897817009152214220	135,20
476049759736780185019863320426	136,20
187917986476809636206106816182	137,20
72296011925221868321867754432	138,20
27100536552675623815421033973	139,20
9895643448212087885657720900	140,20
3518809469215246674471137445	141,20
1218188518954373491963941778	142,20
410469644257777841397889905	143,20
134578728883610362687948956	144,20
42922326202321018477964134	145,20
13313350850537483688040795	146,20
4014948521124898160046136	147,20
1176956063192583969693850	148,20
335304025749188321037861	149,20
92820047987119934843961	150,20
24964141568631475698713	151,20
6522866460655978035463	152,20
1655841933087491554371	153,20
408428773643995666809	154,20
97915078529053938182	155,20
22825160245224236477	156,20

Table B.2: Numbers of unlabeled 3–edge–connected blocks by number of edges m and nodes n (continued).

Number	**m,n**
5177268405698579126	157,20
1143725640006467101	158,20
246397974391751011	159,20
51854408408921164	160,20
10683644651354675	161,20
2160902468773771	162,20
430526581202360	163,20
84832494165305	164,20
16608816150719	165,20
3247646989637	166,20
637725170121	167,20
126457606707	168,20
25458149343	169,20
5228952926	170,20
1100528496	171,20
238253063	172,20
53233294	173,20
12313532	174,20
2957523	175,20
739775	176,20
193295	177,20
52932	178,20
15214	179,20
4613	180,20
1476	181,20
497	182,20
177	183,20
68	184,20
26	185,20
11	186,20
5	187,20
2	188,20
1	189,20
1	190,20
58017904	32,21
15784066241	33,21
1421142524673	34,21
67208502451502	35,21
2041497264716565	36,21
44541184989504720	37,21

Table B.2: Numbers of unlabeled 3–edge–connected blocks by number of edges m and nodes n (continued).

Number	**m,n**
748868298916564053	38,21
10177969728564296721	39,21
115736473950291584606	40,21
1129863783952311570743	41,21
9660176164101512621934	42,21
73487327345833958913117	43,21
503808457689190496152050	44,21
3145708792693930155813438	45,21
18046394141939881405570438	46,21
95831279182935946227138047	47,21
474047916066211660070568522	48,21
2196370812842863233129405159	49,21
9576658468052413878542228752	50,21
39458756657227034051920925734	51,21
154195606777624541785391515088	52,21
573315592597904940066517493226	53,21
2033975488452822762927571174132	54,21
6902880666550040315650026377746	55,21
22461253682989421806388617587498	56,21
70216951238371335955548584415770	57,21
211276671307779504641024391385231	58,21
612890160999689818906263648674582	59,21
1716670878487250713412130530395205	60,21
4648963500650577574814072325862275	61,21
12187867884135969638840662612435080	62,21
30966478594632977072246277583618732	63,21
76330055213605022252506320302495091	64,21
182703886567079400499092167699565674	65,21
425033013413664273964071240300840758	66,21
961752282230045877721764065147486137	67,21
2118284343342427733091771034830554397	68,21
4544383168106079899289888375765962035	69,21
9501697604917019874530915920664193111	70,21
19373486062554713979673110951338179020	71,21
38540740244459918024787371202893356775	72,21
74842065512903535132960304229954965543	73,21
141930738667115103697060986085564990464	74,21
262959496453586818460892732202712381656	75,21
476152469593271731738154611573995906089	76,21
842943690094514031235894644928940229038	77,21

Table B.2: Numbers of unlabeled 3–edge–connected blocks by number of edges m and nodes n (continued).

Number	**m,n**
145943730713657678379956881288671150567 7	78,21
24719236630883839146827095619872503385 36	79,21
40969998501671339132089849701259708850 20	80,21
66464171118923556772191258075439276805315	81,21
105560407615109587187405702703134342741 34	82,21
164172203174476504481235305027026216496 63	83,21
250074471908087232979713463422275205016 50	84,21
373155305076408749953304490037886358237 73	85,21
545548712202933919814172587800658907252 38	86,21
781570995924437362460640470268972715672 18	87,21
1097377991161082722729829170377996326426 47	88,21
1510267429763308492318568026104469070765 63	89,21
2037574262479798297353525237110295018898 16	90,21
2695149070172610201643127092505553024002 53	91,21
3495469715956888392257529348798386878300 66	92,21
4445508933000815849605878189953848509898 57	93,21
5544555941333771611574161959736885071983 75	94,21
6782255046425370404917122580683140931986 87	95,21
8137163282482335416490000663352724207680 08	96,21
9576128424247675510584364830235304622871 87	97,21
11054738563711834223341108444709513447915 11	98,21
12518993165382226086904348710181779231478 14	99,21
13908201160214958936308819143860063073385 66	100,21
15158942322678309968937498473623078170585 29	101,21
16209759881197587039922391796382633471709 87	102,21
17006114566852400714220215440653297721748 91	103,21
17505050415647815442736976036109520461704 77	104,21
17679019729723199299207965267060906368086 30	105,21
17518394888835819087532116126252409050822 30	106,21
17032349268277221947298202735118910129078 68	107,21
16247995301768621728464804009943710027247 12	108,21
15207891665083610461232755452235607004855 28	109,21
13966237014668269400706352138482289762311 84	110,21
12584221745637903094944996503417514596055 01	111,21
11125088965642493824677646005011106396116 77	112,21
9649452610287541132886885163053552392665 44	113,21
8211340736601388008112035216465964301688 05	114,21
6855294720044700336150315223066994720942 15	115,21
5614687608824925147072760883867054971998 26	116,21
4511256630360955649897888994270874502456 40	117,21

Table B.2: Numbers of unlabeled 3–edge–connected blocks by number of edges m and nodes n (continued).

Number	m,n
355570156119677755729585437156966473752992	118,21
274910024235247779486017550699188622092850	119,21
208484285138812285953696220794441943671772	120,21
155078577510250641069718958161968698879706	121,21
113136454440245427271384244092298230911051	122,21
80946931224044292333569440752414576900874	123,21
56796074521457395211257656473405648905902	124,21
39077575913912231357523739559116083366260	125,21
26363175703656169647350499188337967778256	126,21
17437983460361152847364664645133657230188	127,21
11308082711641982142177334903847176651760	128,21
7188523859910655077982382662038706230558	129,21
4479302587884848464028742657653438102757	130,21
2735652703203560646130551683645735911500	131,21
1637379399938189028294959817807440988203	132,21
960353982217188359412583696736012889566	133,21
551900480413958732228461923707994124353	134,21
310734310010687505790444238314729876558	135,21
171382328485096633176566274350079726959	136,21
92584521908386920062129825127320602168	137,21
48983641259493111258479445755389696198	138,21
25377363924338042308672232396910799498	139,21
12872553901613023313457079571451839635	140,21
6392094911464266467571605375493227696	141,21
3106826218654762988921219790052307062	142,21
1477809036813430961325106938769928473	143,21
687823450464129718869639326299782629	144,21
313198676401325722555828187314507072	145,21
139499079313051243423887708909077801	146,21
60764878947501543916013040903799403	147,21
25881076741468352892319186435921738	148,21
10776459613523346454957104615589123	149,21
4385778471838770749773373962854985	150,21
1744230856711452561811861380893809	151,21
677727220465347109368202688006907	152,21
257219432187397722353513477889686	153,21
95334913442502776761805277553511	154,21
34498358046782251466073436527946	155,21
12185395419967405368556839271263	156,21
4200199966227428316187514405048	157,21

Table B.2: Numbers of unlabeled 3–edge–connected blocks by number of edges m and nodes n (continued).

