Self-Number

Santanu Bandyopadhyay
Department of Energy Science and Engineering
Indian Institute of Technology Bombay
Powai, Mumbai, 400076, India
14 July 2020

Let's start with a number, say 29. If we add digits of this number to itself, we obtain 29 + (2 + 9) = 40. We call 29 is the *generator,* and 40 is the *generated number* (Kaprekar, 1963). We express this symbolically as: $a_{n+1} = a_n + ds(a_n)$, where a_n is the generator, a_{n+1} is the generated number, and $ds(a_n)$ stands for the sum of all the digits of a_n. In 1949, D. R. Kaprekar studied these and related numbers (Gardner, 1975).

Start with a number and produce the generated number. Continue the process to produce a series of such generated number, each one of them (obviously except the first one) is generated from the previous generator. If we start at 23, we generate the following series:

23 → 28 → 38 → 49 → 62 → 70 → …

Kaprekar observed an interesting property of such a series. Consider any term (except the first one) of the series. Now determine its generated number and subtract the first term from it. For example, let's us consider 70. Its generated number is 77 (= 70 + 7 + 0) and subtracting 23 from it, results in 54 (= 77 – 23). This is equal to the sum of digits of all the numbers in the series up to the chosen number. For our example, (2 + 3) + (2 + 8) + (3 + 8) $+ (4 + 9) + (6 + 2) + (7 + 0) = 54$, as expected. A simple proof of this result is included in the adjacent box (←). Kaprekar wrote to Gardner, "Is this not a wonderful new result? …The Proof of all this rule is very easy and I have completely written it with me. But as soon as the proof is seen the charm of the whole process is lost, and so I do not wish to give it just now" (Gardner, 1975).

$$
\begin{aligned}
a_2 &= a_1 + ds(a_1) \\
a_3 &= a_2 + ds(a_2) \\
&\vdots \\
a_n &= a_{n-1} + ds(a_{n-1}) \\
a_{n+1} &= a_n + ds(a_n) \\
\hline
a_{n+1} &= a_1 + ds(a_1) + ds(a_2) + \ldots + ds(a_n)
\end{aligned}
$$

Let us start the *digit-addition process* with 149, a prime number. 149 generates 163 (= 149 + 1 + 4+ 9), another prime number. 163 generates 173 (163 + 1 + 6 + 3), another prime number. The next generated number, 184 (= 173 + 1 + 7 + 3) is not prime, but a composite.

149 → 163 → 173 (→ 184)

Kaprekar conjectured that the number of consecutive primes numbers in such a digit-addition process cannot be more than four (Athmaraman, 2004). Vishal Joshi, from Jamnagar, India, disproved this conjecture by producing counterexamples; primes 37783, and 516493, generate five and six consecutive primes (Tikekar, 2006).

37783 → 37811 → 37831 → 37853 → 37879 (→ 37913 =31 × 1223)

516493 → 516521 → 516541 → 516563 → 516589 → 516623 (→ 516646)

A simple search (also not very efficient) produces a counterexample with eight consecutive primes:

286330897 → 286330943 → 286330981 → 286331021 → 286331047 → 286331081 → 286331113 → 286331141 (→ 286331170)

A number without any generator is called a *self-number* (Kaprekar, 1963). These numbers are also known as *Colombian numbers*, based on a problem proposed by Recamán (1974). Some of the initial self-numbers or Colombian numbers are 1, 3, 5, 7, 9, 20, 31, 42, 53, 64, 75, 86, 97, ... (A003052 in OEIS). Some of the self-numbers may be generated from the following relation (Bange, 1974): $S_{k+1} = 8 \times 10^k + S_k + 8$ with S_1 = 9. This relation produces following self-numbers: 9, 97, 905, 8913, 88921, 888929, 8888937, 88888945, 888888953, … (A232229 in OEIS). For every *k*, it produces a *k*-digit self-number and thus, establishes the existence of infinite self-numbers.

