## ON SOME MATHEMATICAL PROBLEMS CONNECTED WITH PATTERNS OF GROWTH OF FIGURES

BY

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1. Introduction. This note will contain a brief discussion of certain properties of figures in two or three dimensional space which are obtained by rather simple recursion relations. Starting from an initial configuration, one defines in successive "generations" additions to the existing figure, representing, as it were, a growth of the initial pattern, in discrete units of time. The basic thing will be a fixed division of the plane (or space) into regular elementary figures. For example, the plane may be divided into squares or else into equilateral triangles (the space into cubes, etc.). An initial configuration will be a finite number of elements of such a subdivision and our induction rule will define successive accretions to the starting configuration.

The simplest patterns observed, for example in crystals, are periodic and the properties of such have been studied mathematically very extensively. The rules which we shall employ will lead to much more complicated and in general non-periodic structures, whose properties are more difficult to establish, despite the relative simplicity of our recursion relations. The objects defined in that way seem to be, so to say, intermediate in complexity between inorganic patterns like those of crystals and the more varied intricacies of organic molecules and structures. In fact, one of the aims of the present note is to show, by admittedly somewhat artificial examples, an enormous variety of objects which may be obtained by means of rather simple inductive definitions and to throw a sidelight on the question of how much "information" is necessary to describe the seemingly enormously elaborate structures of living objects.

Much of the work described below was performed in collaboration with Dr. J. Holladay<sup>1</sup> and Robert Schrandt.<sup>2</sup> We have used electronic computing machines at the Los Alamos Scientific Laboratory to produce a great number of such patterns and to survey certain properties of their morphology, both in time and space. Most of the results are empirical in nature, and so far there are very few general properties which can be obtained theoretically.

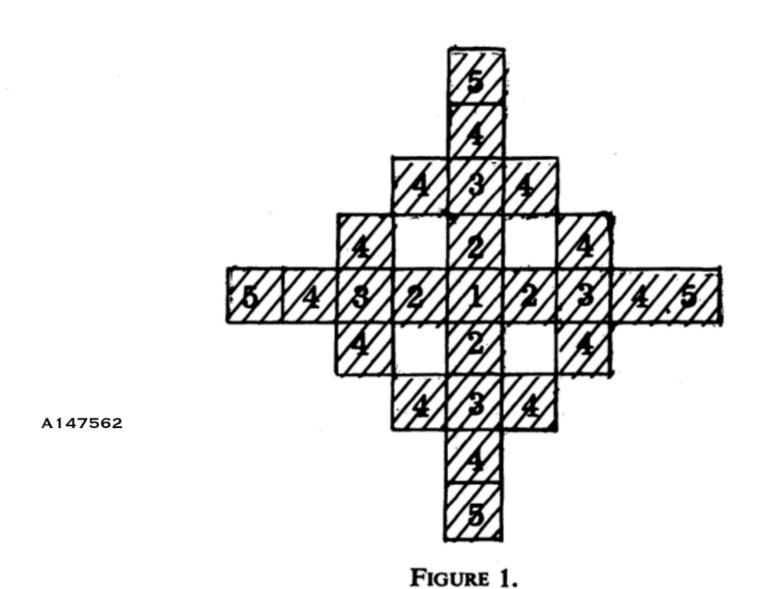
2. In the simplest case we have the subdivision of the infinite plane into squares. We start, in the first generation, with a finite number of squares and define now a rule of growth as follows: Given a number of squares in the nth generation, the squares of the (n + 1)th generation will be all those which are adjacent to the

<sup>&</sup>lt;sup>1</sup> Holladay, J. C. and Ulam, S. M., Notices Amer. Math. Soc. 7 (1960), 234.

<sup>&</sup>lt;sup>2</sup> Schrandt, R. G. and Ulam, S. M., Notices Amer. Math. Soc. 7 (1960), 642.

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existing ones but with the following proviso: the squares which are adjacent to more than one square of the nth generation will not be taken. For example, starting with one square in the first generation one obtains the following configuration after five generations.



