The average number of splitters in a random permutation

If $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n)$ is a permutation of $[n] := \{1, 2, \ldots, n\}$, then we say that "$\sigma$ splits at j" or "j is a splitter in σ" if $\sigma_i < \sigma_j$ for all $j < i$ and $\sigma_j < \sigma_k$ for all $j < k$ (so that necessarily $s_j = j$). Splitting properties of random permutations are of interest because of the appearance of that concept in the *quicksort* algorithm (see e.g. section 2.2 in H. Wilfs book *Algorithms and Complexity*, Prentice-Hall, 1986).

In this note I give a very short proof of the fact that the average number of splitters in a random permutation of n elements behaves as $2/n$ for large n.

- Counting splitters is easy: j appears as a splitter in precisely $(j-1)!\,(n-j)!$ permutations of $[n]$, so that there is a total of

$$s_n := \sum_{j=0}^{n} (j-1)!\,(n-j)! = (n-1)! \sum_{j=0}^{n-1} \binom{n-1}{j}^{-1}$$

  splitters in permutations of $[n]$.

- In order to deal with the reciprocals of binomial coefficients it is useful to remember (the β-integral)

$$\frac{1}{a+b+1}\binom{a+b}{a}^{-1} = \int_0^1 t^a\,(1-t)^b dt$$

  for integers $a, b \geq 0$, which can be proved by simple induction using partial integration.

- Let us now consider

$$b_n := \sum_{k=0}^{n} \binom{n}{k}^{-1} = (n+1)\int_0^1 \sum_{k=0}^{n} t^k (1-t)^{n-k} dt = (n+1)\,a_n$$

  Note that the a_n satisfy a very simple recursion:

$$a_{n+1} = \frac{a_n}{2} + \frac{1}{n+2} \quad , \quad a_0 = 1$$

  This can be seen as follows:

$$\begin{aligned} a_{n+1} - \frac{a_n}{2} &= \int_0^1 t^{n+1} dt + \int_0^1 \sum_{k=0}^{n} \left[t^k (1-t)^{n+1-k} - \frac{1}{2} t^k (1-t)^{n-k} \right] dt \\ &= \frac{1}{n+2} + \int_0^1 \left[\sum_{k=0}^{n} t^k (1-t)^{n-k} \right] \left(\frac{1}{2} - t \right) dt \end{aligned}$$

  where the last integral vanishes for reasons of symmetry!

- Now $\overline{s}_n = s_n/n! = b_{n-1}/n = a_{n-1}$, the average number of splitters in permutations of $[n]$, satisfies the recursion

$$\overline{s}_{n+1} = \frac{\overline{s}_n}{2} + \frac{1}{n+1} \quad , \quad \overline{s}_1 = 1$$

  from which it follows immediately that $n \cdot \overline{s}_n \to_{n\to\infty} 2$.

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