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PRECOMPLETE CLASSES IN k-VALUED LOGICS

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In this note we shall consider problems connected with the problem of completeness in k-valued logics.*

1. As is well known [1], one of the ways of solving the completeness problem is the determination of all so-called precomplete classes of a given finite-valued logic P_k. In 1921 there appeared a report of the results of E. Post, which was published in detail only in 1941 [2]. Post constructed, in particular, all precomplete classes for P_2. In 1954, all precomplete classes in P_3 were described in a paper by S. V. Jablonskiĭ [3]. At approximately the same time, A. V. Kuznecov pointed out the theoretical possibility of constructing all precomplete classes in P_k and established that the number $\pi(k)$ of these classes is not greater than 2^{k^k} [1]. In this connection, the problem arose of giving an explicit description of all precomplete classes in P_k. Various sets of precomplete classes were constructed in 1953 by Kuznecov and Jablonskiĭ and were described in detail in [1]. Other sets of precomplete classes were constructed by V. V. Martynjuk [4] in 1959, by Lo Čžu-kaĭ, Pan Jun-cze, Van Sjan-hao and Lju Sjuĭ-hua [5-10] in 1962–1964, and by E. Ju. Zaharova [11] in 1965. In 1965 there appeared a note by J. Rosenberg [12] in which it was reported that all precomplete classes in P_k consisted of six families, of which four were already completely known and two were partially known (the proof of Rosenberg's theorem has not yet been published).

At the same time, there arose the problem of a bound on the number of precomplete classes.** In 1952 Kuznecov announced the result $\pi(k) \geq 2^{2^{k(1-\epsilon)}}$, where $\epsilon \to 0$ as $k \to \infty$, which was reported in [13]. However, it subsequently became clear that Kuznecov did not have a proof of this fact.

The present note is devoted to bounds on the number of precomplete classes in P_k. The proofs given for the propositions below are based upon Rosenberg's Theorem, which the authors have been able to prove.

2. We introduce the following notation for families of precomplete classes described in [12]: M is the family of classes of monotonic functions, S of self-dual functions, L of quasilinear functions preserving a partition of the set $E_k = \{0, 1, \cdots, k-1\}$, C of functions preserving central predicates, and B the family of classes of functions which are homomorphic inverse images of classes of functions preserving elementary predicates. Let us recall the definition of an h-place central predicate which plays an important role in our reasoning. Let us consider a completely symmetric predicate $\rho(y_1, \cdots, y_h)$ defined on E_k^h and possessing the property of reflexivity, i.e. ρ becomes 1 on all sequences all terms of which are the same. A nonempty set $\mathfrak{C} \subset E_k$ is called the center of ρ if, for all $c \in \mathfrak{C}$ and arbitrary $\alpha_2, \cdots, \alpha_h \in E_k$, $\rho(c, \alpha_2, \cdots, \alpha_h) = 1$. A predicate possessing a center is called central.

* All concepts not defined here can be found in [1].

** The problem was posed by Jablonskiĭ in lectures on many-valued logics at Moscow University in 1954.

Theorem 1. $\pi(k) \sim \delta(k) k 2^{C_{k-1}^{[(k-1)/2]}}$, *where* $\delta(k) = 1$ *if* k *is odd and* $\delta(k) = 2$ *if* k *is even.*

Proof. One can show that different precomplete classes are classes preserving different predicates; hence, the problem reduces to a bound on the number of predicates for the families $M - B$. A rough calculation gives an upper bound $l(k)$ – the number of predicates defining the families $M, S, L, U, B,$ – equal to 2^{k^2}. For $c(k)$, the number of central predicates, one establishes an asymptotic bound. Comparison of the asymptotic bound $c(k)$ and the upper bound $l(k)$ shows that $\pi(k) \sim c(k)$.