It is obvious that with this rule of growth the figure will continue increasing indefinitely. It will have the original symmetry of the initial configuration (1 square) and on the four perpendicular axes all the squares will be present—these are the "stems," from which side branches of variable lengths will grow.

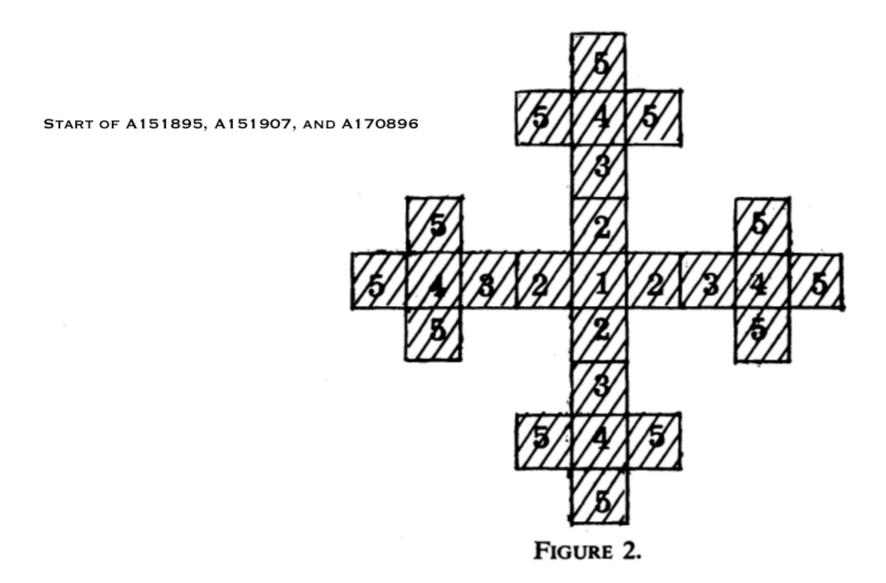
We can consider right away a slightly modified rule of growth. Starting again with a single square and defining the (n + 1)th generation as before to be squares adjacent to the squares of the *n*th generation, we modify our exclusion proviso as follows: we will not put into existence any square for the (n + 1)th generation if another prospective candidate for it would as much as touch at one point the square under consideration. With this second rule we obtain after five generations the figure shown on the next page. With this rule we will again notice immediately that the "stem" will continue indefinitely but now the density of the growing squares will be less than in the previous case. In this case again one can calculate which squares will appear in the plane and which will remain vacant.

A general property of systems growing under the rules (and even somewhat more general ones) is given by a theorem due to J. Holladay. At generations whose index number n is of the form  $n = 2^k$ , the growth is cut off everywhere except on the "stems," i.e., the straight lines issuing from the original point.

The old side-branches will terminate and the only new ones will start growing from the continuation of the stems.

One of the most interesting situations arises when the plane is divided into equilateral triangles and starting from one initial triangle we construct new ones,

generation by generation. We can again have the analogue of the first rule, i.e., for the (n + 1)th generation we consider all triangles adjacent to a triangle of the nth generation. As before we shall not construct those which have two different parents in the nth generation. The system which will grow will have the six-fold



symmetry of the original figure. There will appear a rather dense collection of triangles in the plane. The second way is to take the analogue of the second rule of "conflict," i.e., do not construct a triangle in the (n + 1)th generation if it would so much as touch at one point another prospective child of some other element in the nth generation. (We of course allow two prospective children to touch on their base from two adjacent parents.) This rule will lead to a pattern which has fewer elements and a smaller density in the plane than the one constructed under the first conflict definition.

One can prove easily that the initial hexagonal symmetry will persist and that the growth will continue indefinitely with the "stems" increasing in each generation by one element, i.e., forming continuous lines. The side branches have variable lengths and get "choked off" at variable times (generations). The author did not manage to prove that there will exist infinitely long side branches. It is possible to demonstrate that there will be arbitrarily long ones. The figure<sup>3</sup> shows a segment of the growing pattern. It represents one-half of a sixty degree section. The other half is obtained by a mirror image. The other sections are obtained by rotation.

