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double poles at these points. For $n=1$, the inequality (33) quickly reduces to a homogeneous form of the known inequality $(u-1)^2 \geqq u(\log u)^2$. For $n=2$, if we divide (33) by $(x_1-x_3)^2$ and take the limit as $x_1 \to x_3$ we obtain the (unproved) statement

$$(34)\qquad \begin{aligned} &[(t-1)u^t - tu^{t-1}]\log^2 u + 2(u^t - u^{t-1})\log u - u^{2(t-1)}(u-1)^2 \\ &\qquad \leqq \left(\frac{1}{t^2} + \frac{1}{(t-1)^2}\right)(tu^{t-1} - (t-1)u^t - 1)^2. \end{aligned}$$

Note that (34) is unchanged if u and t are replaced by u^{-1} and $1-t$. If we denote the left hand side of (33) by $\sigma = \sigma(x_1, \cdots, x_{n+1}\,; t)$ then

$$\sigma(x_1, \cdots, x_{n+1}\,; t) = (x_1 \cdots x_{n+1})^{2(n-1)}\sigma(x_1^{-1}, \cdots, x_{n+1}^{-1}\,; n-1-t).$$

It is easy to see that (33) is true for $|t|$ large; we also mention that

$$\lim_{t\to 0} \sigma(x_1, x_2, x_3\,; t) = 0$$

and hence also $\lim_{t\to 1} \sigma = 0$ when $n=2$.

6. Comment. We note that the mean $u(x,y;\alpha)$ bears some resemblance to the functions denoted by $G_t(x,y)$ and $A_t(x,y)$ in Carlson's paper ([3], p. 616); note especially his remark there that "$1/G_t$ is log convex in t".

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AN EXTENSION OF TRIGG'S TABLE

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In a recent issue of this MAGAZINE, Charles W. Trigg asks in [1] some interesting questions about the prime factorization of

$$Q(p_k) = (p_1p_2p_3 \cdots p_k) + 1$$

where p_i denotes the ith prime. He tabulated the prime factorization of $Q(p)$ for $2 \leqq p \leqq 19$; we include and extend his work as summarized in Table 1, page 93. We also show data on the closely related

$$R(p_k) = (p_1p_2p_3 \cdots p_k) - 1\,.$$

We conjecture that

$$xyz \cdot U_0(x, y, z) \leqq U_0^2(xy, xz, yz); \tag{26}$$

if U_0 is replaced by the arithmetic mean in (26), a known inequality is obtained (see, e.g., [1], p. 11, inequality (6)).

At this point we attempt to generalize $u(\alpha)$ as far as possible. Let $x_1, \cdots, x_{n+1}$ be positive numbers and set

$$a_i(t) = x_i^t \prod_{1 \leqq j < k \leqq n+1}{}' (x_j - x_k) \tag{27}$$

where the prime mark indicates that every factor involving x_i is deleted. Let $(\alpha)_n = \alpha(\alpha - 1) \cdots (\alpha - n + 1)$. Define

$$\begin{aligned} u(\alpha, \beta) &= u(x_1, \cdots, x_{n+1}; \alpha, \beta) \\ &= \left\{ \left[(\beta)_n \sum_{i=1}^{n+1} (-1)^{i+1} a_i(\alpha) \right] \Big/ \left[(\alpha)_n \sum_{i=1}^{n+1} (-1)^{i+1} a_i(\beta) \right] \right\}^{1/(\alpha-\beta)}. \end{aligned} \tag{28}$$

It is easy to see that this reduces to the earlier definitions when $n = 1$ or 2. Our main problem is to show that $u(\alpha, \beta)$ cannot decrease if either α or β is increased. Once again we have (15) where now

$$\log u(t, t) = \frac{d}{dt} G_n(t) \text{ and} \tag{29}$$

$$G_n(t) = \log \left| \frac{\sum_{i=1}^{n+1} (-1)^{i+1} a_i(t)}{(t)_n} \right|. \tag{30}$$

Conjecture. The function $G_n(t)$ is convex.