Number	**m,n**
1412479228112105458563993058523	158,21
463305295907767761934709928529	159,21
148189077776132113718213230157	160,21
46208258626159718240369937991	161,21
14043304707092534821281407296	162,21
4158718636016445794041395486	163,21
1199743516368252962439654060	164,21
337101016171244360330669873	165,21
92233165547281183448409076	166,21
24569361399372831586518134	167,21
6371185700122594781833379	168,21
1608151031034850927647979	169,21
395094405676087676633744	170,21
94485278158154079328429	171,21
21997955027063323742190	172,21
4987428524155481576699	173,21
1101636978510361014274	174,21
237217205834682351871	175,21
49840893370642343302	176,21
10229983214534518774	177,21
2054418717776179686	178,21
404472802852673182	179,21
78261845571386159	180,21
14927233150189303	181,21
2816622710798436	182,21
527939073705635	183,21
98748032364212	184,21
18521618443886	185,21
3501007982998	186,21
670143401754	187,21
130480742687	188,21
25945377023	189,21
5287050488	190,21
1107432710	191,21
239083371	192,21
53336122	193,21
12326876	194,21
2959362	195,21
740045	196,21
193337	197,21

Table B.2: Numbers of unlabeled 3–edge–connected blocks by number of edges m and nodes n (continued).

Number	**m,n**
52939	198,21
15215	199,21
4613	200,21
1476	201,21
497	202,21
177	203,21
68	204,21
26	205,21
11	206,21
5	207,21
2	208,21
1	209,21
1	210,21
5909292	33,22
5274624215	34,22
868755846653	35,22
62976533375735	36,22
2688560119147580	37,22
78302251497532197	38,22
1698563127563828317	39,22
29076611078616773333	40,22
409061697726784456782	41,22
4872955794215162303939	42,22
50286704118487817782046	43,22
457653193275710790207593	44,22
3726262757561068334338312	45,22
27463559470441553082416324	46,22
185017656209561541165464095	47,22
1148671849995191165891509512	48,22
6617985642385662730952382579	49,22
35595478464589671144696489636	50,22
179657736143209445527254510211	51,22
854739767414822203482462895038	52,22
3848340348541415548581646349933	53,22
16454216565998416875964326085878	54,22
67017191950391542275770008426243	55,22
260732410662265993178015816767999	56,22
971341317275465401649790640501668	57,22
3472767381571225149272684320003149	58,22
11939062051711725250408777663954514	59,22

Table B.2: Numbers of unlabeled 3–edge–connected blocks by number of edges m and nodes n (continued).

Number	**m,n**
39539912512169155540318756172554511	60,22
126351119301021455588075405110727032	61,22
390159737139992138751391441862130989	62,22
1165763119151825167676650652586035966	63,22
3374550851985377787723864776161130230	64,22
9474296081657277671327798606509593447	65,22
25825556509272025736110676985203856436	66,22
68412224819396224957269083156547645671	67,22
176268725001363525650678654241742659770	68,22
442099950136323201811449917566688658704	69,22
1080160193215874359872451489209393952901	70,22
2572600976248577705153753407211241174194	71,22
5976470038506333099996207489016771882127	72,22
13550522145526768959823355364540917724352	73,22
30001183193658329596916337758685425019220	74,22
64894319792426347196921963065586644238648	75,22
137201867733222546271047557002530708627232	76,22
283650127310908918054574698583815663223121	77,22
573649744981563953657999224373321629348705	78,22
1135299492236090306021015564179469199442334	79,22
2199487611619340884413149344056604384126363	80,22
4172700277423434804943880725297335433035985	81,22
7753984640861531323379404089476835042657374	82,22
14117678389965797570161818706646611596171246	83,22
25190829256706743864100655165153116660474700	84,22
44062081777424983102014206470013139422010818	85,22
75565980035172955561784436402650437608106989	86,22
127091099664666817558359693879350643937065170	87,22
209659257419463641885659606698895166163539047	88,22
339312502669367348067157836634629924445703820	89,22
538820439199048016220810543787797779032542602	90,22
839678644140378455287338746813560143896219899	91,22
1284306852060323382139506562791924653568809782	92,22
1928273230272699858231597148344048631865934165	93,22
2842272919255552189715701283070034288490305684	94,22
4113485233102488452887786613367477647814885390	95,22
5845841147181995261364457047964000174298169202	96,22
8158673504756022587044719242719438728791494105	97,22
11183222996415226092245114926524105859878270078	98,22
15056559170190654689717659141598265684917244501	99,22

Table B.2: Numbers of unlabeled 3–edge–connected blocks by number of edges m and nodes n (continued).

Number	m,n
19912666929241354781888619561926284752556129589	100,22
25870751441253827476477244553179398978044980699	101,22
33021214905549721390293809623964472208634637093	102,22
41410220360392908493418020879479736493522655498	103,22
51024220077896158688411874663875154450768157793	104,22
61776210015670134302511571985079830046598369244	105,22
73495690734070372121276398305919602556125781503	106,22
85924290900099390654528950956281588954593417716	107,22
98718690013405893913027292574948708920444938160	108,22
111461853079723300500190461207881703572987435925	109,22
123682705867215502034041083212289560593279311217	110,22
134883334123915427153845007066872261026992796196	111,22
144571727296503809105388576632128160401582854353	112,22
152297173771931757581431189863195032219767877545	113,22
157684811548885631357025181308134410332895562978	114,22
160465669367852540176780790686306617799855149369	115,22
160498858296609834077470584425789220160379322586	116,22
157783371873110902884973259881426199084748134141	117,22
152458122261094249380709589246798684087866254514	118,22
144790212859240928257129922968416227995261754457	119,22
135152820172142563079840203780557411753956032412	120,22
123995225437603944436710042074247181544610859759	121,22
111808332404321389463035774519185082295531369955	122,22
99089330573366507452699360561277248047936302507	123,22
86308993139238839452169731506471745995145040348	124,22
73884495903657677912629406982238095327289106726	125,22
62159731595665986168415331802657007421355014719	126,22
51394034199778706524728674221849433951048988645	127,22
41759186440150549362948336990520699846512415422	128,22
33343703349708673777484839812369554549632202790	129,22
26162764124863240068650083303467152338669558254	130,22
20171846864626387198737932381687343661806395308	131,22
15282096759956093474320023611525310145662755385	132,22
11375675868881479328781614689545634620961456893	133,22
8319723856614603367973482524676471322773698588	134,22
5978017996570269801763504952597613007082447827	135,22
4219878875315855594317471338444984928562270179	136,22
2926265620236089181540604540019019559418820910	137,22
1993304664724653900201372460616579469833141989	138,22
1333686086068997301643138333945231324501355805	139,22

Table B.2: Numbers of unlabeled 3–edge–connected blocks by number of edges m and nodes n (continued).

Number	**m,n**
876447665398356318014172950932618470422120866	140,22
565668254351571412343700265271499531622265140	141,22
358534254138631161803565846115910253214782783	142,22
223151928548115107853255820794756930480259989	143,22
136376140045517815630818130338739371687168496	144,22
81829329437784872164627722078616603839912285	145,22
48203210848582819495419844192788352456731581	146,22
27874125724691139470831489995658837126609530	147,22
15821408898645630873530432924519579304527715	148,22
8813863121963792935133062208463332124473677	149,22
4818608238191650963293420497991693767508339	150,22
2585034116944276806812409374580485143534695	151,22
1360672671970105471175012875828629285691566	152,22
702643947202360315460615385907976867139079	153,22
355926953378303666689879908477579471566564	154,22
176839016452207824498538710686918539165530	155,22
86165357708891556332460462361735040914499	156,22
41168818781991668683376522890954749949615	157,22
19285320476819015968625755155956408661761	158,22
8856203013044179389781770193714120048496	159,22
3986280389459769944446392153781830133104	160,22
1758420741417204557687099584135714701215	161,22
760055402151602199766466156345373491917	162,22
321859032751046267676225713001582987658	163,22
133509537857293960121233084428608194478	164,22
54238886399758377815853001501092769262	165,22
21576670603745798090191517237451678868	166,22
8403383081691225070996253298207947867	167,22
3203600611618896486205739664777317827	168,22
1195234204030436505977325447415682949	169,22
436326547441903858554614261224891278	170,22
155820792780296625721491901786736510	171,22
54425597147688027390153277441587822	172,22
18588815376289929116463715158859566	173,22
6206912554957372729188178017295617	174,22
2025712611400960526528216653276627	175,22
646040460122717923364800819858404	176,22
201289699434579932377497312437131	177,22
61258021324678078612797924906320	178,22
18204702078253857802828174569148	179,22

Table B.2: Numbers of unlabeled 3–edge–connected blocks by number of edges m and nodes n (continued).