Upper bound. Every central h-plane predicate is completely determined by the subset of those combinations of k elements taken h at a time which occur in the truth set of the predicate. For example, a predicate, the center of which contains an element α, must contain C_{k-1}^{h-1} sequences; the

Table 1*

k	M	S	L	U	C	B	Total Number
2	1 1	1 1	1 1	0 0	2 1	0 0	5 4
3	3 1	1 1	1 1	3 1	9 3	1 1	18 8
4	18 2	1 1	1 1	13 3	40 7	7 2	~~80~~ 16
5	190 4	24 1	12 1	50 5	355 19	36 4	~~667~~ 34
6	3285 ~~3375~~ 12	11 2	0 0	190 9	11 490 77	171 7	15 ~~337~~ 107
7	88851 ~~93 684~~ 39	720 1	360 1	917 13	7 758 ~~233~~ 205	813 11	7 854 ~~784~~ >1500

* The upper figure in a square is the number of classes; the lower figure is the number of types.

other sequences can vary. The number of these other combinations is equal to C_{k-1}^{h}. From this we find the number of h-place predicates with center α, and taking into account the bounds on the variation of h and α, we obtain

$$c(k) \leqslant k \sum_{h=1}^{k-1} 2^{C_{k-1}^h} \leqslant \delta(k) k 2^{C_{k-1}^{[(k-1)/2]}}.$$

Lower bound. As we have noted, the number of h-place predicates, the center of which contains an element α and possibly other elements, is equal to $2^{C_{k-1}^{h}}$. The number of h-place predicates, the center of which contains α, β and possibly other elements, is equal to $2^{C_{k-2}^{h}}$. From this, the number of h-place predicates, the center of which contains only α, is not less than

$$\sum_{h=1}^{k-2} (2^{C_{k-1}^h} - (k-1)\, 2^{C_{k-2}^h}) + 1.$$

Taking into account the fact that α can assume k values, we obtain the result that the number of all predicates, the center of which consists of one element, is not less than

$$k\left[\sum_{h=1}^{k-2}\left(2^{C^h_{k-1}}-(k-1)2^{C^h_{k-2}}\right)+1\right]$$

$$\geqslant k\left[\delta(k)2^{C_{k-1}^{[(k-1)/2]}}-(k-1)(k-2)2\cdot 2^{C_{k-2}^{[(k-2)/2]}}+1\right]\geqslant \delta(k)k\cdot 2^{C_{k-1}^{[(k-1)/2]}}$$

Thus, "almost all" precomplete classes are determined by central predicates.

We denote by $\tau(k)$ the maximal number of pairwise nonisomorphic [1] precomplete classes in P_k (i.e. the number of types of precomplete classes).

Theorem 2. $\tau(k) \sim \pi(k)/k!$.

The proof of this assertion is a further development of the idea of the proof of Theorem 1.

3. Let us evaluate the effectiveness of tests of completeness formulated in terms of precomplete classes, for small values of k. A detailed study of families of precomplete classes permits us to construct Table 1. We remark that all precomplete classes for $k = 4$ were known before the appearance of Rosenberg's paper.* From Table 1 it is evident that a test for completeness in terms of precomplete classes is practicable for $k \leq 4$ and impractical for $k \geq 5$, and, in terms of precomplete classes, respectively, for $k \leq 6$ and $k \geq 7$. We note that Theorems 1 and 2 for $k = 8$ still permit us to compute $\pi(k)$ and $\tau(k)$ with a relative error of 10^{-5}.**

In conclusion we give lists of predicates characterizing the types of precomplete classes for $k = 4, 5$. For writing the predicates, we use the logical operations $\vee$, &, $-$, the predicates $\leq$, $=$, $g_i(y) = \overline{\operatorname{sgn}|i-y|}$, and the function $x + y \pmod k$. (Sometimes we shall omit the symbol &.) For each type of precomplete classes we write a predicate determining one class of the type.

$k = 4$. Family M. $\rho_1(y_1y_2) = (y_1 \leq y_2)$, $\rho_2(y_1y_2) = (y_1 \leq y_2)\,\&\,(\bar g_1(y_1) \vee \bar g_2(y_2))$,

Family S.

