619 Greythorne Road

Wynnewood, Pa. 19096

December 25, 1974

Mr. N. J. A. Sloane

Mathematics Research Cehter

Bell Telephone Laboratories, Inc.

Murray Hill, New Jersey 07974

Dear Mr. Sloane:

First, I would like to commend you for your excellent book A Handbook of Integer Sequences. This work is the best collection of sequences that I have ever seen. As a number theory devotee, I am sure that your book will provide many hours of pleasure.

Second, I would appreciate it if you would include me when any new supplements are issued to add to the Handbook.

Third, I was surprised to find that although the continued fraction expansion of e^2 was included (sequence 1811), the continued fraction for e itself was not. I hope that in the future, this sequence will be included.

Sincerely yours,

Jeffrey Shallit

Jeffrey Shallit

Sent

add to list

Ent'd

N 85

f91

SHALLIT 2949

619 Greythorne Road

Wynnewood, Pa. 19096

February 10, 1975

Mr. N. J. A. Sloane

2C-363 Bell Laboratories

600 Mountain Avenue

Murray Hill, N. J. 07974

Dear Sir:

I am writing to you about sequence 1294.5 which appears in the first supplement of your book A Handbook of Integer Sequences. I believe that the sequence as you have given it is incorrect from the 30th term onwards. The sequence is for the continued fraction expansion for $6^{1/3}$.

The sequence should be, as confirmed by two independent calculations have made, as follows:

Sequence 1294.5 2949

1, 4, 2, 7, 3, 508, 1, 5, 5, 1, 1, 1, 2, 1, 1, 24, 1, 1, 1, 3, 3, 30, 4, SSC
10, 158, 6, 1, 1, 2, 12, 1, 10, 1, 1, 3, 2, 1, 1, 89, 1, 1, 2, 1, 1, 1,
3, 1, 2, 1, 7, 1, 2, 18, 1, 17, 2, 2, 10, 14, 3, 1, 2, 1, 2, 1, 5, 1, . . .

My method of calculation of these partial quotients was the algorithm described in "Continued Fractions for some Algebraic Numbers", which appeared in Journal fur die reine und angewandte Mathematik, V. 255 (1972) pp. 112-134. The computation was done on an IBM 370 computer. I believe my calculations are correct, and hope this information will be of some use to you.

Sincerely yours,

Jeffrey Shallit

Jeffrey Shallit

f91

Bell Laboratories

600 Mountain Avenue
Murray Hill, New Jersey 07974
Phone (201) 582-3000

February 18, 1975

Mr. Jeffrey Shallit
619 Greythorne Road
Wynnewood, Pennsylvania 19096

Dear Mr. Shallit:

Thank you for your letter of February 10, enclosing a corrected version of Sequence 1294.5. That is a very kind gesture and I greatly appreciate it.

Have you checked any of the other continued fraction sequences attributed to "HPR"? I am afraid that some of them may also be in error - the ones I was able to check against the values given in the J. R. A. M. article you mention were also in error from about the 30th place. But I have not checked Numbers 19.5 ($5^{1/5}$), 1031.8 ($4^{1/5}$), 1411.5 ($3^{1/5}$), or 1677.5 ($2^{1/5}$). If you felt like checking these it would be a great help.

I don't know what your other interests are, but the enclosed offprints may amuse you.

Thank you again for your letter.

Yours sincerely,

MH-1216-NJAS-mv

N. J. A. Sloane

Enc.
As above

f New sequence

2954

= N418.5

MATHEMATICS OF COMPUTATION, VOLUME 25, NUMBER 116, OCTOBER, 1971

A Note on Chowla's Function*

By M. Lal and A. Forbes

Abstract. Iterates of a number-theoretic function, defined by $L(n) = \sigma(n) - (1 + n)$, are investigated empirically, for $n \leqq 10^5$. This search has yielded 9 reduced amicable pairs.