For the construction with triangles under the first rule Holladay's cut-off property holds for generations with index of the form  $2^k$ . Under the second rule it was not even possible to prove the value or indeed the existence of a limiting

<sup>&</sup>lt;sup>3</sup> See Example 5 at the end.

density of the triangles obtained by the construction (relative to all the triangles in

the whole plane).

In the division of the plane into regular hexagons and starting with, say, again one element, one can obtain the analogues of the two patterns. Again the analogue of the more liberal construction has the cut-off property. For the more stringent rule it was, so far, impossible to predict the asymptotic properties.

3. The construction of the elements of the (n + 1)th generation is through a single parentage: each element attempts to generate one new one in the next generation. In the division of the plane into squares, triangles, hexagons, etc., one could adopt a different point of view: in the case of, say triangles, one can consider instead of the areas of the triangles, their vertices only and imagine that each pair of vertices produces a new vertex—namely, the one forming the triangle with the two given vertices as their sides. Actually, the origin of the above-

mentioned constructions is due to this point of view:

In a paper, "Quadratic Transformations" (Los Alamos Laboratory Report LA-2305, 1959), P. R. Stein and the writer have considered problems of "binary" reaction systems. Mathematically, these involve the following situation: a great number of elements is given, each element being one of, say, three types. These elements combine in pairs and produce, in the next generation another pair of elements whose types are unique functions of the types of the two parents. The problem is to determine the properties of the composition of the population, as time goes on. If x, y, z denote the proportions of elements of the three types in the nth generation, then the expected value of the numbers of particles of each type in the new generation will be given by a quadratic transformation. For example, the rule could be that an x type and a y type particle together produce an x type, the (x + x) a z type, (x + z) a y type, (y + y) an x type, (y + z)a z type and (z + z) a y type. (Actually there are more than ninety possible and different such rules—we assume, however, that once a rule is chosen it is valid for all time.) The rule above would lead to the new proportions x', y', z' given, as follows:  $x' = 2xy + y^2,$ 

 $x' = 2xy + y^2,$   $y' = z^2 + 2xz,$  $z' = y^2 + 2yz.$ 

This is a transformation of a part of the plane into itself. We have three variables, but x + y + z = 1 = x' + y' + z'. By iterating this transformation one obtains the expected values of the numbers of elements of each type in the subsequent generations. In the above-mentioned study some properties of the iterates of the transformation were established. In particular, in some cases there may be convergence to a stable distribution, in other cases there is a convergence to an oscillating behavior, etc.

These studies concerned a random mating (or collisions) between pairs of elements. The question arose as to the behavior of such systems if the binary production were not a random one but instead subject to some constraints, say

due to geometry. A most stringent one seemed to be to imagine, for example, that the elements form the vertices of a division of a plane into regular triangles, each vertex being of one of the three possible "colors." Then consider an initial configuration as given and assume the production of new elements by pairs of vertices forming sides of the triangular division. In the simplest case one can start with one triangle whose three vertices are all different in type. The next generation will be formed then by the three pairs as parents and each side of the given triangle will produce a new vertex whose color is a function of the two colors of the parents. We shall obtain then a second generation and continue in this fashion. It is immediately found, however, that the construction cannot be uniquely continued. After a small number of generations it will appear that two pairs of vertices forming two sides of the configuration will have a single vertex as completing the two triangles to be constructed. Which color to assign to the new vertex? It may be that the two sets of parents will give a conflicting recipe for the color of the new point.