Number	**m,n**
528182217844592419124202871404 4	180,22
1495773089184134272789905388646	181,22
413365804412929260710910797736	182,22
111455106624032773016297972095	183,22
29314208461826321948082504398	184,22
7519590876074345688442276218	185,22
1880983662857196227051864246	186,22
458778548433480787130004094	187,22
109099081767447768262096131	188,22
25295161355414540781384175	189,22
5718517006717131482493979	190,22
1260757529648469275669830	191,22
271147936955371863629318	192,22
56911033510978680239093	193,22
11664618715780725516391	194,22
2336654242930683318845	195,22
457985830796493151109	196,22
87956458643518885368	197,22
16581744039044664943	198,22
3075494059140576243	199,22
562732256992832574	200,22
101902233466560329	201,22
18329830635095390	202,22
3288516318340236	203,22
591023331901959	204,22
106884565967396	205,22
19535841032487	206,22
3623530476513	207,22
684550272166	208,22
132140242322	209,22
26134330801	210,22
5308573469	211,22
1109921553	212,22
239380403	213,22
53373313	214,22
12331825	215,22
2960069	216,22
740154	217,22
193355	218,22
52942	219,22

Table B.2: Numbers of unlabeled 3–edge–connected blocks by number of edges m and nodes n (continued).

Number	**m,n**
15216	220,22
4613	221,22
1476	222,22
497	223,22
177	224,22
68	225,22
26	226,22
11	227,22
5	228,22
2	229,22
1	230,22
1	231,22
1064020440	35,23
383118363555	36,23
45417776539662	37,23
2813397116148746	38,23
111339372400773896	39,23
3148741704342410344	40,23
68301462153114293217	41,23
1192671408904027971096	42,23
17360133045057598874086	43,23
216223214548580980863771	44,23
2351719946873373117706206	45,23
22699122620530349098241788	46,23
196994353487491879383402861	47,23
1553831786661278935411527097	48,23
11240269703086785037903737045	49,23
75142675646583397442390970960	50,23
467269738278151382633945845718	51,23
2718092215172509400613401474548	52,23
14862820746651523831926945477022	53,23
76725335381326473237535469004336	54,23
375331310361562296361666712664451	55,23
1745739713015000346665906045026130	56,23
7743248850074877071798332555621368	57,23
32839690656066248256267367232406927	58,23
133487791594479826365841984045417110	59,23
521173358079859427049434631661805618	60,23
1958224811144821480525508296105897398	61,23
7093246697914629034310259267802461109	62,23

Table B.2: Numbers of unlabeled 3–edge–connected blocks by number of edges m and nodes n (continued).

Number	**m,n**
24809758635002608492183453124901135370	63,23
83912375079977534347443445680545480705	64,23
274809423775819939079679662815386316303	65,23
8725021215614877491012705651130361205 52	66,23
26885236992355717984176117467304596003 93	67,23
80485487119037937631853581369403223278 29	68,23
234307354488680035136165469126897105836 64	69,23
663888458888571685995391433188057157633 27	70,23
1832282056996522388200376036666893465081 67	71,23
4929448362900862254348035331183605736920 55	72,23
12936293513512504856247187917765122889104 66	73,23
33136132428743578609629923092336803978623 09	74,23
82895138327112661366186642110442894637006 34	75,23
202641388509714494471872269589075991353053 58	76,23
484304905288360490857400242492158551645587 45	77,23
1132154150523463409419535888220343340979867 18	78,23
2589872671634549210132506100123978384846159 06	79,23
5799820171117798705660130321473769864519003 431	80,23
12719761482129251003943134393239103643377276 13	81,23
27329120774946168601022815381208362226733621 85	82,23
57543760494130633865833562191516562964425151 28	83,23
118776578661709954937724956879521837678305346 37	84,23
240407850788264119614887727219868933766607349 36	85,23
477275145474384079120988079975589765341416164 69	86,23
929610012724056498037525179114642958151510379 21	87,23
1776834405466238368520500585584556942663918523 48	88,23
3333520324073390170781276046064430848168645228 40	89,23
6139866370118580241536309023665188919301299452 36	90,23
11104482999439414787521528134817558145974085568 33	91,23
19724227394478726357392239384865767481654196522 70	92,23
34414188370520448413505862002671839240520787826 04	93,23
58990207508603103161765473991019652179999165414 46	94,23
99355477314496900225620164171575358543316047555 86	95,23
164449957467705337028638627110109088005769956307 30	96,23
267523862360550890792450767033743402207890563375 20	97,23
427790627991238361392953518981655081428652107318 44	98,23
672496144707543162033706675358120679988357468590 33	99,23
1039404962549314283660470787533190757570705414255 74	100,23
1579646551326609778104232183418350751498788848317 78	101,23
2360783273565576868190033498989162983109386376527 16	102,23

Table B.2: Numbers of unlabeled 3–edge–connected blocks by number of edges m and nodes n (continued).

Number	**m,n**
346985761085537680955286339438431321874875506214647	103,23
501605544103125843902627835837407034327446164740959	104,23
713249714475120525964527648689832499679180346155845	105,23
997655822410852970484691167431115628044361121084522	106,23
1372805129760931901681775111334518962392261758364837	107,23
1858460092593503405429664426125182855047222083346196	108,23
2475364037391189228745657506385931695784753327490290	109,23
3244067737759655760057167158016658029563517160664488	110,23
4183382323312493363217896514495670850139049214892362	111,23
5308504514188132340601237254692235489452781273200533	112,23
6628915021816495671406212166146718115041365207443453	113,23
8146207966321209471334354067236448420011116773608796	114,23
9852060030012316531585172703117786179342674066374548	115,23
11726583266303179574831142999908917025139580863182286	116,23
13737315727211987236408970958455552122612269211104870	117,23
15839082104511053964053970900167225027618497037122830	118,23
17974898974414184990068893774449119517781688182560065	119,23
20078007821926012194809132564893593045534653715013542	120,23
22075001543809710190440678417312684259065399436651398	121,23
23889879869760731732286562175811657245624938211937280	122,23
25448743395309847451372868073784919953880283836700161	123,23
26684733478851290878804095533985531238693818484014819	124,23
27542763280359097851462054520708938699658776660173322	125,23
27983576079708932189784712200149059024945348918714301	126,23
27986715729721175820172038326620235318045990141673538	127,23
27552096970705388966664934800023241425877031325210019	128,23
26700008093302081880587008015375876288481865742306851	129,23
25469546002659618948361635877776622903209577484749068	130,23
23915651260025041012695569142624604725434279453617745	131,23
22105055319785638561975637940632993235431542502819916	132,23
20111554890566848471052202243646732416748830044856999	133,23
18011076906400168421794248789324008782346093739023730	134,23
15876988341938417803388426982414123594032012374841433	135,23
13776043102560769360362428603523663649334151307040532	136,23
11765255853300353038972509387833416497904248910255100	137,23
9889867076905213537549868306061065557500675896210642	138,23
8182433636485249559370270655450037437946419500212222	139,23
6662961915434481077840470634886858514129410545369291	140,23
5339909448935431658333007446791694358854296986028091	141,23
4211823547323289494568117103657958322704925909988293	142,23

Table B.2: Numbers of unlabeled 3–edge–connected blocks by number of edges m and nodes n (continued).

