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ON THE NUMBER OF PLANE 2-TREES

EDGAR M. PALMER† AND RONALD C. READ

1. *Introduction*

Since higher dimensional analogues of trees were introduced in [**8**] as acyclic simplicial complexes, a number of papers have dealt with various characterizations and enumeration problems associated with these tree-like structures [**1**, **2**, **3**, **4**, **10**]. As defined in [**8**], a 2-tree is a certain 2-dimensional simplicial complex and the techniques and formulas which serve to obtain generating functions for these structures also serve to enumerate those which can be imbedded in the plane. The latter, it turns out, correspond precisely to triangulations of the disk, and the question of how many there are has been studied and answered by many mathematicians including Euler (see [**5**] for a brief history of the problem, and [**6**]).

In this paper, a 2-tree will be a graph, as defined in [**1**], and these 2-trees are simply the 1-skeletons of the 2-dimensional simplicial complexes of [**8**]. These graphs are easily seen to be planar and we shall provide formulas for the numbers of labelled and unlabelled, plane 2-trees as well as some asymptotic results. From now on in this paper the adjective "plane" will usually be omitted. For graph theoretic terminology not provided here see [**7**]. Appropriate information on generating functions may be found in [**11**].

We now define 2-trees inductively. The complete graph on 3 vertices is a 2-tree, and a 2-tree with $p+1$ vertices is obtained from a 2-tree with p vertices by adding a new vertex adjacent to each of two adjacent vertices. It follows quickly that if a 2-tree has p vertices, q edges and r triangles, then

$$q = 2p-3 \tag{1}$$

and

$$r = p-2. \tag{2}$$

It is also clear that each of these graphs is planar, i.e. it can be imbedded in the plane so that no two edges intersect. Note that although every 2-tree as defined here is planar, its corresponding 2-dimensional simplicial complex as considered in [**4**] and [**8**] is not necessarily planar. It remains to be seen how many different imbeddings there are for those with a given number of vertices. We emphasize that two imbeddings of a 2-tree will be regarded as equivalent if and only if one can be brought into coincidence with the other by an orientation-preserving homeomorphism of the plane, i.e. a distortion of the plane not involving a reflection.

For each $n = 3, 4, \ldots$, let P_n be the number of labelled, plane 2-trees with n vertices so that

$$P(x) = \sum_{n=3}^{\infty} P_n \frac{x^n}{n!} \tag{3}$$

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is the exponential generating function for labelled, plane 2-trees. Furthermore, let Q_n be the number of unlabelled, plane 2-trees on n vertices and set

A3093

$$Q(x) = \sum_{n=3}^{\infty} Q_n x^n, \tag{4}$$

the ordinary generating function for unlabelled, plane 2-trees. We seek to determine formulas for P_n and Q_n. To find $P(x)$ we shall first establish formulas for various classes of plane 2-trees rooted at edges. Then $Q(x)$ can be expressed in terms of $P(x)$. Note that Figure 1 shows $Q_3 = 1$, $Q_4 = 2$ and $Q_5 = 10$, while $P_3 = 2$, $P_4 = 36$ and $P_5 = 1200$.

