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ON COVERING SPACES OF KNOTS

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To each representation ϱ of the group of a knot k onto a transitive permutation group P there corresponds a branched covering space M of S^3, with branch link $\bar{k}$ over k. Consider the set of all such spaces for a given k and a given P. Then any topological invariant of each element (or of the placement of $\bar{k}$ within it) is, taken as a set over all possible ϱ, a placement invariant of k in S^3. Among such invariants are the linking numbers, where defined, between identifiable components of $\bar{k}$ and the fundamental groups of their complements in M [5]. Algorithms for computing the latter (algebraically) have been set forth by R. H. Fox in varying degrees of generality [2; 3, § 8; 5, § 4]. However, no general algorithm appears to have been published for computing these linking numbers, examples of which have appeared in the literature from time to time [1, § 8; 2, § 5; 4; 5, § 5; 6, Ch. III, § 15]. It is the purpose of this paper to fill this gap. As background, we discuss the known algorithms for covering groups.

Our approach is intuitive in the strict sense of the word. Avoiding the apparent (and possibly inherent) algebraic intractability of noncyclic coverings, we simply take a careful look at the geometric model.

1. The fundamental groups

There is an obvious, Wirtinger-like algorithm for calculating the fundamental group of any covering space (cyclic or noncyclic, with or without particular components of $\bar{k}$) of a tame knot (or link) arising out of a representation onto any transitive permutation group. Simply write the permutation corresponding to each Wirtinger meridian to one side (say the right) of each oriented segment of a knot diagram (in regular projection) and imagine n copies of S^3 woven together along the 2-cells lying beneath the knot in Reidemeister's halfcylinder [6, pp. 53 and 55], where n is the number of sheets in the covering. Since $M - \bar{k}$ is connected, $n - 1$ of these 2-cells must be removed to create a maximal cave and they may always be removed from beneath a minimal set of generating

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segments of the knot diagram. Thus if g is the bridge number of k (Cf. [7, p. 606]), there will be at most $ng - n + 1$ generators for $\pi(M - \bar{k})$. Adjoining to $M - \bar{k}$ any branch curve (component of $\bar{k}$) will further reduce the number of generating 2-cells by adjoining relations corresponding to the one or more* pre-images in M of the circle shown in Figure 1 which surround that particular branch curve. Adjoining them all, to get M, will clearly eliminate a total of cg generators, where c is the number of cycles in a meridian permutation. Thus we obtain (geometrically) the well known formula $g' = ng - cg - n + 1$ for the maximum number of non-trivial generators of $\pi(M)$ [2; 4]. Note that in choosing generators for this Wirtinger-like presentation one need only look at 2-cells which lie beneath generating segments of k, since the remaining 2-cells may be defined in terms of these in essentially the same manner as one works out the Wirtinger algorithm on a knot diagram (Cf. [3, p. 129]). Specifically, n relations are obtained at each crossing by considering, as Wirtinger-like relations, the n pre-images in M of the circle shown in Figure 2. One only has to pay attention to which 2-cells are connected to which along the n vertical lines below the crossing. This may easily be done on a knot diagram (with meridian permutations written in, as above) by associating with each letter of each permutation the 2-cells in M which can be »seen« on the right hand side of that segment from that particular copy of S^3.

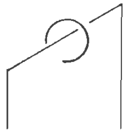

Figure 1

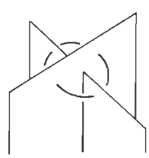

Figure 2

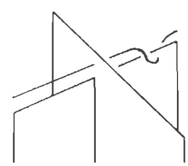

Figure 3

2. Linking numbers between branch curves

There is a similar geometric algorithm for constructing (where possible) a 2-chain in M which bounds (i. e., has as its only boundary) any component k_i of $\bar{k}$, or a multiple thereof. One merely assigns variable coefficients to the 2-cells in the cellular decomposition of M induced by Reidemeister's half-cylindrical cellular decomposition of S^3 and obtains from each pre-image in M of the circles shown in Figures 1 and 2 a linear equation relating these

* Cf. the footnote at [5, p. 200]. Actually, this possibility was suggested by Fox. For an example, see Figure 4.

variables. These equations are obtained in the obvious way to reflect the fact that along the vertical 1-cells in M (Figure 2) the coefficients assigned to the 2-cells which meet thereat must cancel each other out, while at the horozontal 1-cells (Figure 1) their boundaries (summed together) must result in only a unit contribution to the component of $\bar{k}$ which one is attempting to bound. One then attempts to solve these equations. If they have a solution in integers, the resulting coefficients, assigned to their respective 2-cells, form a 2-chain with boundary k_i. If they have a solution in rational numbers, simply multiply the solution vector by an appropriate integer d to obtain a 2-chain with boundary d times k_i. If they have no solution, no multiple of k_i bounds in M and its linking number with any other branch curve is undefined.