$\rho_3(y_1y_2) = g_0(y_1)g_1(y_2) \vee g_0(y_2)g_1(y_1) \vee g_2(y_1)g_3(y_2) \vee g_2(y_2)g(y_1)$.

Family L. $\rho_4(y_1y_2y_3y_4) = (y_1 = y_2)(y_3 = y_4)\,\&\,(y_1 = y_3)$

$$\vee\ (y_2 = y_4)(y_1 = y_4)(y_2 = p_3)\ \vee \mathop{\&}_{i<j} \overline{y_i = y_j}.$$

Family U $\rho_5(y_1y_2) = g_0(y_1)g_0(y_2) \vee [g_1(y_1) \vee g_2(y_1) \vee g_3(y_1)][g_1(y_2) \vee g_2(y_2) \vee g_3(y_2)]$,

$\rho_6(y_1y_2) = [g_0(y_1) \vee g_1(y_1)][g_0(y_2) \vee g_1(y_2)] \vee [g_2(y_1) \vee g_3(y_1)][g_2(y_2) \vee g_3(y_2)]$,

$\rho_7(y_1y_2) = g_0(y_1)g_0(y_2) \vee g_1(y_1)g_1(y_2) \vee [g_2(y_1) \vee g_3(y_1)][g_2(y_2) \vee g_3(y_2)]$.

Family C. $\rho_8(y_1) = g_0(y_1)$, $\rho_9(y_1) = g_0(y_1) \vee g_1(y_1)$,
$\rho_{10}(y_1) = g_0(y_1) \vee g_1(y_1) \vee g_2(y_1)$,
$\rho_{11}(y_1y_2) = (y_1 = y_2) \vee g_0(y_1) \vee g_0(y_2)$,
$\rho_{12}(y_1y_2) = (y_1 = y_2) \vee g_0(y_1) \vee g_0(y_2) \vee g_1(y_1)g_2(y_2) \vee g_1(y_2)g_2(y_1)$,
$\rho_{13}(y_1y_2) = (y_1 = y_2) \vee g_0(y_1) \vee g_0(y_2) \vee g_1(y_1) \vee g_1(y_2)$,
$\rho_{14}(y_1y_2y_3) = (y_1 = y_2) \vee (y_1 = y_3) \vee (y_2 = y_3) \vee g_0(y_1) \vee g_0(y_2) \vee g_0(y_3)$.
Family B. $\rho_{15}(y_1y_2y_3) = (y_1 = y_2) \vee (y_1 = y_3) \vee (y_2 = y_3)$

$$\vee\ [g_2(y_1) \vee g_2(y_2) \vee g_2(y_3)][g_3(y_1) \vee g_3(y_2) \vee g_3(y_3)],$$

$$\rho_{16}(y_1y_2y_3y_4) = \bigvee_{1\leqslant i<j\leqslant 4}(y_i = y_j).$$

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* In 1966 A. I. Mal'cev reported to one of the authors that he had a proof of completeness of the indicated list of precomplete classes for $k = 4$.

** For $k = 8$ the number of classes in C is equal to 549,758,283,756. X wrong

$k = 5$. Family M. $\rho_1(y_1y_2) = (y_1 \leqslant y_2)$, $\rho_2(y_1y_2) = (y_1 \leqslant y_2)[\bar{g}_1(y_1) \vee \bar{g}_2(y_2)]$,

$\rho_3(y_1y_2) = (y_1 \leqslant y_2)[\bar{g}_1(y_1) \vee \bar{g}_2(y_2)g_3(y_2)]$,

$\rho_3(y_1y_2) = (y_1 \leqslant y_2)[\bar{g}_1(y_1) \vee \bar{g}_2(y_2)g_3(y_2)]$,

$\rho_4(y_1y_2) = (y_1 \leqslant y_2)[\bar{g}_1(y_1)\bar{g}_2(y_2) \vee \bar{g}_1(y_1)\bar{g}_3(y_2) \vee \bar{g}_2(y_1)\bar{g}_3(y_2)]$.