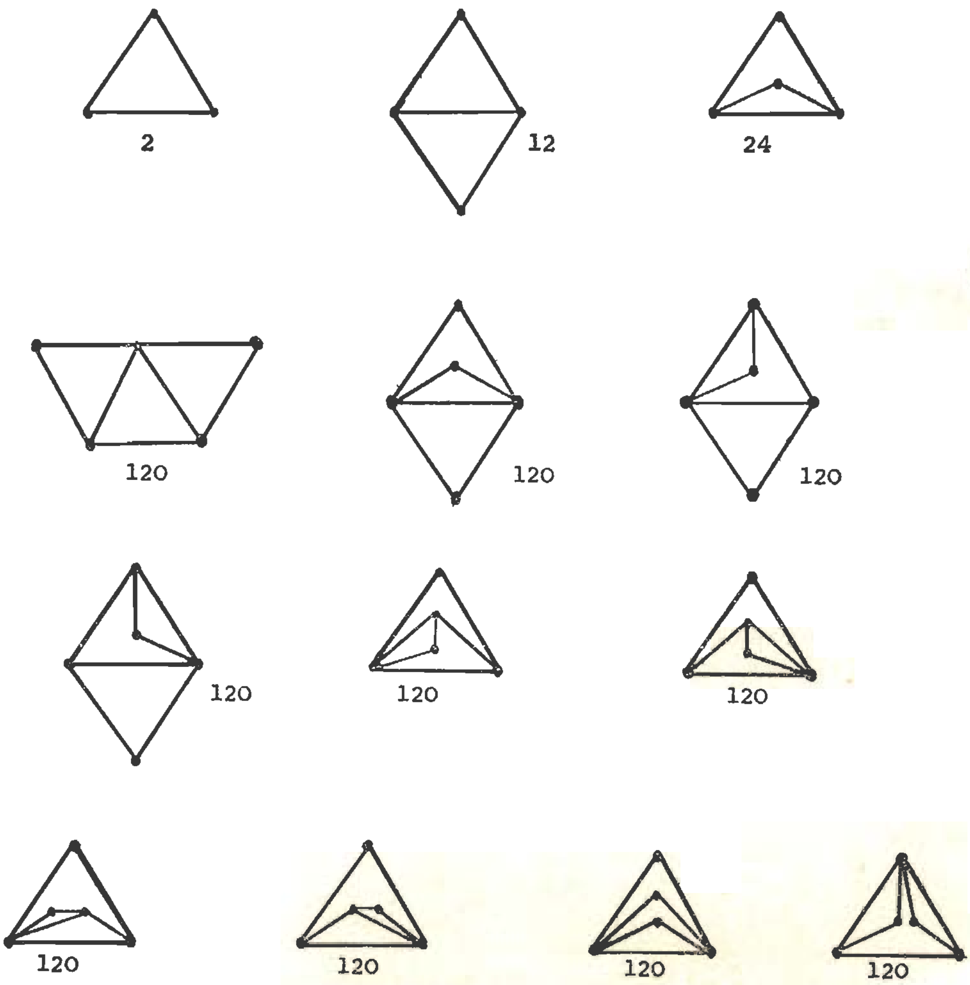
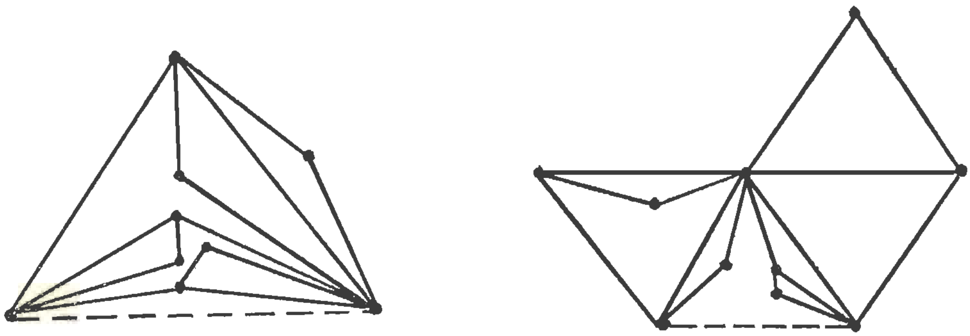


Fig. 1.
The plane 2-trees on 3, 4, and 5 vertices and the number of ways that each can be labelled.

2. *Externally-rooted 2-trees*

A *rooted 2-tree* has one of its edges (the root edge) distinguished. A *labelled rooted 2-tree* has all of its vertices labelled *except* those of the root edge. In an *externally-rooted 2-tree* the root edge is an exterior edge. A *simply-rooted 2-tree* is an externally rooted 2-tree whose root edge belongs to exactly one triangle. See Figure 2(*a*) and (*b*).



(*a*) externally-rooted (*b*) simply-rooted

Fig. 2.
Rooted 2-trees.