1. Introduction. Professor Chowla defined a number-theoretic function, $L(n)$, for $n > 1$.

$$L(n) = \sigma(n) - (1 + n) \tag{1}$$

where $\sigma(n) = \sum_{d|n} d$. That is, $L(n)$ denotes the sum of the divisors of n except n and unity.

For n prime, $L(n) = 0$. The rth iterate of $L(n)$ is denoted by

$$L_r(n) = L(L_{r-1}(n)); \qquad L_1(n) = L(n). \tag{2}$$

Professor Chowla conjectured that the sequence of iterates defined above takes only a finite number of different values, and Nasir [1] verified the conjecture for $n \leqq 100$. Furthermore, he found that the sequence converges to zero except for $n = 48$, 75 and 92.

Lehmer, in his review [2] of Nasir's paper, defined the pair (48, 75) as amicable, because $L(48) = 75$ and $L(75) = 48$. In order to avoid confusion, we shall call these pairs as reduced amicable pairs and the pairs defined by the function $S(n) = \sigma(n) - n$ as amicable pairs. In this brief note, we intend to investigate empirically certain properties of $L(n)$ and to provide some evidence for the question whether the number of such reduced amicable pairs is finite.

Results. For $n \leqq 10^5$, it was found that there are only 9 reduced amicable pairs. These pairs are given in Table 1.** If $A(n)$ is the number of reduced amicable pairs of which the smaller number is less than n, then the distribution of $A(n)$ is as follows: $A(n) = 1$ for $n \leqq 10^2$, $A(n) = 2$ for $n \leqq 10^3$, $A(n) = 8$ for $n \leqq 10^4$ and $A(n) = 9$ for $n \leqq 10^5$. It is of interest to note that the number of amicable pairs for $n \leqq 10^5$ is 13, which is comparable to that found for the reduced amicable pairs.

Regarding the function $L_r(n) = 0$, it was found that in most cases $L_r(n) = 0$ and there are only 2151 values for which the iterates converge to a member of a reduced amicable pair. The frequency with which a given reduced pair is reached while

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AMS 1969 *subject classifications*. Primary 1003, 1043; Secondary 1007, 1061, 1063.

Key words and phrases. Reduced amicable numbers, number-theoretic function.

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** See editorial note at end of paper.

TABLE 1

Frequencies of Various Reduced Amicable Pairs $n \leqq 10^5$

Amicable Pairs		Frequency
(48,	75)	1138
(140,	195)	430
(1050,	1925)	42
(1575,	1648)	226
(2024,	2295)	156
(5775,	6128)	70
(8892,	16587)	63
(9504,	20735)	22
(62744,	75495)	4

TABLE 2

Lowest Values of n, for a Given r such that $L_r(n) = 0$

r	n	r	n
1	2	19	3344
2	4	20	3888
3	8	21	5360
4	15	22	8895
5	12	23	11852
6	27	24	25971
7	24	25	23360
8	36	26	38895
9	90	27	35540
10	96	28	35595
11	245	29	36032
12	288	30	53823
13	368	31	47840
14	676	32	62055
15	1088	33	59360
16	2300	34	83391
17	1596	35	70784
18	1458		

iterating $L_r(n) = 0$ is given in Table 1. Out of the total number of 2151 values where $L_r(n) \neq 0$, for some finite r, the smallest pair (48, 75) is reached 1140 times. The frequency with which these pairs appear decreases rapidly. The frequency for the pair (1050, 1925) is unexpectedly low.

It would be of interest to find reduced amicable triplets or groups of higher order. A reduced amicable triplet is defined to be a set of three distinct positive integers n, m, p, such that $L(n) = m$, $L(m) = p$ and $L(p) = n$. Similarly, groups of higher order are defined. For $n \leqq 10^5$, there are no triplets or groups of higher order.