Number	**m,n**
32693635380288119776275276285832375790862699572347 62	143,23
249746348097904585389175470004213229774420658292536 0	144,23
18774272930922556192964281538112568054270578611447 47	145,23
13887988883614301015633490901328384430348167132415 23	146,23
10109067084012522376505529077348586327158734436393 69	147,23
724036620633793898920275966780194042023796963791407	148,23
510233510613855606997883556061861478818328608111331	149,23
353766465823859551024006296516442873080195861941933	150,23
241314162347489406982539064309936728135323360552066	151,23
161936820238898779122287379746610863870231745104916	152,23
106901102183499671798992071576464008235944867759131	153,23
69417440186618487200550100654506110687209226719666	154,23
44338378634357234703034827666056281030132617028914	155,23
27854134944488073960487032754372873749813931145541	156,23
17209605596605475303317259504531948358906304281346	157,23
10456710857826903235001690462205334729866018933477	158,23
6247878964139095245558553034223182216391021446112	159,23
3670731562177964100302838685238811280353710508216	160,23
2120425945040707783002687913308922821213971131025	161,23
1204233647213326824281639694921158609871094383813	162,23
672327671112366611548898746944197299790592782044	163,23
368975812192907603422778072618529533663374864332	164,23
199032615802997929919444091079817138406746683288	165,23
105516755663069706429222566149424323764037655046	166,23
54972996161469429798257072380744220393377847785	167,23
28142761919790190360456992492414820866169132618	168,23
14155669304168890767286419288495254957209921504	169,23
6995133685944201166439104261595642462063756569	170,23
3395607859020104133787110426082152119100754376	171,23
1619001797287277354001325666310870646148118383	172,23
758116897442945380569293663311395731804789601	173,23
348604627828445294503003434200876019343019462	174,23
157393071398626509713493369587247956771196063	175,23
69765252337190829600664105033121493473160060	176,23
30355516770823559070059617145762683932214523	177,23
12963520429454829941419169641931227272595895	178,23
5432943941267374725053554803090302169687336	179,23
2234151319374168207314435696337112586418033	180,23
901345356300688780136583812243565817457992	181,23
356701826306428288341667143926882688133031	182,23

Table B.2: Numbers of unlabeled 3–edge–connected blocks by number of edges m and nodes n (continued).

Number	**m,n**
138448118714744545171959088710299292685618	183,23
526946462565986372844106036325809470328 29	184,23
196639899370039542433453981491632072982 05	185,23
71933040423191453145708063207785048959 91	186,23
25790551107563745665532255454365421479 63	187,23
9061288843053280614684399825241191110 68	188,23
3119149001699242954957560007369471038 12	189,23
1051760098444790865787771145315192142 64	190,23
347334047672909367860333847763474912 02	191,23
112316082563597703663481527890523643 48	192,23
35555973532408415309252921879354345 44	193,23
11017164266414645485986454624935424 41	194,23
3340588651499407412847933983978105 35	195,23
991015468032119007768526100668816 21	196,23
287574168720179823127044473123029 44	197,23
81609177339669698539330564126737 64	198,23
22644151290362549885001223524054 11	199,23
6142011033070496160244071128775 26	200,23
1628225412766092579034357225084 78	201,23
421777381757999788324088938104 58	202,23
106742490163725111541730810813 34	203,23
26387850497242621503081613741 69	204,23
6371180011726687825954466559 76	205,23
1502219680216200794955754105 77	206,23
345866685570956420658554053 73	207,23
77755203832793980580067544 13	208,23
17068852859764469417174720 18	209,23
3659092446499821306115857 65	210,23
766153474833998730142034 65	211,23
156732564544529014409331 59	212,23
3133945007675287713366 883	213,23
612876640318602605639179	214,23
117315174235965049108265	215,23
22003479576368976274257	216,23
4049183958190459118159	217,23
732338763865185014995	218,23
130442714791463081122	219,23
22938566759500607972	220,23
3994053534715768844	221,23
690885245614759655	222,23

Table B.2: Numbers of unlabeled 3–edge–connected blocks by number of edges m and nodes n (continued).

Number	**m,n**
119163215536491528	223,23
20574737024921888	224,23
3570601421661404	225,23
625308911680681	226,23
110923277562820	227,23
19998343352439	228,23
3675258986536	229,23
690238318720	230,23
132760735659	231,23
26202267608	232,23
5316143530	233,23
1110793092	234,23
239485595	235,23
53386783	236,23
12333670	237,23
2960339	238,23
740196	239,23
193362	240,23
52943	241,23
15216	242,23
4613	243,23
1476	244,23
497	245,23
177	246,23
68	247,23
26	248,23
11	249,23
5	250,23
2	251,23
1	252,23
1	253,23
98101019	36,24
115159388113	37,24
24713302866893	38,24
2323238727900672	39,24
127977425246247028	40,24
4785492965141146848	41,24
132664172450608663737	42,24
2890012911055483092129	43,24
51544360933956836954021	44,24

Table B.2: Numbers of unlabeled 3–edge–connected blocks by number of edges m and nodes n (continued).

Number	**m,n**
7757977159818669967 09567	45,24
100845651097805931740 61523	46,24
1152952800733969300715 50040	47,24
117642694829382649460522 2632	48,24
1084222774777051340883031 4642	49,24
9115812483176363680982935 0111	50,24
70507561328166262711059163 3618	51,24
505287506413041740983822842 2446	52,24
3375730511927557959957206495 3867	53,24
2113645243774093268630578241 51641	54,24
1246084041535156273258309654 332934	55,24
6945202636989953506539428991 772745	56,24
36729125668894249640505904024 420070	57,24
184890856649675980419646235239 644363	58,24
888466722987101818813614644928 538270	59,24
4086023282770709922586926978093 464613	60,24
18025860167555774425047613416751 5723361	61,24
76441819706547457117912770075174 610383	62,24
312194407114669255578271589470037 023038	63,24
1230049451034052487391418537373228 178669	64,24
4682754262470292452274152549838350 185653	65,24
17249664644466467208443117113259675 673723	66,24
61564044608356799286175960054074558 754266	67,24
213137878657244084567457352320873499 455049	68,24
716570402388574695864207423028599451 524172	69,24
2341864029581422903149588745418141729 317945	70,24
7446886525049876038850443778410737874 386883	71,24
23060715442387956506359296741096944352 234034	72,24
69598913787596303903953256811966342696 811287	73,24
204873096389233192575555925940249165044 002952	74,24
588597027765684720259228000690041797395 701902	75,24
1651499694991136283531274894784176713888 726708	76,24
4528177348508274611142335676248936852953 317133	77,24
12139260546313307045599349756619758149541 554982	78,24
31835225262322599779550964582586337633759 231463	79,24
81710614208385822629946294996291468089279 516802	80,24
205351545541424225370430677353126909479709 411002	81,24
505530937823158973815846052195777291776316 660829	82,24
1219543395542666532960397105751908335189629 080297	83,24
2884065371935523904710357873428709386728633 233775	84,24

Table B.2: Numbers of unlabeled 3–edge–connected blocks by number of edges m and nodes n (continued).

Number	m,n
6688361032409394859643660475762887052450759990975	85,24
15215286784371844104559699756112625317367836445034	86,24
33963837521319797174979640127849586839306286210538	87,24
74413512028243450435105352589311218082310861626214	88,24
160066340490992187117291678714157126661584347873079	89,24
338118645766056565192697009280175533746659267519458	90,24
701552718871539689410106902986701990826351286910231	91,24
1430107540464713562856568145103147335804153436564597	92,24
2864730015095685101710610286005325125820952117730438	93,24
5640134147571337516353657139820729996212185501401038	94,24
10916038428125398517271083520496392110257565065214845	95,24
20772323470479136281582201521129149327237086947133720	96,24
38870458395766278361290000211336396386774127628682638	97,24
71537688928797907761693967649670358892243356169233002	98,24
129506952605831121214517718355683769353401397759250377	99,24
230649698991989394648711070412647239185369121237941357	100,24
404175199912604893461845311936733170496771381994517214	101,24
696940280147167835931520649991228354703164944889986557	102,24
1182714183844512967771995286642355115492443898208668977	103,24
1975462484793506252746902728440634581647466325786732393	104,24
3247927177813295589113211611436851918251364566990128298	105,24
5256946868792034808278026181515644694986167166916029540	106,24
8377021918717348313364246400426796764166591387719596690	107,24
13143505714207768926452034788746805100396943348466861858	108,24
20306400255364784225240992876764599956654175884438313169	109,24
30894950529118097409099569046784344888562943912089420197	110,24
46291975568401177813097528937237172545178450869856842559	111,24
68315085343506528626758896093315422166446241927589827067	112,24
99299614613333799760160071957590285206833467896739301900	113,24
142175354908920545737199681707910040362714261870372958712	114,24
200526204566687804417906242531966497487803452435588349765	115,24
278619045691486577107698523555071581540701359530041883434	116,24
381385995434875130486599355752095276140218260553665026250	117,24
514343271691701136441779362602305615338268478798265926703	118,24
683430902682809589710595427352912176967863999391257415291	119,24
894760973441248705715117424275233713244302011322138397964	120,24
1154268423005584445829850857801444455980006875259336134763	121,24
1467267633662801569491588055143643643597121366625895350894	122,24
1837929781334985498744650371243078669937595921294095090955	123,24
2268709206839568417252764371386848343148311999302440872990	124,24

Table B.2: Numbers of unlabeled 3–edge–connected blocks by number of edges m and nodes n (continued).