Let T_n be the number of labelled simply-rooted 2-trees with n labelled vertices (i.e. $n+2$ vertices altogether); and let $T(x)$ be the corresponding exponential generating function. Thus

$$T(x) = \sum_{n=1}^{\infty} T_n \frac{x^n}{n!} \quad . \tag{5}$$

The following simple result will be used several times in the course of the paper.

Lemma 1. *The function for labelled externally-rooted 2-trees is*

$$\sum_{k=0}^{\infty} T^k(x) = (1-T(x))^{-1}. \tag{6}$$

Proof. An externally-rooted 2-tree in which the root edge belongs to exactly k triangles can be obtained by identifying the root edges of k simply-rooted 2-trees. Any sequence of k such 2-trees determines a unique externally-rooted 2-tree, the first in the sequence being the innermost. These sequences are enumerated by $T^k(x)$. Summing over all values of k we obtain the result of the lemma.

Note that (6) contains a constant term 1. This corresponds to the *degenerate* 2-tree consisting only of the root edge, whose inclusion at this point will simplify later work.

The next lemma provides a functional equation for $T(x)$ from which an explicit formula for T_n is easily derived.

Lemma 2. *The generating function $T(x)$ for labelled simply-rooted 2-trees satisfies*

$$T(x) = x(1-T(x))^{-4}, \tag{7}$$

and

$$T_n = \frac{(5n-2)!}{(4n-1)!} \quad . \tag{8}$$

Proof. Let S be any simply-rooted 2-tree; let uw be the root edge, and let v be the other vertex of the unique triangle to which the root edge belongs. S determines four externally-rooted 2-trees, namely those on either side of each of the edges uv and vw. Conversely, given a sequence of four externally rooted 2-trees, we can construct a unique simply-rooted 2-tree. These tetrads of externally-rooted 2-trees will be enumerated by $(1-T(x))^{-4}$, by Lemma 1. However, S has one more labelled vertex, namely v, than occur in these four 2-trees, so this generating function needs to be multiplied by x. Thus (7) is established, and (8) follows from a routine application of Lagrange's inversion formula **[12]**. This formula states that if $T = xg(T)$, then

$$f(T) = f(0) + \sum_{n=1}^{\infty} \frac{x^n}{n!} D^{n-1} [f'(T)g^n(T)]_{T=0}. \tag{9}$$

Here we take $g(T) = [1-T]^{-4}$ and $f(T) = T$.

3. *Rooted 2-trees*

Let $W(x)$ be the (exponential) generating function for those labelled 2-trees which are rooted at an edge, say uw, which is in only one triangle, say uvw, in which u and v are exterior points but w is not. For an example, see Figure 3.

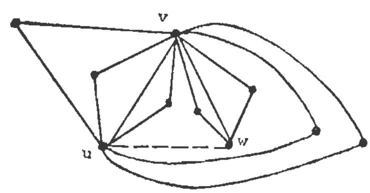


Fig. 3.

If we consider any simply-rooted 2-tree and let the vertices of the root edge have labels u and w and let the vertex adjacent to u and w be called v, then 2-trees enumerated by $W(x)$ can be obtained by identifying with the edge uv the root edge of any non-degenerate externally-rooted 2-tree. Furthermore, all 2-trees enumerated by $W(x)$ can be obtained in this manner. The generating function for non-degenerate, externally-rooted 2-trees is

$$\sum_{k=1}^{\infty} T^k(x) = T(x)(1-T(x))^{-1}; \tag{10}$$

(compare Lemma 1). On multiplying (10) by $T(x)$ we have the generating function for the number of those 2-trees for which the exterior vertex of the root edge is on the left as in Figure 3. There is an equal number when it is on the right. Hence we have the following lemma.

LEMMA 3. *The generating function $W(x)$ for labelled 2-trees rooted at an edge which is in exactly one triangle uvw and where u and v are exterior vertices and w is interior, is given by*

$$W(x) = 2T^2(x)/(1-T(x)). \tag{11}$$

Now let $X(x)$ be the generating function for those rooted 2-trees for which at least two vertices of the unique triangle that contains the root are interior. Examples are shown in Figure 4.