Bounds on the Number of Iterations. For a given r, the smallest values of n, such that $L_r(n) = 0$, were recorded. A table of such values of n suggest, for any n,

$$1 \leqq r \leqq c \ln(n); \qquad c = 3.2 \quad \text{for } n \leqq 10^5.$$

For the purpose of detailed comparison of the function $L(n) = \sigma(n) - (1 + n)$ with $S(n) = \sigma(n) - n$, it would be of interest to compare the upper bound for the iterations of $S(n)$ such that $S_r(n) = 1$. It is known that $S_r(n)$ is bounded for $2 \leqq n \leqq 275$. The verification for higher n is very tedious. For $n = 276$, $S_r(n) = 1$, for $r > 119$ [3] and $S_{189}(936) = 1$. This suggests that the corresponding value of c for the iterations of $S_r(n)$ is considerably higher.

Department of Mathematics
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St. John's, Newfoundland, Canada

1. A. R. Nasir, "On a certain arithmetic function," *Bull. Calcutta Math. Soc.*, v. 38, 1946, p. 140. MR **8**, 445.
2. D. H. Lehmer (Reviewer), *Math. Rev.*, v. 8, 1948, p. 445.
3. H. Cohen, "On amicable and sociable numbers," *Math. Comp.*, v. 24, 1970, pp. 423–429.

Editorial note. While this paper was in press, we learned of a short note by Mariano Garcia, "Números Casi Amigos y Casi Sociables," that appeared in Revista Annal, año 1, October 1968, *Asociacion Puertorriqueña de Maestros de Matematicas.* Garcia's table on page 7 therein gives the same nine pairs listed in Table 1, and no others. The frequencies in Table 1 and the data in Table 2 are not given.

619 Greythorne Road
Wynnewood, Pa. 19096
February 25, 1975

4 new sequences
[JSH]

Mr. N. J. A. Sloane
2C-363 Bell Laboratories
600 Mountain Avenue
Murray Hill, N. J. 07974

Dear Mr. Sloane:

Thank you for your letter of February 18, and the offprints that you included.

At your suggestion, I checked sequences 19.5, 1031.8, 1411.5, and 1677.5 with the aid of a computer. I regret to say that they are all inaccurate after a point, that point being the 35th, 31st, 43rd, and 43rd partial quotient, respectively.

I recalculated the first fifty partial quotients using the algorithm mentioned in the J. R. A. M. article. Then I checked these results by actually calculating the fifth roots of the four numbers to about sixty decimal places and applying the traditional algorithm for determination of the partial quotients to a continued fraction. Both results, completely independent, agreed perfectly to the fiftieth partial quotient for all four numbers.

I am enclosing the true values of the four sequences on an enclosed page.

I thought I should mention that I am a senior at Lower Merion High School and will graduate in June of this year. I did this work at the IBM Philadelphia Scientific Center, where I have been fortunate enough to have been granted unlimited access to the computer.

I am also enclosing an offprint of mine which might amuse you!

Sincerely yours,
Jeffrey Shallit
Jeffrey Shallit

Sequence 1677.5: A2950

1, 6, 1, 2, 1, 1, 1, 3, 25, 1, 4, 3, 3, 7, 52, 1, 2, 3, 2, 15,
2, 2, 4, 16, 2, 7, 1, 1, 1, 10, 21, 1, 1, 1, 141, 2, 4, 1, 4,
2, 1, 1, 17, 1, 3, 3, 4, 1, 3, 1.

Sequence 1411.5: A3117

1, 4, 14, 2, 1, 1, 3, 2, 29, 2, 1, 7, 1, 5, 2, 1, 1, 19, 12,
77, 2, 16, 2, 1, 1, 15, 1, 1, 3, 14, 5, 1, 3, 2, 1, 1, 1, 1, 1,
1, 5, 1, 463, 1, 379, 3, 5, 3, 11, 1.

A3118

Sequence 1031.8:

1, 3, 7, 1, 2, 2, 1, 2, 4, 56, 1, 14, 2, 1, 1, 3, 5, 6, 2, 1,
1, 2, 1, 1, 8, 1, 2, 2, 1, 5, 1, 4, 1, 1, 3, 3, 1, 1, 3, 7, 4,
1, 10, 1, 2, 1, 8, 2, 4, 1.