Number	**m,n**
2759760456998798591540828529623618451810632847191347892846	125,24
3308399246491418050864253992276985706195639058101344350738	126,24
3908668312196664721236095504478233840558283073552201158443	127,24
4551070977895829690974435202792977447778559724489672457674	128,24
5222529688013922537525484192556974083465876895721912235330	129,24
5906613095639665149425331023450771985501694529659812450700	130,24
6584053896756334406100511998817125258465004939094721576769	131,24
7233552121747104849936056968355942702131462651026827840637	132,24
7832827831492229153705031958936583245853435355661669969136	133,24
8359856810189798147668630940197953440068643599344524247367	134,24
8794196995266420438312198010183194741154141215009219313252	135,24
9118295919888045354859414165818358072165530554600221090821	136,24
9318663392794562949878857231059684342779922332386306636957	137,24
9386800617573799007820701847685392409171491808997621949484	138,24
9319796803324279421323480091999517685788527387386628072276	139,24
9120535038965977890583647994312696066632522039137272049252	140,24
8797487180028612524931531092804142597257792245779139042274	141,24
8364118017703796767167973277564514271620813260115580701521	142,24
7837956965250531105341429905084930625447852631054847579949	143,24
7239426197308626873662986476294796261041615451310014605977	144,24
6590534003872842959070759494775779331485208984406723621917	145,24
5913549076477842833818377231188584024194779698564797074095	146,24
5229765379403473548097161795129435414186011651790308712982	147,24
4558449791688379955311829370499965606711702212250401306014	148,24
3916038865396575766883763025507817432461545338685525006001	149,24
3315620714467489802379303375529837174223422505349062922787	150,24
2766707317973527756485264540367717697729114806101475302150	151,24
2275275077839412045916037904691524318828836190794332599429	152,24
1844030112660203235835824611286478126861980768989811595647	153,24
1472841122837616472115433455811960870352443866175730804298	154,24
1159277118050030975152674753727787813541165564784063949592	155,24
899189145094443192066541985875985263920213623868771838361	156,24
687282862217138334259033616928375573839451218053700482757	157,24
517640384061647295122778572351048239857926019337009819824	158,24
384163179693405501956539695539879630487112343719048499699	159,24
280921063703083184291329223808717615279602467633819948819	160,24
202404017533948415129113695976676433599803147814440227490	161,24
143682781340075262769030687781507942325335904058632474888	162,24
100490461663233783025087955730932260620398750337109849620	163,24
69240857295416836570125691926201718830175441456584208252	164,24

Table B.2: Numbers of unlabeled 3–edge–connected blocks by number of edges m and nodes n (continued).

Number	m,n
47000197232155023117986980689772140216131859068402542477	165,24
31428086364760008254136577897311828396752216679611972177	166,24
20701306524756689084278855512076607741970787665128834276	167,24
13431324797679782245695597713913924022893742832928293471	168,24
8583414716599776051785095407313783996059116375476518717	169,24
5402558829346723565775800749497172629672116088034332256	170,24
3348992813468738629986307244498146992674811203824133818	171,24
2044468758072698415310678778585850277534172561585065792	172,24
1229061558314094328674635137703032872068390546164397090	173,24
727558810169029763554546623652702049155040675920933084	174,24
424069813393684561415678480439188419101430699260898218	175,24
243362642764953796343054668263526567325394192190024175	176,24
137495756635938514242156456750034279968034641024268138	177,24
76474039806854319091324659561507516812173754814155743	178,24
41869485733715915010799946465758297495425989364702383	179,24
22563607169583460352401947683552428440746749988184852	180,24
11967794531489925557630740460826373119538230514727152	181,24
6247132171458582937919399624202269620510422622380326	182,24
3209027234425339540609707397206753800865293427497901	183,24
1622020218800040617445461925704041872623406686625719	184,24
806662956128090880784691719246407705235666107003461	185,24
394677697050170059544377576943006352856854661270341	186,24
189962587525846037804048724313972022130430281933059	187,24
89934788839647781197386300831597641092019100339196	188,24
41877365172058633759461383591971025202409740010837	189,24
19176947179376352642609600709440512335328417463590	190,24
8635415362839613933702652074017400841259807972300	191,24
3823352812413325547695150928441525761635006947949	192,24
1664237176883800849000168301077955960588581209507	193,24
712109224458741449948269924776636451120358876280	194,24
299493843212050448017687873309797126605270148993	195,24
123790378145141735327091605911351627715736576632	196,24
50279368862085001050136100024020622364799447867	197,24
20065098994702128991082059652969491658752950694	198,24
7866550531563446300590461940085418179353986147	199,24
3029425025792042125692805636668386825514195827	200,24
1145799198243410729478820892200527135308626841	201,24
425566177420715066628603668629326708690109481	202,24
155192977789927800929655060330479916159794343	203,24
55559419465103835518829570222577942624427364	204,24

Table B.2: Numbers of unlabeled 3–edge–connected blocks by number of edges m and nodes n (continued).

Number	m,n
195234470681358708686698490178235097445314 01	205,24
67328532846880023904123900365516271529929 93	206,24
22783189682397539930548236539314377210282 75	207,24
7563619892614619671308022515870085357005 90	208,24
2463031431543386393881201764888370275126 42	209,24
786608911471003536327798502963127914200 77	210,24
246330568085966702546121041493792773074 93	211,24
75625558952220103769878658187729915022 49	212,24
22757710590144608684873275790287151891 40	213,24
6711440738560965353079794645028831396 60	214,24
1939307138360631871180744190131609822 06	215,24
548954898679571677447723228156959038 01	216,24
152195228279137726020090334547642118 92	217,24
41319463666666972056973497448464663 01	218,24
10982796979926763430604926380443294 58	219,24
2857539824044189908997045648455727 70	220,24
727630552527321016934100004297079 30	221,24
181296698850902040544757265261445 80	222,24
44193074842105928721603264067237 67	223,24
10537421567582939010911953118170 93	224,24
2457355491701409086824975399845 39	225,24
560405280412085003213417315796 19	226,24
124967257392701375719877315353 59	227,24
27247370541325577261063878002 92	228,24
5808730036222034404577586871 19	229,24
1210832642584072570882850010 21	230,24
246820694227166334230134796 34	231,24
49210632078050723466308992 10	232,24
9599473576987078905483717 40	233,24
1832880428005215602015762 42	234,24
342747417659775939837384 89	235,24
62820951206940441098880 02	236,24
11296903865280701924908 91	237,24
1995665217611677423218 05	238,24
346872377380770384273 43	239,24
59433845999566742909 54	240,24
10061638231640366635 31	241,24
1687441414141199162 45	242,24
281208391895297037 22	243,24
46721971303976854 32	244,24

Table B.2: Numbers of unlabeled 3–edge–connected blocks by number of edges m and nodes n (continued).

Number	**m,n**
776716104531272512	245,24
129674134254464207	246,24
21821126714748075	247,24
3713914014298840	248,24
641322779161851	249,24
112668088475924	250,24
20184618368209	251,24
3694876952543	252,24
692294900837	253,24
132977762147	254,24
26225625391	255,24
5318743428	256,24
1111096424	257,24
239523128	258,24
53391749	259,24
12334378	260,24
2960448	261,24
740214	262,24
193365	263,24
52944	264,24
15216	265,24
4613	266,24
1476	267,24
497	268,24
177	269,24
68	270,24
26	271,24
11	272,24
5	273,24
2	274,24
1	275,24
1	276,24
21113212825	38,25
9810024194846	39,25
1494395581465328	40,25
118402773746489318	41,25
5964829721149072775	42,25
213754747386418955628	43,25
5850595594755484535639	44,25
128414486485746230261074	45,25

Table B.2: Numbers of unlabeled 3–edge–connected blocks by number of edges m and nodes n (continued).