In practice certain variables (those assigned to any set of 2-cells which may be removed to create a maximal cave in M) may be set equal to zero at the outset, since if k_i bounds in M it bounds on the walls of a maximal cave. The variables assigned to 2-cells lying beneath generating segments may then be expressed in terms of each other for each cycle of the corresponding permutations, and the remaining variables worked out through each crossing, so as to reduce drastically the number of equations and unknowns. This parallels the algorithm for $\pi(M - k_i)$, viewing each generator as a variable and obtaining linear equations from the exponents of a relation. Where $g' = 0$, this will circumvent the need for any equations or unknowns whatsoever. One may simply plug in unit coefficients for the 2-cells bounding portions of k_i along generating segments and they will work themselves out to bound it all.

Once these coefficients are obtained, it is a simple matter to compute the linking number between the branch curve which they bound and any other. Simply follow any path in M lying ε-close to that other branch curve (Figure 3), noting the copies of S^3 through which it passes and the 2-cells in M which it intersects, and sum the resulting intersection numbers (the coefficient times the sign of the intersection) until one returns to the point (and copy number) where one started, dividing the total by d. Note that it may be necessary to go around the knot more than once (as for the *one* curve of branching index 1 in Figure 4).

3. Visualization of covering linkage

Linking numbers in M may sometimes be easily read from a knot diagram. In Figure 5, a self-intersecting disc is exhibited which bounds k and lifts to the copy numbers of S^3 indicated in Roman numerals to bound the component of $\bar{k}$ of branching index 2. The linking number between branch curves in this 3-sheeted noncyclic covering of the trefoil is (obviously) the sum of the intersections of the projected disc with its boundary.

This method works for a great many 3-, 4-, and 5-sheeted coverings of tabled knots, where meridians are sent into transpositions, transpositions and 3-cycles (i. e., reps onto S_3 [5, p. 200], S_4 and A_5 [7, p. 613]). For example, one may easily distinguish the square knot, the granny and R. Riley's »favourite knot« [8, p. 239] by their linking numbers with respect to S_3 — (2, 0, 0, —2), (2, 2, 4, 4) and (0, 0, 2, 4), respectively.

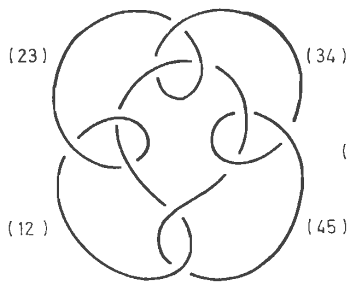

Figure 4

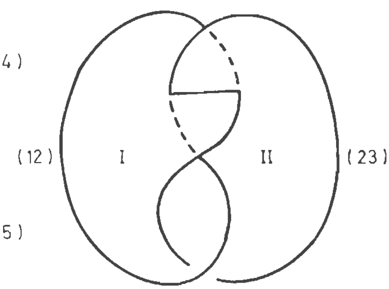

Figure 5

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O PROSTORIMA NATKRIVANJA UZLOVA

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Sadržaj

Neka je k uzao, tada svakoj reprezentaciji ϱ grupe uzla na tranzitivnu grupu permutacija P odgovara prostor natkrivanja s grananjem M nad S^3, sa spletom grananja $\bar{k}$ nad k. Promatrajmo skup svih ovakvih prostora za dani k i danu grupu P, tj. skup svih M koje dobivamo variranjem ϱ. Proizvoljna topološka invarijanta svakog elementa tog skupa je invarijanta smještenja od k u S^3. Među takve invarijante spadaju i spletni brojevi (linking numbers) između komponenata od $\bar{k}$ (koje se mogu poistovjetiti) i fundamentalnih grupa njihovih komplemenata u M (za preciznu definiciju spomenutih spletnih brojeva v. ref. [5]). U radu je dan opći algoritam za izračunavanje tih spletnih brojeva.