One way out of this dilemma would be not to consider a point for which a conflicting determination of color may be given and leave its position vacant. This recipe extended to points which are doubly determined by two sides of previously constructed triangles gave rise to the study mentioned in the previous paragraphs. Actually the patterns mentioned above could be considered as consisting of points which are of three different kinds (imagining, for example, that the new ones arise in a "molecule" as a result of a double bond, etc.). As it is R. Schrandt and the writer have considered also other recipes for determining the color of points which were given conflicting determinations by the two pairs of parents. One rule (1) was to choose the type not involved in the conflicting determination: since there are three types, if the two determinations for the new points differ, one may choose the third one. Another rule (2) was also considered: to decide, at random with equal probability, which of the two contrasting determination should be chosen. Still another rule (3) was to choose, in case of such a conflict, a fourth color whose proportion will be denoted by w and such that an x type + w type produces x; y + w produces y and z + w produces z and w + wproduces w in subsequent combinations. This could have an interpretation of representing a molecule of a type which cannot propagate except in combination with itself. We have studied experimentally, on a computing machine, the propagation of such systems. The Rule No. 2 in particular involves sometimes a random determination of points somewhat similar to the study in LA-2305 mentioned above on random mating. Under all these rules, there seems to be a convergence of the number of particles of different types to a steady distribution (in contrast to the behavior given by iteration of the quadratic transformations where in many cases there is an oscillatory limit or even more irregular ergodic asymptotic behavior). In some cases the convergence seems to take place to a fixed point (i.e., a definite value of x, y, z), and under Rule No. 2 to values, numerically not too different from the fixed point of the corresponding quadratic transformation. It has not been possible to prove the existence of a limiting distribution

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but the numerical work strongly indicates it. It should be noted that all the initial configurations were of the simplest possible type, e.g., consisted of one triplet of points. A detailed description of this work will appear in a report by R. Schrandt.

4. We return now to our discussion of growing patterns where we do not label the new elements by different colors but merely consider, as in paragraph 2, the geometry of the growing figure. The problem arose of considering the properties of growth of such figures with a rule of erasure or "death" of old elements: suppose we fix an integer k arbitrarily and to our recursive definition of construction of new elements add the rule that we erase from the pattern all elements which are k generations old. In particular, suppose k = 3 and consider the growth from squares, as in the first rule in paragraph 1, with the additional proviso that after constructing the (n + 1)st generation, we shall erase all points of the (n - 1)st generation. (The construction allows the configuration to grow back into points of a previous generation of index l where l is less than n-1.) In this construction, starting say with two squares to begin with, one will observe a growth of patterns, then a splitting (due to erasures) and then later recombinations of the pattern. A search was undertaken for initial patterns which in future generations split into figures similar or identical with previous ones, i.e., a reproduction at least for certain values of the index of generation. It was not possible, in general, even in the cases where a growth pattern without erasure could be predicted, to describe the appearance of the apparently moving figures which in general exhibit a very chaotic behavior. In one starting configuration, however, one could predict the future behavior. This configuration consists of two squares touching each other at one point and located diagonally. Under our Rule No. 1 with erasure of the third oldest generation, this pattern is reproduced as four copies of itself in every  $2^p$ th generation (p = 1, 2, $3 \cdot \cdot \cdot$ ), displaced by  $2^p$  units from the original pattern. The same behavior holds for starting patterns of say four squares located diagonally, or 8 points or 16 points, etc.

In case of a triangular subdivision the behavior of growth with a rule of erasure for old elements was also experimentally investigated. The process of growth was considered as follows: given a finite collection of vertices of the triangular subdivision of the plane—some labeled with the index n-1 and others with n—one constructs the points of the (n+1)th generation by adding vertices of the triangles whose sides are labeled either with n-1 and n or n and n—again, however, not putting in points which are doubly determined. One then erases all points with the index n-1. In case of squares our rules of growth enable the pattern to exist indefinitely, starting with any non-trivial initial condition. This is not always the case for triangles. In particular a starting pattern of two vertices with the same generation terminates after ten generations—that is to say, all possible points of growth are conflicting ones and these are not allowed by our rule of construction. One has to point out here that in the case of the "death" rule which operates by erasure of all elements that are k generations old the initial configuration has to specify which elements are of the 1st and which of the 2nd

generation. Two vertices, one labeled 1st and the other 2nd generation, will give rise to a viable pattern.