Family S. $\rho_5(y_1y_2) = g_0(y_1)g_1(y_2) \vee g_1(y_1)g_2(y_2) \vee g_2(y_1)g_3(y_2)$

$$\vee\, g_3(y_1)g_4(y_2) \vee g_4(y_1)g_0(y_2).$$

Family L. $\rho_6(y_1y_2y_3) = (2y_1 = y_2 + y_3)$.

Family U. $\rho_7(y_1y_2) = g_0(y_1)g_0(y_2) \vee [g_1(y_1) \vee g_2(y_1) \vee g_3(y_1) \vee g_4(y_1)]$
$\& \, [g_1(y_2) \vee g_2(y_2) \vee g_3(y_2) \vee g_4(y_2)]$.

$\rho_8(y_1y_2) = [g_0(y_1) \vee g_1(y_1)][g_0(y_2) \vee g_1(y_2)]$
$\vee [g_2(y_1) \vee g_3(y_1) \vee g_4(y_1)][g_2(y_2) \vee g_3(y_2) \vee g_4(y_2)]$,

$\rho_9(y_1y_2) = g_0(y_1)g_0(y_2) \vee g_1(y_1)g_1(y_2)$
$\vee [g_2(y_1) \vee g_3(y_1) \vee g_4(y_1)][g_2(y_2) \vee g_3(y_2) \vee g_4(y_2)]$,

$\rho_{10}(y_1y_2) = g_0(y_1)g_0(y_2) \vee [g_1(y_1) \vee g_2(y_1)][g_1(y_2) \vee g_2(y_2)]$
$\vee [g_3(y_1) \vee g_4(y_1)][g_3(y_2) \vee g_4(y_2)]$,

$\rho_{11}(y_1y_2) = g_0(y_1)g_0(y_2) \vee g_1(y_1)g_1(y_2) \vee g_2(y_1)g_2(y_2)$
$\vee [g_3(y_1) \vee g_4(y_1)][g_3(y_2) \vee g_4(y_2)]$.

Family C. $\rho_{12}(y_1) = g_0(y_1)$, $\rho_{13}(y_1) = g_0(y_1) \vee g_1(y_1)$,

$\rho_{14}(y_1) = g_0(y_1) \vee g_1(y_1) \vee g_2(y_1)$,

$\rho_{15}(y_1) = g_0(y_1) \vee g_1(y_1) \vee g_2(y_1) \vee g_3(y_1)$,

$\rho_{16}(y_1y_2) = (y_1 = y_2) \vee g_0(y_1) \vee g_0(y_2)$,

$\rho_{17}(y_1y_2) = (y_1 = y_2) \vee g_0(y_1) \vee g_0(y_2) \vee g_1(y_1)g_2(y_2) \vee g_1(y_2)g_2(y_1)$,

$\rho_{18}(y_1y_2) = (y_1 = y_2) \vee g_0(y_1) \vee g_0(y_2) \vee g_1(y_1)g_2(y_2)$
$\vee g_1(y_2)g_2(y_1) \vee g_3(y_1)g_4(y_2) \vee g_3(y_2)g_4(y_1)$,

$\rho_{19}(y_1y_2) = (y_1 = y_2) \vee g_0(y_1) \vee g_0(y_2) \vee g_1(y_1)g_2(y_2)$
$\vee g_1(y_2)g_2(y_1) \vee g_1(y_1)g_3(y_2) \vee g_1(y_2)g_3(y_1)$,

$\rho_{20}(y_1y_2) = (y_1 = y_2) \vee g_0(y_1) \vee g_0(y_2) \vee g_1(y_1)g_2(y_2) \vee g_1(y_2)g_2(y_1)$
$\vee g_1(y_1)g_4(y_2) \vee g_1(y_2)g_4(y_1) \vee g_2(y_1)g_3(y_2) \vee g_2(y_2)g_3(y_1)$,

