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DEPARTMENT OF MATHEMATICS

22 Oct. 1973

Dr. N. J. A. Sloane, Mathematics Research, Bell Laboratories, Murray Hill, N.J.

Dear Dr. Sloane,

I thank you kindly for your letter of 11 October, just received today (mail service amazes me).

The identity

(1)
$$\sum_{k=0}^{\infty} {c + k - 1 \choose k}^2 \quad x^k = (1 - x)^{1-2c} \sum_{k=0}^{c-1} {c - 1 \choose k}^2 \quad x^k$$

which you ask about is just relation (3.141) in my book.

You will find that I give it in a more general form, however:

(2)
$$(1-x)^{z+1} \sum_{k=0}^{\infty} {\binom{-z-1}{k}}^2 x^k = (1-x)^{-z} \sum_{k=0}^{\infty} {\binom{z}{k}}^2 x^k$$
,

from which your form (1) follows when we restrict z to be an integer with z = c - 1, c = 1, 2, 3, ...

You must also use the trivial identity $\binom{-c}{k} = (-1)^k \binom{c+k-1}{k}$ to transform (2) into(1). This is a very useful little relation, mentioned on page iv of my book as among those things used ad libitum to derive and transform things.

The real beauty of my form (2) is that this may be thought of (as I say in my book, page 40) as a property of the Legendre polynomial $P_n(x)$ when n is allowed to be a real number. Indeed, (2) says really then that

(3) If
$$f(z) = P_z(\frac{1+x}{1-x})$$
, then $f(z) = f(-z-1)$.

I hope this answers your query. I should be interested in how you arrived at (1). I was motivated by my formula (3.134) and the idea of looking at $P_n(x)$.

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WEST VIRGINIA UNIVERSITY

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DEPARTMENT OF MATHEMATICS

3099

1 January 1974

Dr. N.J.A. Sloane, Mathematics Research, Bell Telephone Laboratories, Mountain Avenue, Murray Hill, N.J. 07974

Dear Dr. Sloane,

I want to thank you so very much for the copy of your new book; it arrived here 27 Dec. 1973. This is a very wonderful collection of sequences and references. Really, it is a fine piece of scholarship, destined to be a major reference work. I can see that you are going to be 'Mr. Sequence' from now on. Such designations are common ... I have often signed my own name as (Gould ... you may see why.

I have checked and found all my very favorite sequences in your handbook. I have even found a few mistakes in some listings in my own files, at least it appears very likely.

At this time I want to sit down and send you some other sequences that ought to fit in very well in a supplement.

Here is an unusual one in which I am specially interested, and for which I can give no published reference. I have studied the sequence, defined by a peculiar binomial summation, but know no way to give it in any closed form. I have a conjecture about it, in case you want more things to study. Here it is:

Let
$$S_n = \sum_{k=0}^{\lfloor \sqrt{n} \rfloor} {n \choose k^2}$$

For $n = 0, 1, 2, 3, \dots$ we obtain the following values for S_n : 1, 2, 3, 4, 6, 11, 22, 43, 79, 137, 231, 397, 728, 1444, 3018, 6386, 14708, 26725, 51852, ...

= N 209.5 You may wish to check my values and generate others. What I would like to settle is whether there are any other solutions to $S_{n+1} = 2S_n - n$ besides things mention n = 2, 7, 12. the equation

In a similar vein I am interested in higher differences of binomial coefficients where a quadratic appears in the upper slot. These two are of interest:

(2)
$$T_n = \sum_{k=0}^{n} (-1)^{n-k} {n \choose k} {k^2 \choose n}$$
, $= N1788.5$ which runs (for n=0,1,2,...) 1, 1, 6, 36, 1322,

(3)
$$U_{n} = \sum_{k=0}^{n} (-1)^{n-k} {n \choose k} {(k+1)^{2} \choose n}, \qquad 32.36$$

$$\sqrt{1258.5}$$

Extend

which runs 1, 3, 24, 220, 6122,

and these could easily be generated on a computer I suppose.

No closed summation formulas are known for but a trivial kind of sum involving things like

$$\sum_{k} \begin{pmatrix} ak^{2} + bk + c \\ dk^{2} + ek + f \end{pmatrix} f(k) , \quad \text{with}$$

and I would like to think that listing a couple of theme kinds of sums might lead to links with other parts of science where, hopefully, such sequences appear in some entirely different way.

Hence I suggest these two I set down above.

Professor Leroy Meyers, Math.Dept., Ohio State University, sent me some sequences in 1965 that bear on <u>linguistics</u>. He was working with POLA (Program on Linguistic Analysis) there at OSU and trees come into the study of grammatical structure of Chinese, etc. In a letter dated 16 Aug.1965 he listed 8 sequences of interest to him. I found that all but the last are in your book now. This last sequence runs as follows:

but the last are in your book now. This last sequence runs as follows:

(4) 1, 2, 3, 6, 10, 20, 37, 76, 152, 320,

and is evidently related to sequence 298 (Wedderburn-Etherington numbers)

listed in your book. Sequence 298 was one of the others Roy Meyers had

mentioned to me. These sequences were discussed by Roy and me because of

my correspondence with him about the Catalan numbers. Some of the

items by Meyers in which such sequences occur are listed in my Catalan-Bell

bibliography (GO4 by your referencing).

I suppose Roy could tell you more about the sequence and others he is

interested in. You should write to him about it. He has helped me

tremendously with old references about Catalan numbers also.