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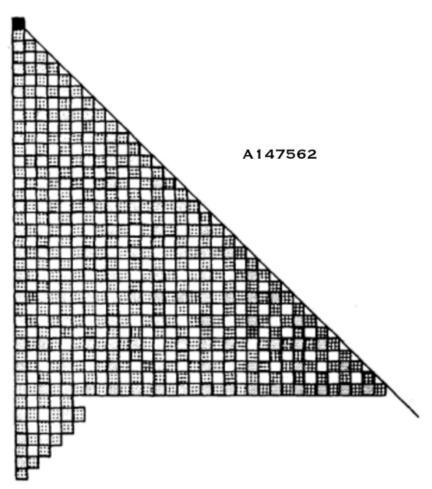
5. In three-dimensional space a similar experimental study was made of growth of patterns on a cubical lattice. The rules of growth can be considered in a similar way to the recipes used in two dimensions. Starting with one cube one may construct new ones which are adjacent to it (have a face in common). Again one will not put in new cubes if they have a face in common with more than one cube of the previous generation. The analogue of the first rule gives a system whose density in space tends to 0. This is in contrast to the situation in the plane where a finite density was obtained for this case.

R. Schrandt has investigated on a computer the growth of system with a rule for erasure of old elements. The case of erasure of elements three generations old was followed. The patterns which appear seem to be characterized by bunches of cubes forming flat groups. These groups are connected by thin threads. Description of these patterns and a few general statements one can make about them will be also contained in Schrandt's report.

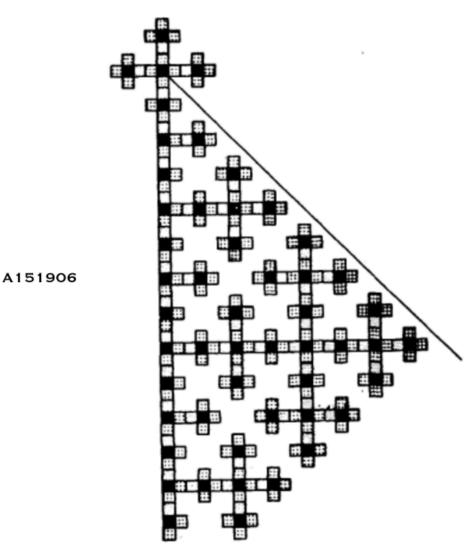
These heuristic studies, already in two dimensions, show that the variety of patterns is too great to allow simple characterizations. The writer has attempted to make corresponding definitions in one dimensions with the hope that some general properties of sequences defined by analogous recursive rules would be gleaned from them. Suppose we define a sequence of integers as follows: starting with the integers 1, 2 we construct new ones in sequence by considering sums of two previously defined integers but not including in our collection those integers which can be obtained as a sum of previous ones in more than one way. We never add an integer to itself. The sequence which starts with 1 and 2 will continue as follows: 1, 2, 3, 4, 6, 8, 11, 13, 16, 18, 26, 28, · · ·. The integer 5 is not in it because it is a sum of two previous ones in two different ways. The next integer which is expressed in one and only one way of the sum of previous ones is 6; 7 has a double representation but 8 is uniquely determined. 11 is the next and so on. Starting with 1 and 3 one obtains the following sequence: 1, 3, 4, 5, 6, 8, 10, 12, 17, 21, · · · . Unfortunately, it appears to the writer that even here it is not easy to establish properties of these "unique sum sequences." For example, the question of whether there will be infinitely many twins, i.e., integers in succession differing by two, seems difficult to answer. Even a good estimate of density of these sequences relative to the set of all integers is not easily made.

The aim in presenting these disconnected empirical studies was to point out problems attending the combinatorics of systems which, in an extremely simplified and schematic way, show a growth of figures subject to simple geometrical constraints. It seems obvious that, before one can obtain some general properties in "auxology," a great deal of experimental data have to be surveyed, It was possible to study the effect of many variations in our rules on the computing machines. A scope attached to the machine allows one to survey the resulting patterns visually—their computation takes only a very short time. This work is continuing and perhaps some more general properties of their morphology will be demonstrable.