Kaprekar proposed an elegant algorithm to test whether any given number is a self-number or not. First, determine the *digital root* of the given number, *N*. Digital root of a number is obtained by summing up all its digits, then summing up the digits of the resultant number and continue the process until only a single-digit remains. If the digital root of the given number, *N*, is an even number, then divide it by 2 to obtain another number; let’s denote it by *C*. Otherwise, add 9 to the digital root of *N* and divide by 2 to obtain *C*. Now, subtract *C* from *N* and check whether the resultant number (i.e., *N* – *C*) generates *N*. If it fails to generate *N*, subtract 9 from the last answer (i.e., *N* – *C* – 9) and examine again. Continue subtracting 9's, each time examining whether the result can generates *N*. Follow this process *k* times, where *k* is the number of digits of *N*. If this process fails to yield a generator for *N*, then it is a self-number. Let’s see for one million, 1000000. Digital root for one million is 1, and therefore, *C* is 5 (= (1+9)/2). We have to check whether any of the following numbers generate one million or not: 999995, 999986, 999977, 999968, 999959, 999950, and 999941. These numbers generate the following numbers: 1000045, 1000036, 1000027, 1000018, 1000009, 999991, and 999982. It should be noted that one million is not generated, and hence, it is a self-number. According to Kaprekar, millionaires are important persons as “one million is a self-number.”

A self-number does not have any generator. On the other hand, a generated number may have more than one generator. For example, 101 has two generators: 91 (91 + 9 + 1 = 101) and 100 (100 + 1 = 101). 91 and 100 are called *co-generators,* and 101 is a *junction number* (Kaprekar, 1963). First few junction numbers are 101, 103, 105, 107, 109, 111, 113, 115, 117, 202, 204, 206, 208, 210, … (A230094 in OEIS). Interestingly, junction number 10000000000001 (or $10^{13} + 1$ in short) has three co-generators. To present such a number with repeated digits, we may use a subscript notation: $(d)_n$ represents digit d repeated n times. The junction number 10000000000001 or $1(0)_{12}1$ has three co-generators: $1(0)_{13}$, $(9)_{11}01$, and $(9)_{10}892$. Junctions numbers with a higher number of co-generators also exist. For example, $10^{13} + 102$ has four co-generators, $10^{(1)_{11}21} + 102$ has five, $10^{(2)_{12}4} + 10^{13} + 2$ has six, … (Alekseyev, 2016).

In 1962, Kaprekar was seriously ill and discovered an interesting result to determine junction numbers with four generators from two junction numbers with two generators. If d-digits numbers, $N - 1$ (with d-digits generators P and Q) and $N + 1$ (with d-digits generators R and S) are two junction numbers, then the junction number $1(0)_{(1)_d}N$ has four generators: $(9)_{(1)_d}R$, $(9)_{(1)_d}S$, $1(0)_{(1)_d}P$, and $1(0)_{(1)_d}Q$. Interested readers may verify this. Fearing his death, he initially called it as *Kaprekar’s Last Theorem*. After his recovery, he finally named it *Junction Combination Theorem* (Athmaraman, 2004).

References

Alekseyev, M.A., Johnson, D. and Sloane, N.J.A., 2016. On Kaprekar’s Junction Numbers. http://neilsloane.com/doc/colombian.pdf

Athmaraman, R., 2004. *The Wonder World of Kaprekar Numbers*. The Association of Mathematics Teachers of India, Chennai, India.

Bange, W.D., 1974. Solution to problem E2408, *American Mathematical Monthly*, 81(4) 407.

Gardner, M., 1975. Mathematical Games. *Scientific American,* 232(3), 112-117.

Kaprekar, D.R., 1963. *The Mathematics of New Self-Numbers*, Devlali, India.

OEIS, The On-Line Encyclopedia of Integer Sequences (https://oeis.org/).

Recamán, B., 1974. Problem E2408, *American Mathematical Monthly*, 81(4) 407.

Tikekar, V.G., 2006. Some interesting mathematical gems. *Resonance*, 11(9), 29-42.