Number	**m,n**
23413524893830120293 69800	46,25
3641548571994646541470 4453	47,25
493202249660068950126744545	48,25
5913036408359127807538695956	49,25
63596137230809343724523642565	50,25
620390562369934504861059028673	51,25
5540082477940937948619554607341	52,25
45643143426685000903280227798652	53,25
349258466764111047893658675870104	54,25
2496541774432357028394668791305701	55,25
16754640497142612863736668986797325	56,25
106035258581942677897839319890567799	57,25
635291638456392318153971548794046660	58,25
3615775280642106331293299611079277556	59,25
19609666402725992056596946210208649929	60,25
101618958139791523359595114948164067758	61,25
504416781937327083000024435590548945802	62,25
2403721073690073732331232023336640253949	63,25
11018845267119095079492232929718860196487	64,25
48679112462342789740019393718213670265949	65,25
207601121005616764079191219305684418970240	66,25
855972997614593163060022591446321636026479	67,25
3416964704349022715378748366850339130229491	68,25
13222926624223729090593734133604605156060248	69,25
49663010910473072198631207970211163226745949	70,25
181229205194334802208328921817983690067133171	71,25
643202793743800476299885211235563171431892093	72,25
2222256852846101684427271099508869831140454990	73,25
7480650137150517071493932258195770588135910797	74,25
24554338517304200435800323066439613636801925951	75,25
78646842805899541398599868243306187214498781414	76,25
245978029701974382369849763719315943887120933458	77,25
751710318653112417309340782636970070710833227020	78,25
2245960830113120875736687983912904815367534204738	79,25
6564358306548156468788446077405402631356009506770	80,25
18777848340644238351856124924561695077239636990618	81,25
52598474492916541049598663788413449169484721640749	82,25
144334787412096823148990694755864385022661096342804	83,25
388171538762156675607638273259481366745330662619374	84,25
1023537699658983667252759960622332985262455029118536	85,25

Table B.2: Numbers of unlabeled 3–edge–connected blocks by number of edges m and nodes n (continued).

Number	m,n
2647118805806136511532365059815922115705288220954624	86,25
6717141032031174692961935422527740015184145304624826	87,25
16729375075136846124540742815652801875494144975167982	88,25
40906599648324534933877333867445771494035192424303343	89,25
98231700297986067419131701285574391716429316873916996	90,25
231725236017450471319219598048439607306281807106287262	91,25
537118654670547859337717735490251104481153271713490535	92,25
1223623313656804461690910047606195029800819552019939125	93,25
2740344651362146881325058057719698608333989694957559151	94,25
6034430056873716722025303493329029681462286584979907306	95,25
13068609021106673896227964562408688781950046331120669468	96,25
27839919680203617009912705548905189528209720688344400280	97,25
58348666717832278424631698731461891624875377956881141327	98,25
120335216513262076338015011474870834395636510842738268252	99,25
244243959482093537023655656805763037652963467361486637455	100,25
487967406450945774931001003672847363981077889015110223956	101,25
959746900310922359262451189925446536793409904692249759853	102,25
1858579736249824021716356973931207768054361865306301297224	103,25
3544220539228606186686260516548513609032913847895733508484	104,25
6656237968041095075585304225837083397589606660443341713613	105,25
12312782161478356516604360637480484109481786818210611446778	106,25
22436279984346345090020931287142552365645249717212609155348	107,25
40277114984529960914607820805750627239680078645565037833348	108,25
71239670663063923364477439374504788165137552408833482292522	109,25
124160218571951025869569280231743874299658868315218571023055	110,25
213245032885126426194266694459323179423106053366596989100936	111,25
360950923525097132195538509002371073951602536525610703869441	112,25
602178406618750242159380836249217231636807812128900530764231	113,25
990246275670728692634104600071380866840991211623186851586764	114,25
1605217263832784280528096092507563596763000562747216355214323	115,25
2565233710300299091746587084278254520681406700439769247355938	116,25
4041578310540690614105027424301997087195357857438228893041518	117,25
6278168832220188979223310328554721187030534884617826744361499	118,25
9616090328492307831482400536483537282830141945393407518354719	119,25
14523521874362258904311566166358171939888968590837905664478457	120,25
21630984596222167364828878066182455827576233692371318162127488	121,25
31771187928454870319527112760914046059277325821867830260768129	122,25
46021862580818447685961635590871855603554750768250179234553092	123,25
65748851437257995307626337796176596210660262662921775182467132	124,25
92645434727189047158013908070948734536051585023690798002783679	125,25

Table B.2: Numbers of unlabeled 3–edge–connected blocks by number of edges m and nodes n (continued).

Number	m,n
128762496656745440863326448157787276191146117202527290980712535	126,25
176522858636432248556483270677359064655838468397440839701273171	127,25
238712126331696420345344738696480764839864325796008755150479739	128,25
318437982961287139607555370775492492265114853239795603324615977	129,25
419050283407251130523769923810197068968073426501015784116007291	130,25
544015811168879470484051479251872646009698288047196893301969761	131,25
696744325307460356099085281836682404208078821848935689021396793	132,25
880366588756941481489961124273302307555480706018898273485119176	133,25
1097470293612892346317193129866386113991692267662442170345310704	134,25
1349805830127993088853512678559808659693525592875708661974616078	135,25
1637980111207866965554325148978299693211969154297027375460224719	136,25
1961162401565617784477536422182716636414059475902596992236588309	137,25
2316830432192350726954068310680810220717874751416607251998208170	138,25
2700587122236614364018702646918389000740388661906362760065249989	139,25
3106077229056292903569100422366221482129597153389663715724389393	140,25
3525028727265524611415563406337963826494310922655460414644451845	141,25
3947435608293583453622964431372728553649349851194141784012743826	142,25
4361887512102507307699700502941903603142513086758911499964515301	143,25
4756038078170061640008296842840934050247435513658428745049575895	144,25
5117189503035404235071350686806136238792526711999502782830705131	145,25
5432957183924645420246586130512680802988544767164793957281790203	146,25
5691967264383819916115769478269420109198576739887577942827916736	147,25
58845329642360969394366306900009605457432308545991098035105984320	148,25
6003253942254560370525490427987618078057625423841273134602631961	149,25
6043487154557137066048157903145609986635680265378472157059153036	150,25
6003647541880784079971136358094658356276276317817382432909197738	151,25
5885311458657399842229449265656918390395626948526485047673810433	152,25
5693113453880840106009808681981637112150178528058063339238284249	153,25
5434445800108479403939020845048100210889128116692813957012158986	154,25
5118987861075139050368356714339614585061789770194885095653964283	155,25
4758106962925850534785131562907541800759568795869118653447313306	156,25
4364182297814598416323723794750766582298219123075717685733410600	157,25
3949907597056349188998085320595598307602657599618917502811870254	158,25
3527626672558789396994525619111908065516865017288654617411022375	159,25
3108748989878084099502291915287009166792066422625960164511347330	160,25
2703281374269179143862736907930645955111969063128309462953125416	161,25
2319498348326203599832091865382641553030971839473175268463066683	162,25
1963759207775246937813830140331706050199884563506956472583728860	163,25
1640466426867535146542695908554793358573624862493840842254138458	164,25
1352148714613119986505764126921336140272080442491919819941773846	165,25

Table B.2: Numbers of unlabeled 3–edge–connected blocks by number of edges m and nodes n (continued).

