SIAM Rev 14 Jan 1972 Sequence 2968 = N1901.5

One is lead to the solution above, by noting that the given equation for $F'(x)$ suggests, that F, as a function of two variables (*par abus langage* denoted by g and h), satisfies

$$\frac{\partial F}{\partial g} = -e^{h^2-g^2}\cos(c-2gh) \quad \text{and} \quad \frac{\partial F}{\partial h} = e^{h^2-g^2}\sin(c-2gh),$$

and hence, that F satisfies the Laplace equation.

Also solved by the proposer.

Problem 71–4*, *Noncomplete Residue System*, by ROBERT SPIRA (Michigan State University).

Given the numbers $1, 2, \cdots, 2n$, form n pairs of them (a_i, b_i), and the corresponding sums and differences $s_i = a_i + b_i$, $d_i = a_i - b_i$. It is conjectured that the numbers s_i and d_i together do not form a complete residue system (mod $2n$).[2]

Remarks by the proposer. The problem originated in the problem of Shen and Shen [1], which was: Is it possible for $n \geqq 3$ to group the set $\{1, 2, \cdots, 2n\}$ into n pairs (a_i, b_i) with $b_i > a_i$ so that all the numbers $b_i - a_i$, $b_i + a_i$, $i = 1, \cdots, n$, are distinct? Using the 7072 computer at Duke University the proposer of the present problem and J. Fink counted the number of such pairings for $3 \leqq n \leqq 10$, obtaining respectively 1, 8, 22, 51, 342, 2609, 16896, 99114. D. A. Klarner [2] showed the connection between the Shens's problem with the queens problem, introducing the notion of a more powerful queen able to reflect via a reflecting strip at one end of the board. The problem of the Shens is also equivalent to placing n queens on half a $2n$-board above the main diagonal.

J. L. Selfridge [3] abstracted a solution of Shens's problem, but never published, stating that the proof could be recovered from the abstract. Klarner [2] stated a problem which might be amenable to Selfridge's process:

Observing the large number of solutions to the Shens's problem, Klarner suggested adding restrictions to reduce the number, the obvious restriction being to study it modulo $2n$. Then, after computing the cases $n \leqq 16$, it seemed that no solutions would arise, although it appears that we can obtain, in certain cases, all residues but one, and hence the problem.

REFERENCES

[1] M. SHEN AND T. SHEN, *Research Problem 39*, Bull. Amer. Math. Soc., 68 (1962), p. 557.
[2] D. A. KLARNER, *The problem of reflecting queens*, Amer. Math. Monthly, 74 (1967), pp. 953–955.
[3] J. L. SELFRIDGE, Notices Amer. Math. Soc., 10 (1963), p. 195.

Solution by A. A. JAGERS (Twente University of Technology, Enschede, Netherlands).

[2] Presented at the 1963 Number Theory Conference.

Set $\xi = \sum_{k=0}^{2n-1} k^2$ and suppose that the numbers s_i and d_i together form a complete residue system (mod $2n$). Then

$$\xi \equiv \sum_{i=1}^{n} ((b_i - a_i)^2 + (b_i + a_i)^2) \equiv 2 \sum_{i=1}^{n} (a_i^2 + b_i^2) \equiv 2\xi \pmod{2n},$$

so $\xi \equiv 0 \pmod{2n}$. On the other hand, $\xi = \frac{1}{3}n(2n-1)(4n-1)$ so that $\xi \equiv n/3 \pmod{2n}$ if $n \equiv 0 \pmod 3$, and $\xi \equiv n \pmod{2n}$ if $n \not\equiv 0 \pmod 3$, a contradiction.

Problem 71–5, *Fourier Series*, by ROBERT E. SHAFER (Lawrence Radiation Laboratory).

Find the Fourier series expansion of

(1) $$e^{-z\cos\theta}E_1(z - z\cos\theta), \quad 0 < \theta < 2\pi, \quad |\arg z| < \pi,$$

where

(2) $$E_1(z) = \int_1^\infty \frac{e^{-tz}\,dt}{t}, \qquad |\arg z| \leqq \frac{\pi}{2}.$$

Solution by the proposer.

Let

(3) $$e^{-z\cos\theta}E_1(z - z\cos\theta) = A_0(z) + 2\sum_{n=1}^{\infty} A_n(z)\cos n\theta.$$

Then

(4) $$A_n(z) = \frac{1}{\pi}\int_0^\pi e^{-z\cos\theta}E_1(z - z\cos\theta)\cos n\theta\,d\theta,$$

and with the aid of (2) and an interchange of the order of integration,

(5) $$A_n(z) = \frac{1}{\pi}\int_1^\infty t^{-1}e^{-zt}\left[\int_0^\pi \cos n\theta \exp(z\cos\theta(t-1))\,d\theta\right]dt.$$

Now replace t by $1 + t[z(1-\cos\theta)]^{-1}$ and obtain

(6) $$A_n(z) = \frac{e^{-z}}{\pi}\int_0^\infty e^{-t}\left[\int_0^\pi \frac{\cos n\theta}{t + z(1-\cos\theta)}\,d\theta\right]dt.$$

The inner trigonometric integral is readily evaluated by residue theory and the result is available in a number of sources; see for example [1, vol. 1, (41), p. 299]. Let $t = 2z\sinh^2\varphi/2$. Then

(7)
$$A_n(z) = \int_0^\infty \exp(-z\cosh\varphi - n\varphi)\,d\varphi = K_n(z) - G_n(z),$$

$$G_n(z) = \int_0^\infty \exp(-z\cosh\varphi)\sinh n\varphi\,d\varphi,$$