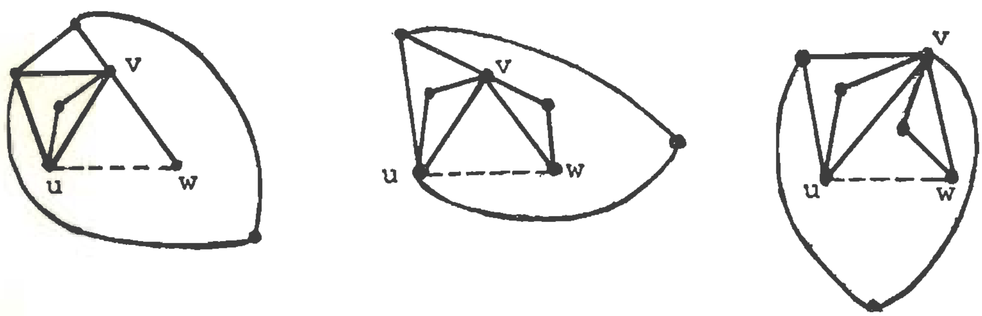


Fig. 4.
2-trees of the type enumerated by $X(x)$.

Note that all rooted 2-trees in which the root edge lies in a single triangle and for which the vertices of the root are not both external are enumerated by $Y(x) = W(x)+X(x)$.

Consider one of the 2-trees enumerated by $X(x)$. If we delete the root edge, the remainder of the 2-tree will consist of two portions having only one vertex in common, namely v; (see Figure 4). One of these portions will lie entirely inside the other. Let us assume for definiteness that it is the portion containing the edge vw. This right-hand portion can only consist of two externally rooted 2-trees (possibly degenerate), one on each side of vw. The other left-hand portion will also contain two such 2-trees on either side of uv, but will contain more besides. If we delete these two 2-trees from the left-hand portion, what is left can be described as follows. It consists of at most two parts having the edge uv in common; there *may* be a part (part A) for which the triangle containing the edge uv has w as an interior point (see triangle uva and the shaded part of Figure 5); there *must* be a part (part B) for which the triangle containing the edge uv is on the other side of uv from w. This is the part that encloses the rest of the 2-tree (see the unshaded part of Figure 5).

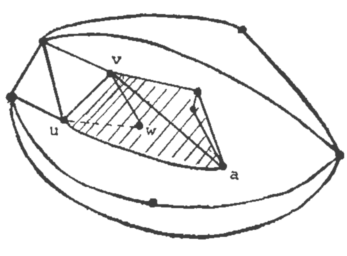


Fig. 5.

We shall enumerate these 2-trees according as part A is absent or present. If part A is absent, then the triangle uvw together with the externally-rooted 2-trees on uv and vw form a simply-rooted 2-tree, enumerated by $T(x)$. To this must be added part B, which will be a 2-tree, rooted at uv, of the type enumerated by $Y(x)$. From any such 2-tree we get *two* different figures according to how part B is joined to the edge uw (see the two different 2-trees in Figure 6) which become identical when the edges uw and vw are deleted. Hence the 2-trees of this type (part A absent) are enumerated by $2T(x)\,Y(x)$.

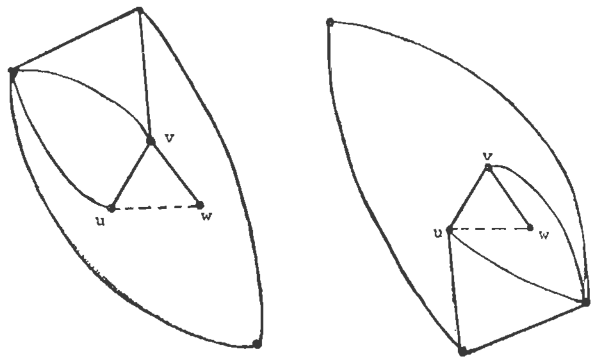


Fig. 6.