Number	**m,n**
1099643942237650114864316835593287229266708621018192953254024629	166,25
882352648302939066398991579487268279743382869987124826325220836	167,25
698531829334679864764822165309013309072107995609096152305496523	168,25
545600764415672444334696407447069414422051680279701366286472159	169,25
420434949590090197804537542444402702842874281920192768289356732	170,25
319629948613929108942014408282396076340302524685645283942444029	171,25
239723223455229219863150895014578661280377690091460897774169565	172,25
177368030506798498216115445985805432310080812552429958438244160	173,25
129458684888427267394296049580030192283624754212000635512060833	174,25
93210553220866373810085780625281608557493725730097813601826776	175,25
66200896284624827255075617892877679772009898449525704341374313	176,25
46378188678518072686601608791065475773461275100541530103528781	177,25
32047965127740314460246593075562788081306955455506929881996989	178,25
21842830436907395287840537749594815238204844430369052375661997	179,25
14683295508199581386772476003211073648108557255043281715981989	180,25
9734823706360916323461741762659719476307493866557823886971384	181,25
6365106629296575516801822862161095177368698685725139715990653	182,25
4104296968100667752304468314463570585977148240420496134355633	183,25
2609811664446089373047085650857212807666574580136026696609893	184,25
1636432057895117696148719456526241469737102784894519584407364	185,25
1011778964496388899396971889538944185890310691903983438590604	186,25
616810712672028726200844462264863705377334609595496491700291	187,25
370745418712873188514904721134372653165234525778472741902037	188,25
219702807869559334127562731470302202596447891606779150116315	189,25
128353855784847251296193769337475738878958660773139400265427	190,25
73921802508534869367654864399829587270681768090080828939217	191,25
41966480292657662672682449506331446253151899716894691482166	192,25
23484114437240591296819741151986023655419020021095255669907	193,25
12952748971523741300798242229928824499342536593019024574605	194,25
7041083474073202324796227477013942700725016165364073248236	195,25
3772069226538327930848952856006082354080698565578522729653	196,25
1991381275482727205538005163566722743937095904982091014204	197,25
1035940726183253946326538679641450718983952857358864708930	198,25
530996124677702259628887246931076688444377307865809026296	199,25
268159434620111321620326723552196472008246066294294616721	200,25
133416178899560086912857229471502006736116268910038994795	201,25
65389054221351154018764166431245635426241149184051795692	202,25
31568157750274265194624954129096891139229519630754005894	203,25
15010893212322104890090841896185867486046152965129390385	204,25
7029774933287482666437745161673625139201861518917233524	205,25

Table B.2: Numbers of unlabeled 3–edge–connected blocks by number of edges m and nodes n (continued).

Number	**m,n**
3242032213098830618697179926378257247260794076689688774	206,25
1472301024850082540390936881556659561335317654646228419	207,25
658325317982103571768219443906046396063803252788954749	208,25
289807289935733734097577826412317104511805363880360433	209,25
125591966655134532273880752095428607447102140716514022	210,25
53574243829389576166983187034955004831268230331573146	211,25
22493057462077691309873836201697724448054255462121288	212,25
9293826813158066953958758629873305145013339093846354	213,25
3778752248440909168864682525425208155850568287960094	214,25
1511690376432797721143101856827958009965314536392806	215,25
594962186461624437275615378774323735609856413428036	216,25
230344640008602482359928326948575033681248761197842	217,25
87715803273334079277780903976464790272833398744544	218,25
32850040347679675797011528506901111541800806509285	219,25
12097571400473997025039932165267937272920375888200	220,25
4380353174988794065774563599281829955581996488248	221,25
1559235207874776015359060296294816318113477243799	222,25
545565765499800137904672855341634102850318233247	223,25
187609756424865624637334646039228768178197580456	224,25
63397898200435624329819323939526030985588865712	225,25
210495182613901866651148874952266250873095812261	226,25
6865820981609579060309183615480812875530524866	227,25
2199679381002168474003241384727322380817968352	228,25
692109652231789942916354233068779644866837741	229,25
213830437846224384843107619652257076927107257	230,25
64859295565226664117000810305428838253568061	231,25
19311257619052379961527520744506825210338215	232,25
5643008398523479078776662380767063522566978	233,25
1618070971009464128747471258251906447577952	234,25
455192360149507452265323320514590616511284	235,25
125610439762338578205056958171316828956206	236,25
33994761247409044345700770615308327947102	237,25
9021430301641134161603491307794114718567	238,25
2347131835848902017606712494894080516717	239,25
598577101646938775149258268494591059344	240,25
149604933311294859653749548282065920297	241,25
36638616677431890590524017575609750703	242,25
8790750237081731356499616318114786370	243,25
2066027585179241278400550025586930303	244,25
475558051055997892026883693023041503	245,25

Table B.2: Numbers of unlabeled 3–edge–connected blocks by number of edges m and nodes n (continued).

Number	**m,n**
107193372969061132562139496246913828	246,25
23657919227099810562629186652654508	247,25
5111945337518392623693066269981296	248,25
1081349037976686133524516007322222	249,25
223923602938778086766473621783323	250,25
45393258367322892594012005452708	251,25
9008803389314811584563437008226	252,25
1750583667777527666421920331679	253,25
333142423834155067479205988689	254,25
62107346546783498689077571902	255,25
11347736479647461148146244987	256,25
2033196720003397868441456838	257,25
357503130374730632470549426	258,25
61748513293444296748439470	259,25
10489133258288657721016776	260,25
1754925413318710858992862	261,25
289705168671547903768202	262,25
47287714866514072240909	263,25
7650653082132399107820	264,25
1230304531375581522398	265,25
197251631985182199762	266,25
31633233520207856135	267,25
5091513030273032786	268,25
825245274148195416	269,25
135124371883115050	270,25
22416076129291810	271,25
3777194536295049	272,25
647904733883207	273,25
113340974676475	274,25
20252697246546	275,25
3701754158676	276,25
692996060764	277,25
133050795411	278,25
26233496486	279,25
5319631820	280,25
1111202523	281,25
239536645	282,25
53393596	283,25
12334648	284,25
2960490	285,25

Table B.2: Numbers of unlabeled 3–edge–connected blocks by number of edges m and nodes n (continued).

Number	**m,n**
740221	286,25
193366	287,25
52944	288,25
15216	289,25
4613	290,25
1476	291,25
497	292,25
177	293,25
68	294,25
26	295,25
11	296,25
5	297,25
2	298,25
1	299,25
1	300,25

APPENDIX C

Numbers of unlabeled minimally 2–edge–connected blocks

Table C.1: Numbers of unlabeled minimally 2–edge–connected blocks by number of nodes n.

Number	n	Number	n
1	3	323778	19
1	4	1019538	20
2	5	3265502	21
3	6	10624881	22
5	7	35065697	23
10	8	117289911	24
19	9	397290416	25
42	10	1362060455	26
92	11	4724272231	27
225	12	16572501190	28
564	13	58782486855	29
1501	14	210782826628	30
4127	15	763991002957	31
11809	16	2798725378190	32
34713	17	10361304926710	33
104904	18	38763328208537	34

Table C.2: Numbers of unlabeled minimally 2–edge–connected blocks by number of edges m and nodes n.

Number	m	n	Number	m	n
1	3	3	52	14	12
1	4	4	74	15	12
1	5	5	58	16	12
1	6	5	19	17	12
1	6	6	11	18	12
1	7	6	1	19	12
1	8	6	1	20	12
1	7	7	1	13	13
2	8	7	10	14	13
1	9	7	82	15	13
1	10	7	185	16	13
1	8	8	155	17	13
3	9	8	92	18	13
4	10	8	25	19	13
1	11	8	12	20	13
1	12	8	1	21	13
1	9	9	1	22	13
4	10	9	1	14	14
7	11	9	12	15	14
5	12	9	135	16	14
1	13	9	400	17	14
1	14	9	483	18	14
1	10	10	282	19	14
5	11	10	141	20	14
17	12	10	30	21	14
10	13	10	15	22	14
7	14	10	1	23	14
1	15	10	1	24	14
1	16	10	1	15	15
1	11	11	14	16	15
7	12	11	200	17	15
27	13	11	871	18	15
32	14	11	1282	19	15
15	15	11	1026	20	15
8	16	11	473	21	15
1	17	11	204	22	15
1	18	11	37	23	15
1	12	12	17	24	15
8	13	12	1	25	15

Table C.2: Numbers of unlabeled minimally 2–edge–connected blocks by number of edges m and nodes n (continued).

