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# On Palindromic Squares of Non-Palindromic Numbers

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#### Introduction

Numbers which are unaffected by reversal of the order of their digits, such as 141 and 15351, are called *palindromes* by analogy with words or sentences having this property. These numbers have been the subject of a considerable recreational problem literature [1, 2, 3, 4, 5, 6, 7, 10, 11, 13, 14, 15] much of which is devoted to an open conjecture that the operation of successively reversing and adding any integer to itself a finite number of times will ultimately generate a palindrome [1, 6, 10, 12].

This paper originated from the observation that although there are infinitely many decimal numbers whose squares, cubes, and fourth powers are palindromes\* [1] a computer search showed that there was no instance of a decimal number n with  $1 < n^k < 2.8 \times 10^{14}$  for 4 < k < 11 such that  $n^k$  was palindromic. Furthermore, in the case of decimal numbers it is not known whether there are finitely or infinitely many non-palindromic n for which  $n^k$ , k = 2, 3, or 4, is palindromic, for example,  $26^2 = 676, 307^2 = 94,249,$  and  $2201^3 = 10,662,526,601$ . In fact, the magnitude of the numbers which have already been examined suggests that for decimal numbers there may be no palindromic kth powers for  $k \ge 5$  and no palindromic powers of non-palindromic numbers for  $k \ge 4$ . This also raises the question of whether these curious representational properties are restricted to decimal numbers or whether similar behavior occurs in other bases.† In this paper it is shown that this is indeed the case—by exhibiting several infinite sets of non-palindromic numbers in bases 3 and 4 whose powers are palindromic.

# Property P

Since all decimal numbers of the form  $1 \ 0 \cdots 0 \ 1$  have palindromic squares, cubes and fourth powers (but not a palindromic fifth or higher power), there are infinitely many palindromic integers  $x^n$  for x > 1 and n = 2, 3 or 4. As the examples

<sup>\* 1</sup>  $0 \cdots 0$  1 is obviously such a number for every *i*.

<sup>†</sup> The author is indebted to H. Hanani for suggesting this extension of earlier work [8].

mentioned in the introduction illustrate, however, there are also examples of palindromic powers of non-palindromic integers for at least squares and cubes. Table 2 of reference [8] gives an exhaustive compilation computed to the limits imposed by a CDC 6600 of the decimal palindromic squares and reveals that out of the 55 such palindromes, 16 are the squares of non-palindromic integers. We shall call this property of a palindromic integer being a power of a non-palindromic integer-property P. In decimal notation instances of palindromes possessing property P are apparently exceedingly rare for powers higher than the square. We have proven [8] and C. W. Trigg has shown independently [11] and communicated to the author the result that the only palindromic cube less than 1,953,125,000,000 whose cube root is not a palindrome is  $10,662,526,601 = (2201)^3$ . As was noted above in this connection, an exhaustive computer examination of all of the palindromes less than  $2.8 \times 10^{14}$  has failed to discover even a single instance of a palindrome greater than 1 whose fifth, sixth, seventh, eighth, ninth or tenth root was an integer [8] and no instance of a fourth power palindrome possessing property P. Hence, for decimal integers (2201)<sup>3</sup> is the only known instance for a power higher than a square of an integer possessing property P.

## Decimal Palindromic Squares

The following four non-palindromic integers

were all shown to have palindromic squares in [8]. Let  $N_n$  represent the *n*th such number and extend this list systematically by the recursion relation  $N_n = N_{n-1} \cdot 10^2 + (-1)^n \cdot 7$ . This procedure generates three new palindromic squares:

 $(306930693)^2 = 94206450305460249$   $(30693069307)^2 = 942064503484305460249$  $(3069306930693)^2 = 9420645034800084305460249$ 

The square of the last number above, with twenty-five digits, is the largest palindrome possessing property P known to the author. The next such number,  $(306930693069307)^2 = 9420645034800\overline{5}1\overline{4}0084305460249$  while almost a palindrome, fails in the two indicated digits, as do the other larger numbers generated by this recursion. The magnitude of the task of direct computation and search with numbers of this size makes it unlikely that this example of a twenty-five digit palindrome with a non-palindromic square root will be improved upon by direct calculation.

#### Binary Palindromic Powers

Table 1 gives the binary palindromic squares,  $N^2$ , for  $1 < N \le 1,234,162$ . The decimal value of the indicated number is given below its binary representation.