Sequence 19.5 A2951

1, 2, 1, 1, 1, 2, 1, 2, 8, 1, 25, 1, 5, 1, 22, 1, 8, 1, 1, 9,
1, 1, 4, 1, 2, 1, 2, 1, 2, 2, 1, 1, 1, 1, 2, 1, 6, 2, 46, 1,
12, 1, 32, 1, 2, 3, 2, 3, 55, 1.

To form the fractions in the intervals $(1,2), (2,3), (3,5), \cdots$, write the reciprocals in reverse order of the fractions in $(1/2, 1)$ in $f \cdot f_{n+1}$, of $(1/3, 1/2)$ in $f \cdot f_{n+2}, \cdots$, respectively. This gives $f \cdot f_n$ as far as we want it.

In fact, one of the purposes of investigating the symmetries of Farey Fibonacci sequences was to develop easy methods to form them.

REFERENCE

1. Krishnaswami Alladi, "A Farey Sequence of Fibonacci Numbers," *The Fibonacci Quarterly*, Vol. 13, No. 1 (Feb. 1975), pp.

****

A SIMPLE PROOF THAT PHI IS IRRATIONAL

JEFFREY SHALLIT

Student, Lower Merion High School, Ardmore, Pennsylvania 19003

Most proofs of the irrationality of phi, the golden ratio, involve the concepts of number fields and the irrationality of $\sqrt{5}$. This proof involves only very simple algebraic concepts.

Denoting the golden ratio as ϕ, we have

$$\phi^2 - \phi - 1 = 0 .$$

Assume $\phi = p/q$, where p and q are integers with no common factors except 1. For if p and q had a common factor, we could divide it out to get a new set of numbers, p' and q'.

Then

$$(p/q)^2 - p/q - 1 = 0$$

$$(p/q)^2 - p/q = 1$$

$$p^2 - pq = q^2$$

$$p(p-q) = q^2 \tag{1}$$

Equation (1) implies that p divides q^2, and therefore, p and q have a common factor. But we already know that p and q have no common factor other than 1, and p cannot equal 1 because this would imply $q = 1/\phi$, which is not an integer. Therefore, our original assumption that $\phi = p/q$ is false and ϕ is irrational.

****

A RAPID METHOD TO FORM FAREY FIBONACCI FRACTIONS

KRISHNASWAMI ALLADI
Vivekananda College, Madras 600004, India

One question that might be asked after discussing the properties of Farey Fibonacci fractions [1] is the following: Is there any rough and ready method of forming the Farey sequence of Fibonacci numbers of order F_n, given n, however large? The answer is in the affirmative, and in this note we discuss the method. To form a standard Farey sequence of arbitrary order is no easy job, for the exact distribution of numbers coprime to an arbitrary integer cannot be given. The advantage of the Farey sequence of Fibonacci numbers is that one has a regular method of forming $f \cdot f_n$ without knowledge of $f \cdot f_m$ for $m < n$. We demonstrate our method with $F_9 = 34$; that is, we form $f \cdot f_9$.

STEP 1: Write down in ascending order the points of symmetry—fractions with numerator 1. (We use Theorem 1.1 here.)

$$\frac{1}{34},\ \frac{1}{21},\ \frac{1}{13},\ \frac{1}{8},\ \frac{1}{5},\ \frac{1}{3},\ \frac{1}{2},\ \frac{1}{1}$$

STEP 2: Take and interval $(1/2, 1/1)$. Write down successively as demonstrated the alternate members of the Fibonacci sequence in increasing magnitude beginning with 2, less than or equal to F_n, for a prescribed $f \cdot f_n$. This will give a sequence of denominators

$$\frac{1}{2},\ \frac{}{5},\ \frac{}{13},\ \frac{}{34}.$$

STEP 3: Choose the maximum number of the Fibonacci sequence $\leqslant F_n$ not written in Step 2, and with this number as starting point write down successively the alternate numbers of the Fibonacci sequence in descending order of magnitude until 1.