Number	m	n	Number	m	n
1	26	15	26	30	18
1	16	16	1	31	18
16	17	16	1	32	18
305	18	16	1	19	19
1721	19	16	24	20	19
3460	20	16	838	21	19
3226	21	16	10729	22	19
1982	22	16	46076	23	19
741	23	16	84963	24	19
292	24	16	85649	25	19
43	25	16	56812	26	19
20	26	16	25859	27	19
1	27	16	9764	28	19
1	28	16	2250	29	19
1	17	17	716	30	19
19	18	17	67	31	19
430	19	17	28	32	19
3327	20	17	1	33	19
8540	21	17	1	34	19
10224	22	17	1	20	20
7053	23	17	27	21	20
3534	24	17	1143	22	20
1111	25	17	18211	23	20
399	26	17	99843	24	20
51	27	17	227280	25	20
22	28	17	278074	26	20
1	29	17	213471	27	20
1	30	17	116683	28	20
1	18	18	45272	29	20
21	19	18	15411	30	20
615	20	18	3078	31	20
6061	21	18	935	32	20
20464	22	18	75	33	20
29975	23	18	32	34	20
25551	24	18	1	35	20
13977	25	18	1	36	20
6007	26	18	1	21	21
1605	27	18	30	22	21
541	28	18	1508	23	21
58	29	18	30138	24	21

Table C.2: Numbers of unlabeled minimally 2–edge–connected blocks by number of edges m and nodes n (continued).

Number	m	n	Number	m	n
206139	25	21	8867397	30	23
582736	26	21	7427196	31	23
853958	27	21	4493430	32	23
777255	28	21	2026590	33	23
483284	29	21	737379	34	23
225413	30	21	193389	35	23
75974	31	21	52084	36	23
23619	32	21	7033	37	23
4125	33	21	1916	38	23
1200	34	21	105	39	23
85	35	21	42	40	23
35	36	21	1	41	23
1	37	21	1	42	23
1	38	21	1	24	24
1	22	22	40	25	24
33	23	22	3280	26	24
1991	24	22	115768	27	24
48292	25	22	1460514	28	24
410492	26	22	7506724	29	24
1423632	27	22	18996203	30	24
2513722	28	22	28000588	31	24
2681078	29	22	27193361	32	24
1946661	30	22	18762947	33	24
1017875	31	22	9743540	34	24
415559	32	22	3855465	35	24
123012	33	22	1268334	36	24
35441	34	22	296315	37	24
5428	35	22	75292	38	24
1529	36	22	8989	39	24
94	37	22	2386	40	24
39	38	22	115	41	24
1	39	22	47	42	24
1	40	22	1	43	24
1	23	23	1	44	24
37	24	23	1	25	25
2556	25	23	44	26	25
75706	26	23	4126	27	25
786619	27	23	173699	28	25
3337394	28	23	2628281	29	25
7056821	29	23	16271677	30	25

Table C.2: Numbers of unlabeled minimally 2–edge–connected blocks by number of edges m and nodes n (continued).

Number	m	n	Number	m	n
49031082	31	25	52	28	27
84789128	32	25	6385	29	27
95116224	33	25	369703	30	27
75257313	34	25	7859367	31	27
44164951	35	25	68962644	32	27
20100014	36	25	291208658	33	27
7063776	37	25	687806218	34	27
2124368	38	25	1027632491	35	27
444164	39	25	1060938331	36	27
107103	40	25	802478031	37	27
11347	41	25	463482596	38	27
2939	42	25	208898873	39	27
127	43	25	76231077	40	27
50	44	25	21640349	41	27
1	45	25	5584056	42	27
1	46	25	943031	43	27
1	26	26	208286	44	27
48	27	26	17518	45	27
5173	28	26	4352	46	27
255454	29	26	151	47	27
4604152	30	26	59	48	27
34042636	31	26	1	49	27
121742422	32	26	1	50	27
246264136	33	26	1	28	28
319166617	34	26	56	29	28
288440985	35	26	7862	30	28
192610260	36	26	526232	31	28
98279669	37	26	13112304	32	28
39814064	38	26	135527040	33	28
12535121	39	26	672752697	34	28
3478620	40	26	1849786337	35	28
652804	41	26	3182025266	36	28
150337	42	26	3745461828	37	28
14169	43	26	3209795849	38	28
3592	44	26	2086475551	39	28
138	45	26	1060972569	40	28
55	46	26	427315478	41	28
1	47	26	141811706	42	28
1	48	26	36468802	43	28
1	27	27	8808255	44	28

Table C.2: Numbers of unlabeled minimally 2–edge–connected blocks by number of edges m and nodes n (continued).

Number	**m**	**n**	**Number**	**m**	**n**
1341065	45	28	3259231737	36	30
285354	46	28	12038222764	37	30
21464	47	28	27288213257	38	30
5245	48	28	41589548929	39	30
163	49	28	45519150344	40	30
64	50	28	37441597392	41	30
1	51	28	23897370720	42	30
1	52	28	12084083788	43	30
1	29	29	4941705523	44	30
61	30	29	1627825475	45	30
9549	31	29	456834456	46	30
738779	32	29	97342482	47	30
21405863	33	29	20938090	48	30
259020238	34	29	2602748	49	30
1503773658	35	29	519001	50	30
4800121628	36	29	31460	51	30
9487647064	37	29	7451	52	30
12718351946	38	29	190	53	30
12324090828	39	29	74	54	30
9021495567	40	29	1	55	30
5136228314	41	29	1	56	30
2330194542	42	29	1	31	31
845957571	43	29	70	32	31
257317778	44	29	13867	33	31
60156608	45	29	1399369	34	31
13677144	46	29	53850679	35	31
1880417	47	29	876656622	36	31
386703	48	29	6861665417	37	31
26083	49	29	29230554420	38	31
6266	50	29	75821051619	39	31
177	51	29	131181951627	40	31
68	52	29	161912714965	41	31
1	53	29	149493232059	42	31
1	54	29	106698936335	43	31
1	30	30	60285875698	44	31
65	31	30	27362812141	45	31
11569	32	30	10169152742	46	31
1022873	33	30	3054884541	47	31
34267036	34	30	795515578	48	31
482299201	35	30	154798443	49	31

Table C.2: Numbers of unlabeled minimally 2–edge–connected blocks by number of edges m and nodes n (continued).

Number	m	n	Number	m	n
31639595	50	31	19622	35	33
3560328	51	31	2531428	36	33
690073	52	31	126544712	37	33
37679	53	31	2711327170	38	33
8804	54	31	28074322539	39	33
205	55	31	157435902697	40	33
78	56	31	531024462441	41	33
1	57	31	1177817286145	42	33
1	58	31	1840732839687	43	33
1	32	32	2131674907233	44	33
75	33	32	1895773868668	45	33
16569	34	32	1328161736848	46	33
1891930	35	32	746242676699	47	33
83209464	36	32	341160777002	48	33
1557952711	37	32	127616746824	49	33
14058076759	38	32	39920531924	50	33
68830400681	39	32	10083341281	51	33
203841152050	40	32	2293905689	52	33
399664315651	41	32	373726643	53	33
555509059170	42	32	69763816	54	33
574808616155	43	32	6452323	55	33
458332054828	44	32	1189768	56	33
288480752940	45	32	53045	57	33
145960293104	46	32	12099	58	33
59950784513	47	32	235	59	33
20383675691	48	32	89	60	33
5606317506	49	32	1	61	33
1361494882	50	32	1	62	33
242287891	51	32	1	34	34
47243147	52	32	85	35	34
4817160	53	32	23177	36	34
909817	54	32	3352602	37	34
44841	55	32	189646445	38	34
10349	56	32	4626926459	39	34
219	57	32	54735161382	40	34
84	58	32	350322018442	41	34
1	59	32	1342432117331	42	34
1	60	32	3362070534561	43	34
1	33	33	5898941759003	44	34
80	34	33	7634108950201	45	34

Table C.2: Numbers of unlabeled minimally 2–edge–connected blocks by number of edges m and nodes n (continued).

Number	**m**	**n**	**Number**	**m**	**n**
7562102817538	46	34	101961234	56	34
5886902819531	47	34	8561514	57	34
3671172793294	48	34	1544243	58	34
1858711800564	49	34	62403	59	34
773336296457	50	34	14090	60	34
264797303973	51	34	250	61	34
76577513861	52	34	95	62	34
17805492689	53	34	1	63	34
3810002352	54	34	1	64	34
568734758	55	34	0	65	34

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