Get

For many years I have tried (elegantly unsuccessfully) to prove that Euler's constant is irrational. I see that you give the decimal values for pi, e, Euler's constant, and some others. I would like to suggest inclusion of at least 3(3), one of the simplest values of Zeta function which we do not know whether is rational or irrational. Many theorems can be adduced relating this number to Euler's constant, etc. Possibly by listing this number in sequence form, other connections can be made.

(5) It starts as $\zeta(3) = 1.2020569032...$ and several tables give rather a large number of decimals.

Problem E 1382, Amer. Math. Monthly, also Problem E 1164, etc. are things that remind me of still another sequence:

One of my old favorite sequences is (let us call it here N) 1, 2, 2, 3, 3, 4, 4, 4, 4, 5, 5, 5, 5, 5, 6, 6, 6, 6, 6, 6, 6, 7, Where the integer k appears exactly k times....

I don't see this in your book. It arises in writing down explicit

1-to-1 correspondences of the rationals with the natural numbers. The sequence is involved in my note about the system of indexing, p. vii of my book Combinatorial Identities.

Extend

What happens is that the sequence is generated by (bracket means integer part as usual)

$$N_n = \begin{bmatrix} 1 + \sqrt{8n+1} \\ 2 \end{bmatrix}$$
 for $n = 0,1,2,3,...$

Prease look at the page I cited. I could give other references in the number theory literature, but the substance is contained there in my note. It is a very useful sequence, and nice to know where to find an explicit formula. See my solution to Problem 571, Math. Mag., 38(1965), 185-187. This makes it just as clear.

A.J.Cole, Cyclic progressive number systems, Mathematical Gazette, 50(1966), May, Whole No. 372, pp. 122-131, gives a most interesting method of numeration for the natural numbers. In the cyclic progressive system, only one digit changes when one passes from a given number A to the next number A+1. Contrast this to the transitions 9, 10; 999, 1000, etc. Here is the beginning of the sequence:

10, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 39, 38, 37, 36, 35, 34, 33, 32, 31, 30, 40, 41, 42,

the pattern is really quite clear. (Would make a nice intelligence test sequence?) The advantage of this notation is that mechanically or electronically, there may be less error when only one digit changes in successive counting.

Another sequence generated by a peculiar summation is the following:

(8)
$$V_n = \sum_{k=1}^n k^{n-k+1} = \sum_{k=1}^n (n-k+1)^k$$
.

N1102.5

=3101

For $n = 1, 2, 3, \dots$ we have $V_n = 1, 3, 8, 22, 65, 209, 732, 2780, 11377,$ 49863, 242768, 1151914 ,....if I am right so far. (I am copying from old notes made twenty years ago.) I don't seem to have a literature reference here. I made a conjecture or two:

 $V_{hn} \equiv 0 \pmod{2}$ and $V_{hn-1} \equiv 0 \pmod{4}$, for any nat.nr. n, but did not see a proof. Perhaps I can pose this anew in a journal.

Here is an interesting sequence I have explored and I have two pertinent references in the literature. Perhaps you can find others. I do not know the general answer and would like to find it.

Problem 159, Pi Mu Epsilon Journal, 3(1964), Spring, No. 10, p. 528: Proposed by David L. Silverman (submitted without a solution)

"If A_n denotes the largest integer divisible by all the integers less than its n^{th} root, show that $A_2=24$, and $A_3=420$. Find a general formula for A_n ."

0d 1.74

The editorial note, Pi Mu Epsilon J., 4(1965), Fall, No.3, p.124: Refers to Nubuo Ozeki, On the problem 1, 2, 3, ..., $\left\lceil n^{1/k} \right\rceil \right\rceil n$, J.Coll.Arts Sci., Chiba Univ., 3(1961/62), 427-431. "It is proved that 720720 is the largest integer which is divisible by all the positive integers that do not exceed the 5-th root of n. Similar results for the k-th roots are proved in the cases $6 \le k \le 10$. The results for k = 2,3,4 are known."

I have a method of attack that yields these numbers in factored form. I have not seen the mediamamma paper of Ozeki, so I do not know if my results agree with his, save for the number 720720 which does agree! I found (1964, unpublished) these values for the sequence:

(9) 2, 24, 420, 27720, 720720, 36756720,

N850.5 = 3102

corresponding to factored forms

$$2, 2^{3}3, 2^{2}3.5.7, 2^{3}3^{2}.5.7.11, 2^{4}3^{2}5.7.11.13, 2^{4}3^{3}5.7.11.13.17, ...$$

I would check the paper of Ozeki before using the sequence (9) as I give it here. I am interested in knowing what else may have been done with this one.

As an outgrowth of Samuel Beatty's original work on complementary sequences, many interesting complementary sequences are known. I published my paper on non-Fibonacci numbers to illustrate. You have my non-Fibonacci sequence as your seq. no. 1319 of course. I know of no other paper on non-Fibonacci numbers per se.

P37 R=2 (N223)

Related hereto, however, at least as early as 1957 Leo Moser*had circulated a formula for the n-th non-k-th power. Let

 $A_n^k = n + \left[\sqrt[k]{n} \right]$

R=3 N178.9 =3103 Then this formula generates the n-th non k-th power. You might consider this kind of sequence also. After all, how many know that an actual nice formula for it can be given? I suggest taking the cases k=2,3,4,5, say. The method is outlined in my paper on non-Fibonacci numbers, seq.1319 for reference citation.

I hope these 10 items might be of interest. If I come across some others, things that seem startling enough to want to see a reference to them in your book, then I'll surely write you. Let me know how these strike you.

Happy New Year and a Happy New Sequence

Sincerely,

*Leo Moser, Notes on Number
Theory, Edmonton, Alb., Canada, c. 1956.

Menry W. Gould July