For binary representations we have been unable to even show whether there are infinitely many palindromic squares or not. It is worth noting that the only value of N greater than 1 and less than 65,000 whose cube is a palindrome is

$$(11)^3 = 11011.$$

TABLE	1	Binary	Palindromie	Sauares
TADDE	т.	Dillary	1 amuutomit	Dyuales

1.1	
3	
0101011	

1000110101011	1001110000010100000111001
4523	20457529
10111011010111	1000100100011111100010010001
11991	143784081
100011100010101	10011101111001010011110111001
18197	331130809
100010100101110011	4004040440040
100010100101110011	1001010110010000010000100110101001
141683	20074072489
0001010101110001001	1000101011110010000010000100111101010001
1092489	1193532215121

# Ternary Palindromic Squares

In ternary representation there are several infinite classes of palindromic squares possessing property P as we shall prove by forming the squares of the corresponding base 3 power series and then using arithmetic in base 3 to reduce the coefficients to normal form.

First consider the ternary integers of the form

$$3^{2k+i-2} + 3^{k+i-1} - 3^{k-1} + 1 (1)$$

where  $k \geq 2$  and  $1 \leq i < k$ .

Simple manipulation shows the square of (1) to be of the form

$$3^{4k+2i-4} + 2 \cdot 3^{3k+2i-3} - 2 \cdot 3^{3k+i-3} + 3^{2k+2i-2} + 3^{2k-2} + 2 \cdot 3^{k+i-1} - 2 \cdot 3^{k-1} + 1 \quad (2)$$

There are two cases to be considered.

Case 1. If  $1 \le i \le k-2$  it is easy to verify that each of the exponents in the seven variable terms are distinct and ordered (in magnitude as shown. The two

$$2 \cdot 3^{3k+2i-3} - 2 \cdot 3^{3k+i-3} = 1 \cdot 3^{3k+2i-3} + 2 \cdot 3^{3k+2i-4} + \dots + 2 \cdot 3^{3k+i-2} + 1 \cdot 3^{3k+i-3}$$

and

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$$2 \cdot 3^{k+i-1} - 2 \cdot 3^{k-1} = 1 \cdot 3^{k+i-1} + 2 \cdot 3^{k+i-2} + \dots + 2 \cdot 3^k + 1 \cdot 3^{k-1}$$

each contributes i+1 terms symmetric about the central term  $0 \cdot 3^{2k+i-2}$ .  $3^{2k+2i-2}$ and  $3^{2k-2}$  each contribute a 1 symmetrically located i spaces from the center also. Therefore, expression (2) is a palindrome for all  $1 \le i \le k - 2$ .

Case 2. If i = k - 1, then

$$3k + i - 3 = 2k + 2i - 2$$

and

$$k + i - 1 = 2k - 2$$

so that expression (2) becomes:

$$3^{6k-6} + 2 \cdot 3^{5k-6} - 3^{4k-4} + 3^{2k-1} - 2 \cdot 3^{k-1} + 1.$$
 (3)

For  $k \ge 2$  the exponents in expression (3) are distinct and ordered as shown. The two pairs of terms

$$2 \cdot 3^{5k-5} - 3^{4k-4} = 1 \cdot 3^{5k-5} + 2 \cdot 3^{5k-6} + \cdots + 2 \cdot 3^{4k-3} + 2 \cdot 3^{4k-4}$$

and

$$3^{2k-1} - 2 \cdot 3^{k-1} = 2 \cdot 3^{2k-2} + 2 \cdot 3^{2k-3} + \dots + 2 \cdot 3^{k} + 1 \cdot 3^{k-1}$$

each contributes k terms symmetric about  $3^{3k-3}$ . Hence, expression (2) is also a palindrome for i = k - 1. Expression (2) is not a palindrome for  $i \ge k$ . The following examples illustrate the foregoing result:

$$k = 4, i = 1$$

$$(10002001)^{2} = 100110101011001$$

$$k = 4, i = 2$$

$$(100022001)^{2} = 10012110001121001$$

$$k = 4, i = 3$$

$$(1000222001)^{2} = 1001222000002221001$$

Using precisely the same method of proof, it is possible to show that expressions (4), (5) and (6) below also have squares which possess property P.

$$3^{k} + 2 \cdot 3^{k-2} + 3^{2} + 2 \qquad k \ge 7 \tag{4}$$

and

$$3^{2k} + 3^k - 1 k \ge 2 (5)$$

and

$$2 \cdot 3^{2k+1} + 3^{k+1} + 3^k + 2 \qquad k \ge 2 \tag{6}$$

The square of expression (4) is

$$(3^{k} + 2 \cdot 3^{k-2} + 3^{2} + 2)^{2} = \begin{cases} 3^{2k} + 3^{2k-1} + 3^{2k-2} + 3^{2k-3} + 3^{2k-4} + 2 \cdot 3^{k+2} + 2 \cdot 3^{k+1} \\ + 2 \cdot 3^{k} \\ 1 + 3 + 3^{2} + 3^{3} + 3^{4} + 2 \cdot 3^{k-2} + 2 \cdot 3^{k-1} \end{cases}$$

which is a palindrome, i.e.,

$$(10200102)^2 = 11111 22222 11111$$
  
 $(102000102)^2 = 11111 0222220 11111$  etc.