APPENDIX: EXAMPLES



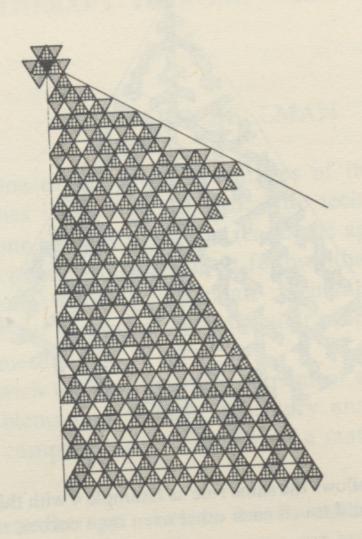
EXAMPLE 1. Starting with the black square as the first generation, each successive generation consists of those squares that are adjacent to one and only one square of previous generations. In most of these illustrations, only cells in certain directions were drawn. Growth in other directions is the same because of symmetry conditions.



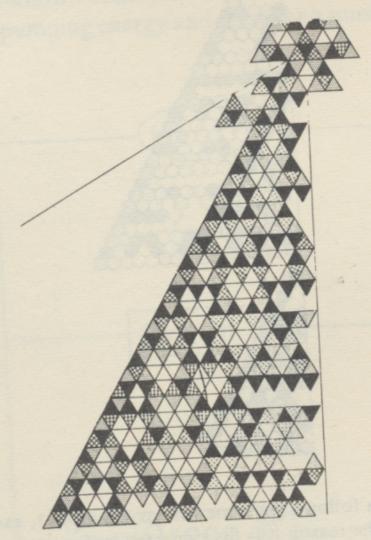
EXAMPLE 2. "Maltese" crosses. The black cells in this pattern are arranged according to the pattern for Example 1, except that they are more spread out. Here we use the same rule as in Example 1 with the following exception: if a cell would touch some other cell (either already grown or being considered for growth in this generation) on either a corner or a side, it is rejected. However, we made two exceptions to this restriction: (1) if the cell touches some other cell by virtue of having the same parent.

(2) In the following case, 1 2 3 4 5. The two starred elements of the fifth generation 2\* 5\*

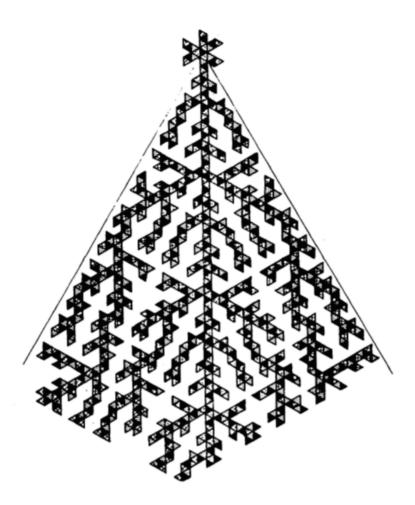
are allowed to touch potential, though previously rejected, children of the third generation. This has to be allowed to enable the growth to turn corners. Note that the children of the third generation were rejected only because of the potential children of the starred members of the second generation.



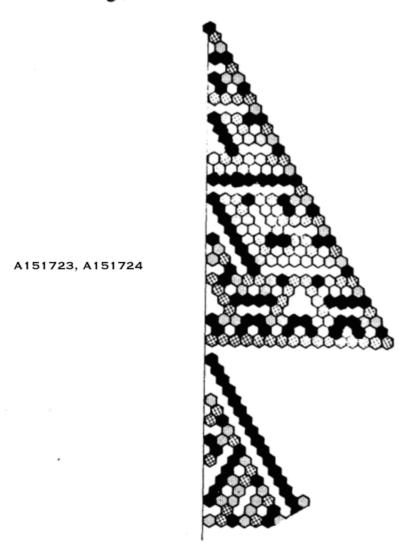
EXAMPLE 3. This pattern follows the same rule as Example 1, except that triangles are used instead of squares.



EXAMPLE 4. This pattern follows the same rule as Example 3 with one exception: if a new cell would touch the corner of some old cell (other than a parent), it is rejected.



EXAMPLE 5. This pattern follows the same rule as Example 4 with this exception: if two new cells (other than siblings) would touch each other even on a corner, they are both rejected.



EXAMPLE 6. This pattern follows the same rule as Example 1, except that hexagons are used instead of squares. The reason it is disconnected is that a triangle of cells is left out. This triangle is the same as that formed in the first few generations for what is drawn plus a mirror image of it.