$\rho_{21}(y_1y_2) = (y_1 = y_2) \vee g_0(y_1) \vee g_0(y_2) \vee g_1(y_1)g_2(y_2) \vee g_1(y_2)g_2(y_1)$
$\vee g_2(y_1)g_3(y_2) \vee g_2(y_2)g_3(y_1) \vee g_1(y_1)g_3(y_2) \vee g_1(y_2)g_3(y_1)$,

$\rho_{22}(y_1y_2) = (y_1 = y_2) \vee g_0(y_1) \vee g_0(y_2) \vee [g_1(y_1) \vee g_3(y_1)][g_2(y_2)$
$\vee g_4(y_2)] \vee [g_1(y_2) \vee g_3(y_2)][g_2(y_1) \vee g_4(y_1)]$,

$\rho_{23}(y_1y_2) = (y_1 = y_2) \vee g_0(y_1) \vee g_0(y_2) \vee g_1(y_1) \vee g_1(y_2)$,

$\rho_{24}(y_1y_2) = (y_1 = y_2) \vee g_0(y_1) \vee g_0(y_2) \vee g_1(y_1) \vee g_1(y_2)$
$\vee g_2(y_1)g_3(y_2) \vee g_2(y_2)g_3(y_1)$,

$\rho_{25}(y_1y_2) = (y_1 = y_2) \vee g_0(y_1) \vee g_0(y_2) \vee g_1(y_1) \vee g_1(y_2)$
$\vee g_2(y_1) \vee g_2(y_2)$,

$\rho_{26}(y_1y_2y_3) = (y_1 = y_2) \vee (y_1 = y_3) \vee (y_2 = y_3) \vee g_0(y_1)$
$\vee g_0(y_2) \vee g_0(y_3)$,

$$\rho_{27}(y_1y_2y_3) = (y_1 = y_2) \vee (y_1 = y_3) \vee (y_2 = y_3) \vee g_0(y_1) \vee g_0(y_2) \vee g_0(y_3) \vee \mathop{\&}_{i=1}^{3} \bigvee_{j=1}^{3} g_i(y_j),$$

$$\rho_{28}(y_1y_2y_3) = (y_1 = y_2) \vee (y_1 = y_3) \vee (y_2 = y_3) \vee g_0(y_1) \vee g_0(y_2) \vee g_0(y_3) \vee \mathop{\&}_{i=1}^{3} \bigvee_{j=1}^{3} g_i(y_j) \vee \mathop{\&}_{i=1,2,4} \bigvee_{j=1}^{3} g_i(y_j),$$

$\rho_{29}(y_1y_2y_3) = (y_1 = y_2) \vee (y_1 = y_3) \vee (y_2 = y_3) \vee g_0(y_1)$
$\vee g_0(y_2) \vee g_0(y_3) \vee g_1(y_1) \vee g_1(y_2) \vee g_1(y_3)$,

$$\rho_{30}(y_1y_2y_3y_4) = \bigvee_{i \neq j} (y_i = y_j) \vee g_0(y_1) \vee g_0(y_2) \vee g_0(y_3) \vee g_0(y_4).$$

Family B.

$$\rho_{31}(y_1y_2y_3) = (y_1 = y_2) \vee (y_1 = y_3) \vee (y_2 = y_3) \vee \bigvee_{i \neq j} [g_2(y_i) \vee g_3(y_i) \vee g_4(y_i)][g_2(y_j) \vee g_3(y_j) \vee g_4(y_j)],$$

$$\rho_{32}(y_1y_2y_3) = (y_1 = y_2) \vee (y_1 = y_3) \vee (y_2 = y_3) \vee \bigvee_{i \neq j} [g_1(y_i) g_2(y_j) \vee g_3(y_i) g_4(y_j)],$$

$$\rho_{33}(y_1y_2y_3y_4) = \bigvee_{i \neq j} (y_i = y_j) \vee \bigvee_{i \neq j} g_3(y_i) g_4(y_j);\ \rho_{34}(y_1y_2y_3y_4y_5) = \bigvee_{i \neq j} (y_i = y_j).$$

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