TABLE 2. Palindromic Squares-–Base 3 For  $N\,<\,575{,}571$  (Base 10)

N	$N^2$	N	$N^2$
1 2 11 101 102	1 11 121 10201 11111	1000000001 1000020001 1000222001 1020000102 1100112011	100000002000000001 1000110010100110001 1001222000002221001 1111100222220011111 1211102000002011121
202 211 1001 1021 2002	112211 122221 1002001 1120211 11022011	2000000002 2000110002 2002212102 2012102202 100000000	11000000022000000011 11001210111101210011 11102202211220220111 11211211111111
10001 10022 11012 12201 20002	100020001 101212101 122111221 1012112101 1100220011	10000022222 10000220001 10012011022 10122002102 10200000102	100001222212222100001 100012101000101210001 101011221101122110101 111012100111001210111 111110002222200011111
100001 100201 200002 201102 1000001	10000200001 10111011101 110002200011 111221122	10200022121 11000120011 12100102001 12200112201 200000000	111112100010001211111 12101111111111111
1000222 1002201 1011221 1101211 1211201	1001221221001 1012200022101 1101202021011 1221221221221 10101111110101	20020122102 20200110202 100000000001 10000200001 100002220001	1110000101001010000111 1122200022002200022211 100000000
1212022 2000002 10000001 10002001 10200102	10110200201101 11000022000011 10000020000001 100110101011001 111112222211111	100010210001 100120011022 101000200101 101122011221 102000000102	1000211211222112112000\) 10101102101010120110101 10201111121212111110201 11012010120202101021011 11111000022222000011111
10201121 11011211 12212101 20000002 20011002	111212202212111 122110101011221 1021101221011201 1100000220000011 1101211111121011	102020121021 110111021211 200000000002 200001100002 200110011	11200102212021220100211 12210001111011110001221 1100000000
100000001 100002222 100022001 102000102 102021021	10000000200000001 10001222122210001 10012110001121001 11111022222011111 11200222122200211	201221020202 1000000000001 1000000222222 1000002200001 1000022220001	112201102002200201102211 1000000000000200000000
102110021 110012011 122010022 200000002 202022112	11212012021021211 12111121112111121 1012120111110212101 110000002200000011 120021201102120021	1001112021201 1001200011022 1001202111201	1010010220112110220100101 1010110010010100100110101 1010122002222222002210101

The square of expression (5) is

$$(3^{2k} + 3^k - 1)^2 = 3^{4k} + 2 \cdot 3^{3k} - 3^{2k} - 2 \cdot 3^k + 1.$$

The difference  $2 \cdot 3^{3k} - 2 \cdot 3^k = 1 \cdot 3^{3k} + 2 \cdot 3^{3k-1} + \cdots + 2 \cdot 3^{k+1} + 3^k$  which upon subtracting  $3^{2k}$  gives a 1 in the central position to form a palindrome, i.e.,

$$(10022)^2 = 101212101$$
  
 $(1000222)^2 = 1001221221001$  etc.

The square of expression (6) requires a slightly different argument to show that it is a palindrome.

$$(2 \cdot 3^{2k+1} + 3^{k+1} + 3^k + 2)^2 = \begin{cases} 3^{4k+3} + 3^{4k+2} + 3^{3k+3} + 2 \cdot 3^{3k+2} + 3^{3k+1} + 3^{2k+3} \\ + 3^{2k+2} \\ 1 + 3 + 3^k + 2 \cdot 3^{k+1} + 3^{k+2} + 3^{2k} + 3^{2k+1} \end{cases}$$
(7)

Case 1. For  $k \ge 3$ : the exponents in expression (7) are all distinct and in the order shown, hence (7) represents a palindromic square in these cases.

Case 2. For k=2:

$$(201102)^2 = 111221122111$$

which is a palindromic square.