$$\frac{}{21},\ \frac{}{8},\ \frac{}{3},\ \frac{1}{1}.$$

STEP 4: Put these two sequences together, the latter written later. (Theorem 1.2 has been used.)

$$\frac{1}{2},\ \frac{}{5},\ \frac{}{13},\ \frac{}{34},\ \frac{}{21},\ \frac{}{8},\ \frac{}{3},\ \frac{1}{1}.$$

STEP 5: Use the fact that $f_{(r+k)n}, f_{(r-k)n}$ have same denominators (Theorem 1.1) to get the sequence of denominators in all other intervals.

$$\frac{}{21}, \frac{1}{34}, \frac{1}{21}, \frac{}{34}, \frac{1}{13}, \frac{}{34}, \frac{}{21}, \frac{1}{8}, \frac{}{21}, \frac{}{34}, \frac{}{13}, \frac{1}{5}, \frac{}{13}, \frac{}{34}, \frac{}{21}, \frac{}{8}, \frac{1}{3}, \frac{}{8}, \frac{}{21}, \frac{}{34}, \frac{}{13}, \frac{}{5}, \frac{1}{2}, \frac{}{5}, \frac{}{13}, \frac{}{34}, \frac{}{21}, \frac{}{8}, \frac{}{3}, \frac{1}{1}$$

STEP 6: Use the concept of factor of an interval to form numerators. The numerators of $(1/2, 1/1)$ will differ in suffix one from the corresponding denominators. The numerators of $(1/3, 1/1)$ will differ by suffix 2 from the corresponding denominators, $\cdots$. Use the above to form numerators and hence the Farey sequence in $[0,1]$. The first fraction is $0/F_{n-1}$.

$$\frac{0}{21}, \frac{1}{34}, \frac{1}{21}, \frac{2}{34}, \frac{1}{13}, \frac{3}{34}, \frac{2}{21}, \frac{1}{8}, \frac{3}{21}, \frac{5}{34}, \frac{2}{13}, \frac{1}{5}, \frac{3}{13}, \frac{8}{34}, \frac{5}{21},$$

$$\frac{2}{8}, \frac{1}{3}, \frac{3}{8}, \frac{8}{21}, \frac{13}{34}, \frac{5}{13}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{8}{13}, \frac{21}{34}, \frac{13}{21}, \frac{5}{8}, \frac{2}{3}, \frac{1}{1}.$$

Bell Laboratories

600 Mountain Avenue
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March 4, 1975

Mr. Jeffrey Shallit
619 Greythorne Road
Wynnewood, Pennsylvania 19096

Dear Mr. Shallit:

Tremendous! Another four sequences finished. Gradually the gaps are being filled in, but it's very slow work. The next issue of the Supplement will include your sequences, for which I am most grateful.

Actually the next Supplement is long overdue, but I'm in the middle of writing a book on error-correcting codes, and the only way to write a book is to give it priority over everything else. A few chapters are finished: if you are interested I could send you a sample.

I read your note on proving that Φ is irrational. That is certainly a very short proof!

If you extend (or connect) any other sequences please let me know. If you would like a suggestion I just noticed that sequence 1021

n	1	2	3	4	5	6	...
a_n	1	1	3	6	15	27	...

A740

satisfies

$$\sum_{d \text{ divides } n} a_d = 2^{n-1} \qquad (1)$$

For example, 1, 2, 3 and 6 all divide 6, and indeed

$$a_1 + a_2 + a_3 + a_6 = 1 + 1 + 3 + 27 = 32 = 2^5$$

(This sequence actually came up two days ago in a problem someone had here.)