If part A is present, then the triangle uvw, the externally-rooted 2-trees on edges uv and vw, and part A, together form a rooted 2-tree of the type enumerated by $W(x)$. Again we have to add part B, enumerated by $Y(x)$, but this time part B can be added in only one way. Hence these are enumerated by $W(x)\,Y(x)$.

All the above was on the assumption that the portion lying inside the other was the right-hand portion; to take account of the other possibility we must multiply by 2. We thus obtain

$$X(x) = 2\big(2T(x) + W(x)\big)\,Y(x). \tag{12}$$

On solving for $Y(x)$ in terms of $W(x)$ and $T(x)$ we have

$$Y(x) = W(x)\,\big(1 - 4T(x) - 2W(x)\big)^{-1} \tag{13}$$

Substituting for $W(x)$ from (11) we have the next lemma.

LEMMA 4. *The exponential generating function $Y(x)$ for labelled 2-trees rooted at an edge whose vertices are not both exterior and which is in exactly one triangle satisfies*

$$Y(x) = 2T^2(x)\big(1 - 5T(x)\big)^{-1}. \tag{14}$$

We shall now remove the condition in this lemma that the root edge is in exactly one triangle. We can add on each side of the root edge an externally-rooted 2-tree (possibly degenerate). We therefore multiply the right side of (14) by $\big(1 - T(x)\big)^{-2}$. There remain those rooted 2-trees for which both vertices of the root edge are exterior. These can be obtained by placing on each side of the root edge an externally-rooted 2-tree, not both degenerate. At first sight the generating function would appear to be $\big(1 - T(x)\big)^{-2} - 1$, but this would count everything twice because of the possibility of a rotation of the figure about the midpoint of the root edge. Hence the required generating function is

$$\tfrac{1}{2}\{(1-T(x))^{-2}-1\}. \tag{15}$$

From these observations we have the next lemma.

LEMMA 5. *The exponential generating function $R(x)$ for labelled rooted 2-trees is given by*

$$\begin{aligned} R(x) &= 2T^2(x)(1-5T(x))^{-1}(1-T(x))^{-2}+\tfrac{1}{2}\{(1-T(x))^{-2}-1\} \qquad (16)\\ &= \frac{T(x)(2-5T(x))}{2(1-T(x))(1-5T(x))}. \end{aligned}$$

4. *Unrooted 2-trees*

To count unrooted 2-trees, we next express $R(x)$ as a sum of partial fractions:

$$R(x) = -\frac{1}{2}+\frac{3}{8}\frac{1}{1-T(x)}+\frac{1}{8}\frac{1}{1-5T(x)} \tag{17}$$

By using Lagrange's inversion formula (9) again, this time with $f(T) = (1-T)^{-1}$ we find that the coefficient of x^n in $(1-T(x))^{-1}$ is

$$\frac{(5n)!}{n!(4n+1)!}. \tag{18}$$

We could expand $(1-5T(x))^{-1}$ in the same way, but it is easier to proceed differently. From equation (7) we readily deduce that

$$xT'(x) = T(x)(1-T(x))(1-5T(x))^{-1}, \tag{19}$$

and thus verify that

$$(1-5T(x))^{-1} = (25/4)\,xT'(x)-(5/4)\,T(x)+1. \tag{20}$$

Hence the coefficient of x^n in $(1-5T(x))^{-1}$ is

$$\frac{25}{4}n\frac{(5n-2)!}{(4n-1)!n!}-\frac{5}{4}\frac{(5n-2)!}{(4n-1)!n!} = \binom{5n}{n} \tag{21}$$

for $n \geqslant 1$.