Note: When k = 1, the square is not a palindrome;

$$(2112)^2 = 20100021$$

hence expression (6) is the best possible.

$$(20011002)^2 = 11012111111121011$$
  
 $(2000110002)^2 = 110012101111101210011 \text{ etc.}$ 

## **Ternary Palindromic Cubes**

The only value of  $1 < N \le 28,800$  for which  $N^3$  is palindromic in base 3 is  $2^3 = 22$ , so that no table is shown for this case.

## **Base Four Palindromic Squares**

In base four an even richer set of identities can be found which generate palindromic squares from non-palindromic factors—and hence palindromic integers which possess property P. Since the method of proof is identical to that used in the previous paragraph, we shall merely state the results without proofs or examples. The following infinite classes of non-palindromic polynomials all possess palindromic squares.

$$4^{2k} + 4^{k+1} + 3 \cdot 4^{k-1} + 1 \qquad k \ge 4$$
 (8)

and

$$4^{2k+1} + 4^{k+1} + 3 \cdot 4^k + 2 \cdot 4^{k-1} + 1 \qquad k \ge 3$$
 (9)

and

$$4^{2k+2} + 4^{2k+1} + 2 \cdot 4^k + 4 + 1 \qquad k \ge 2$$
 (10)

and

$$4^{2k+1} + 2 \cdot 4^k - 1 \qquad k \ge 1 \tag{11}$$

and

$$4^{2k} + 4^k + 2 \cdot 4^{k-1} + 1 \qquad k \ge 2 \tag{12}$$

and

$$4^{2k} + 4^{2k-1} + 4^k + 2 \cdot 4^{k-1} + 4 + 1 \quad k \ge 3$$
 (13)

TABLE 3. Palindromic Squares—Base 4 For  $N\,<\,566,\!449$  (Base 10)

2.12.02	,2 0, 2		· ·
N	$N^2$	N	$N^2$
1	1	1 100010001	10002000300020001
11	121	100012001	10003001010030001
101	10201	100103001	10021212021212001
111	12321	100301113	10121101210112101
1001	1002001	100332101	10200011011000201
1013	1032301	101002101	10202031013020201
1103	1223221	103231203	12010203030201021
10001	100020001	110002011	12101103030110121
10101	102030201	110003323	12102131013120121
10121	103101301	110012011	12103303230330121
10331	120202021	110033011	12121132223112121
100001	10000200001	110111211	12131212121213121
100133	10033233001	1000000001	1000000002000000001
1000001	1000002000001	1000013333	1000033332333300001
1001001	1002003002001	1000132001	1000330100010330001
1001201	1003010103001	1012211213	1031312213122131301
1010301	1021320231201	1033001203	1201210310130121021
1100211	1211130311121	1033020131	1201311320231131021
1100323	1212110112121	1100333011	1212132222222312121
1101211	1213332333121	1102013231	1221113300033111221
10000001	100000020000001		
10001333	100033323330001		
10013201	100331000133001		
10031113	101231222132101		
100000001	10000000200000001		

TABLE 4. Palindromic Cubes—Base 4 For  $N < 28,\!800$  (Base 10)

N	$N^3$
1	1
11	1331
101	1030301
1001	1003003001
10001	1000300030001
100001	1000030000300001
1000001	1000003000003000001
10000001	100000030000030000001

TABLE 5. Palindromic Squares—Base 5 For N < 692,021 (Base 10)

N	$N^2$
2	4
11	121
101	10201
111	12321
231	114411
1001	1002001
1111	1234321
10001	100020001
10101	102030201
11011	121242121
11204	131141131
100001	10000200001
101101	10221412201
110011	12102420121
242204	131441144131
1000001	1000002000001
1001001	1002003002001
1010101	1020304030201
1042214	1143442443411
1100011	1210024200121
2020303	4133144413314
2043122	4342230322434
2443304	13431400413431
10000001	100000020000001
10011001	100220141022001
10100101	102012040210201
11000011	121000242000121
100000001	1000000200000001
100010001	10002000300020001
100101001	10020210401202001
101000101	10201020402010201
110000011	12100002420000121
111103411	12400140104100421

TABLE 6. Palindromic Cubes—Base 5 For  $N \leq 65,000$  (Base 10)

N	$N^3$
11	1331
101	1030301
1001	100 <b>30</b> 03001
10001	1000300030001
100001	1000030000300001
1000001	1000003000003000001

## Conclusion

Based on the several results given here and in spite of our failure to exhibit even a single infinite class in either binary or decimal notation, we conjecture that, independent of the base, there are infinitely many palindromic squares possessing property P. Unfortunately we have been unable to construct an infinite class of cubics with property P irrespective of the base, so that we cannot even conjecture about the behavior to be expected for higher powers with respect to property P.

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