By what is called Mobius inversion, (1) implies

$$a_n = \sum_{d \text{ divides } n} \mu(d) 2^{n/d} , \qquad (2)$$

where $\mu(d)$ is the Mobius function - see any book on number theory.

In fact

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1 \\ (-1)^r & \text{if } n \text{ is product of } r \text{ dirrerent primes} \\ 0 & \text{otherwise} \end{cases}$$

This might be fun to program.

Best regards,

MH-1216-NJAS-rb

N. J. A. Sloane

3423

619 Greythorne Road

Wynnewood, Pa. 19096

March 14, 1975

Mr. N. J. A. Sloane

2C-363 Bell Laboratories

600 Mountain Avenue

Murray Hill, New Jersey 07974

Dear Mr. Sloane:

Thank you for your letter of March 4. After I had sent you the corrected sequences, I suddenly realized that I had not sent enough to fill two lines in the supplement. I am enclosing, therefore, the first hundred partial quotients to the fifth roots of 2, 3, 4, and 5. This should easily fill two lines in your next supplement.

I would also like to suggest that you add the following sequence to your next supplement:

6, 34, 1154, 1331714, 1773462177794, . . . A3423

This sequence is defined by $a_0 = 6$, $a_{n+1} = a_n^2 - 2$. For reference, you could use DI2 376-377 (using the terminology in your book).

I am enclosing the first eight members of this sequence on a separate page. This should easily fill two lines.

If you have any other sequences I could check, I would be happy to oblige.

volume? #

Sincerely yours,

Jeffrey Shallit

Jeffrey Shallit

2950
3117
3118
2951
2949
3423

5q PARTIAL QUOTIENTS TO THE CONTINUED FRACTION EXPANSIONS OF SOME FIFTH ROOTS

$\sqrt[5]{2}$: 2950 SSS

1	6	1	2	1	1	1	3	25	1
4	3	3	7	52	1	2	3	2	15
2	2	4	16	2	7	1	1	1	10
21	1	1	1	141	2	4	1	4	2
1	1	17	1	3	3	4	1	3	1
3	2	1	1	2	33	1	6	6	1
2	4	1	21	1	3	3	8	10	1
46	6	1	10	1	1	1	1	2	11
1	3	1	5	3	4	23	1	4	124
2	1	1	1	2	3	2	30	45	6

$\sqrt[5]{3}$: 3117 SSS

1	4	14	2	1	1	3	2	29	2
1	7	1	5	2	1	1	19	12	77
2	16	2	1	1	15	1	1	3	14
5	1	3	2	1	1	1	1	1	1
5	1	463	1	379	3	5	3	11	1
7	7	1	1	2	1	1	1	2	1
1	1	2	1	46	17	44	1	1	1
2	24	9	1	7	4	1	2	2	1
3	2	7	1	7	1	1	2	1	1
4	1	46	8	2	1	28	1	3	6

$\sqrt[5]{4}$: 3118 SSC

1	3	7	1	2	2	1	2	4	56
1	14	2	1	1	3	5	6	2	1
1	2	1	1	8	1	2	2	1	5
1	4	1	1	3	3	1	1	3	7
4	1	10	1	2	1	8	2	4	1
1	9	2	2	2	1	2	1	1	1
92	1	26	4	31	1	2	4	1	62
8	5	1	1	1	2	1	1	63	1
2	5	4	2	1	2	1	1	4	8
1	19	1	1	2	11	49	1	1	1

$\sqrt[5]{5}$: 2951 SSC

1	2	1	1	1	2	1	2	8	1
25	1	5	1	22	1	8	1	1	9
1	1	4	1	2	1	2	1	2	2
1	1	1	1	2	1	6	2	46	1
12	1	32	1	2	3	2	3	55	1
12	3	8	1	1	11	1	4	1	1
1	2	1	1	7	1	1	4	3	3
3218	1	3	1	2	2	3	1	1	2
11	1	7	57	2	2	2	2	1	1
67	1	2	3	1	1	13	3	3	1