From (18) and (21) it follows that $R_n/n!$, the coefficient of x^n in $R(x)$, is given by

$$R_n/n! = \frac{3}{8}\frac{(5n)!}{(4n+1)!n!}+\frac{1}{8}\frac{(5n)!}{(4n)!n!} = \frac{1}{2}\frac{(5n)!(n+1)}{n!(4n+1)!}. \tag{22}$$

From our definition of R_n, it follows that the number of 2-trees with $n+2$ vertices all labelled, which are also rooted at an edge, is $(n+2)(n+1)R_n$. Now any labelled 2-tree can be rooted at any of its edges. Therefore, the number of labelled 2-trees with $n+2$ vertices, denoted by P_{n+2}, is just $(n+2)(n+1)R_n/(2n+1)$; or equivalently

$$P_n = \frac{n(n-1)}{2n-3}R_{n-2}. \tag{23}$$

Combining (22) and (23) we have the following theorem.

THEOREM 1. *The number P_n of plane, labelled 2-trees is given by*

$$\boxed{P_n = n(n-1)^2\frac{(5n-10)!}{(4n-6)!}\ .} \tag{24}$$

The first few values of P_n are displayed in Table 1.

TABLE 1. The numbers of plane, labelled 2-trees.

1	2	n	3	4	5	6	7	8
0	1	P_n	2	36	1200	57000	3,477,600	157,826,240

2

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We observe next that when 2-trees are imbedded in the plane, most of their symmetries are eliminated. For example, if the exponential generating function $T(x)$, which enumerates simply-rooted 2-trees, is regarded as an ordinary generating function, then it counts unlabelled simply-rooted 2-trees. In particular, from (7) we see that the coefficient of x^2 in $T(x)$ is 4; so there are simply-rooted 2-trees with 4 vertices (see Figure 7). This follows from the fact that the automorphism group of any 2-tree rooted at an exterior edge consists only of the identity automorphism. Hence there are precisely $n!$ ways of labelling the n vertices which are not incident with the two vertices of the root edge (see [**9**]). Similarly it follows that $P_n/n!$ is the sum of the reciprocals of the orders of the automorphism groups of all plane, un-

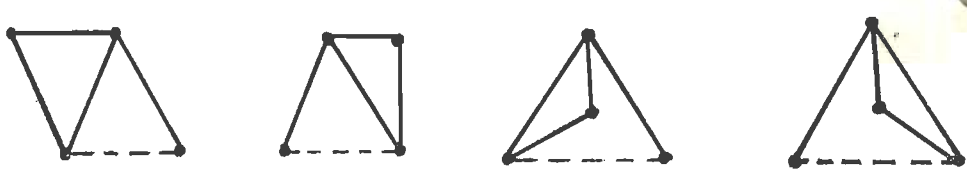

Fig. 7.

labelled 2-trees with n vertices. On the other hand, the orders of these groups are either 1, 2 or 3 (see [**8**]). In Figure 8 we have 2-trees whose groups have orders 2 and 3 respectively. Therefore if $B(x)$ and $C(x)$ are the exponential generating functions

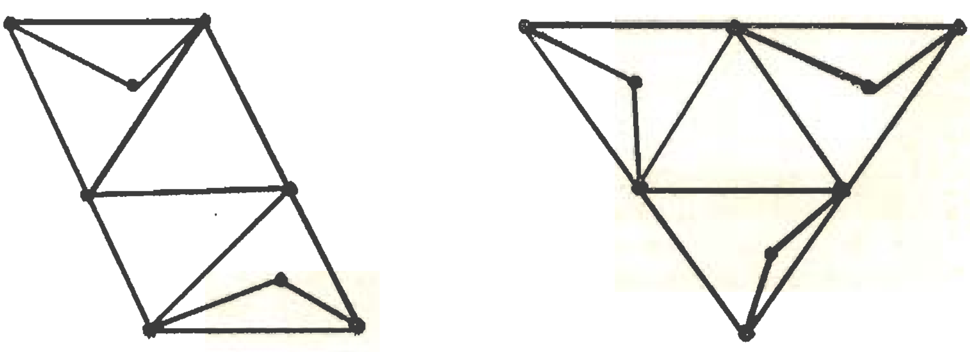

Fig. 8.

for labelled 2-trees whose groups have order 2 and 3 respectively when the labels are removed, then the series $Q(x)$ which counts unlabelled 2-trees is given by

$$Q(x) = P(x)+\tfrac{1}{2}B(x)+\tfrac{2}{3}C(x). \tag{25}$$

We have seen that $(1-T(x))^{-1}-1$ counts non-degenerate labelled externally-rooted 2-trees. Two copies of such a 2-tree with the root edges appropriately identified constitute a 2-tree with a group of order 2. Therefore we double the vertex count in $(1-T(x))^{-1}-1$ by replacing x by x^2 and then on multiplication by x^2 to include the vertices of the root edge we have

$$B(x) = x^2\big((1-T(x^2))^{-1}-1\big). \tag{26}$$

Similarly for $C(x)$ we obtain

$$C(x) = x^3(1-T(x^3))^{-2}. \tag{27}$$

Now we can state the following corollary of Theorem 1 which expresses the generating function for unlabelled 2-trees in terms of those for labelled 2-trees.

COROLLARY 1. *The ordinary generating function $Q(x)$ for plane 2-trees is given by*

$$Q(x) = P(x)+\tfrac{1}{2}x^2\big((1-T(x^2))^{-1}-1\big)+\tfrac{2}{3}x^3(1-T(x^3))^{-2}. \tag{28}$$

If $n = 2u$, then the coefficient of x^n in $\frac{1}{2}B(x)$ is

$$\frac{1}{8u-6}\binom{5u-5}{u-1}, \tag{29}$$

and otherwise it is zero.

If $n = 3v$, then the coefficient of x^n in $\frac{2}{3}C(x)$ is

$$\frac{2}{3(v-1)!}\sum_{k=0}^{v-1}\binom{v-1}{k}\frac{(5k)!(5v-5-5k)!}{(4k+1)!(4v-3-4k)!}, \tag{30}$$

and otherwise it is zero. Furthermore, it can be shown that the expression (30) is equal to

$$\frac{4}{3(v-1)}\binom{5v-4}{v-2} \tag{31}$$

for $v \geqslant 3$.

These results are summarized in the next corollary.

COROLLARY 2. *The number Q_n of (unlabelled) plane 2-trees with n vertices is given by the following formulas.*

If $2 \nmid n$ and $3 \nmid n$, then

$$Q_n = \frac{n-1}{(4n-6)(4n-7)}\binom{5n-10}{n-2}. \tag{32}$$

If $n = 2u$ and $3 \nmid u$, then

$$Q_n = \frac{1}{8u-6}\left\{\frac{2u-1}{8u-7}\binom{10u-10}{2u-2}+\binom{5u-5}{u-1}\right\}. \tag{33}$$

If $n = 3v$, $v \geqslant 3$, *and* $2 \nmid v$

$$Q_n = \frac{3v-1}{(12v-6)(12v-7)}\binom{15v-10}{3v-2} + \frac{4}{3v-3}\binom{5v-4}{v-2}. \tag{34}$$

If $n = 6w$ *and* $w \geqslant 2$

$$Q_n = \frac{6w-1}{(24w-6)(24w-7)}\binom{30w-10}{6w-2}$$

$$+ \frac{1}{24w-6}\binom{15w-5}{3w-1} + \frac{1}{6w-3}\binom{10w-4}{2w-2}. \tag{35}$$

The values of Q_n in Table 2 for small n were calculated using Corollary 2.

TABLE 2. The numbers of plane 2-trees.

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n	3	4	5	6	7	8	9	10
Q_n	1	2	10	83	690	6412	61842	457025

??

Note that if n is not a multiple of 6, then $Q_n = P_n/n!$. The coefficients of $P(x)$ and $Q(x)$ are easily shown to be asymptotically equivalent as stated in the concluding corollary.

COROLLARY 3. *The number Q_n of unlabelled plane 2-trees is asymptotic to $P_n/n!$, i.e.*

$$Q_n \sim P_n/n! \tag{36}$$

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