I have not been able to resolve this conjecture; the second derivative of $G_n(t)$ is unwieldy. I can, however, manipulate

$$G_n''(t) \geqq 0 \tag{31}$$

into a sort of "standard form" with the aid of the identity

$$\left(\sum_{i=1}^{n} A_i B_i^2 \right) \left(\sum_{j=1}^{n} A_j \right) - \left(\sum_{i=1}^{n} A_i B_i \right)^2 = \sum_{i<j} A_i A_j (B_i - B_j)^2. \tag{32}$$

The result is that (31) is equivalent to

$$\begin{aligned} &\left(\sum_{i=1}^{n+1} (-1)^{i+1} a_i(t) \right)^2 \left(\sum_{j=0}^{n-1} \frac{1}{(t-j)^2} \right) \\ &\qquad + \sum_{i<j} (-1)^{i+j} a_i(t) a_j(t) (\log x_i - \log x_j)^2 \geqq 0. \end{aligned} \tag{33}$$

The inequality (33) has a "vague reasonableness" to it. The double sum can be positive or negative, but the other terms are clearly nonnegative. The first sum on the left vanishes when $t = 0, 1, \cdots, n-1$ but the second sum compensates by having

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TABLE 1

p	$Q(p)$	$R(p)$
2	3	1
3	7	5
5	31	29
7	211	11·19
11	2311	2309
13	59·509	30029
17	19·97·277	61·8369
19	347·27953	53·197·929
23	317·703763	37·131·46027
29	331·571·34231	79·81894851
31	200560490131	228737·876817
37	181·60611·676421	229·541·1549·38669
41	61·450451·11072701	304250263527209
43	167·78339888213593	141269·92608862041
47	953·46727·13808181181	191·53835557·59799107
53	73·139·173·18564761860301	87337·326257·1143707681
59	277·3467·105229·19026377261	C_2
61	223·525956867082542470777	1193·C_2
67	C_3	163·2682037·17975352936245519
71	1063·303049·598841·2892214489673	C_3
73	2521·P_3	313·130126775077472920609013813
79	22093·C_3	163·2843·C_3
83	265739·P_2	139·26417·P_2
89	131·1039·2719·64225891884294373371806141	23768741896345550770650537601358309
97	2336993·C_4	66683·P_3

In Table 1, all entries for $Q(p)$ and $R(p)$ are primes or 1, except that C_n denotes a composite number with no more than n prime factors and P_n denotes a number, possibly prime, with no more than n prime factors. None of the C_n and P_n in Table 1 has a prime factor less than 10^7, and all were checked for divisors sufficiently large to establish the validity of the subscript. The P_n satisfy the congruence

$$2^{m-1} \equiv 1 \pmod{m}$$

and thus are quite likely prime; indeed, we were able to establish some of the larger factors prime by an application of a version of a theorem of Lehmer [2]:

THEOREM. *Let b and n be integers exceeding 1. Suppose that $b^{n-1} \equiv 1 \pmod{n}$, and let p be a prime factor of $n-1$. Let $a \equiv b^{(n-1)/p} \pmod{n}$. If $(n, a-1) = 1$, then every prime factor q of n satisfies $q \equiv 1 \pmod{p}$.*

We owe special thanks to Dr. Carl Pomerance of the University of Georgia, who designed an eminently programmable version of this test.

We also obtained data on $Q(p)$ and $R(p)$ for larger values of the prime p, and we obtained coincident results although at the time we were working independently, with different programs, on different computers, each of us unaware of the other's work. With the aid of Table 1, special-purpose programs, and some recent results in the literature, we can answer most of Trigg's questions.

1. Are any $Q(p)$ prime for $p > 19$?

This question was answered by Kraitchik [3], and his results extended by Borning [4], who found that in the range $23 \leqq p \leqq 307$, only $Q(31)$ is prime. Borning also found that for $p \leqq 307$, $R(p)$ is prime only for $p = 3, 5, 11, 13, 41$, and 89. We have confirmed these results for $p \leqq 97$, and Table 1 also gives complete or partial factorizations not given by Kraitchik or Borning.

2. The prime $p_7 = 17$ and the least prime factor $p_8 = 19$ of $Q(17)$ are twin primes. Does this case of twin primes, or even of consecutive primes, occur again?

Yes; $Q(1459)$ is divisible by $p_{233} = 1471$, but the latter and $p_{232} = 1459$ are not twin primes. In the range $19 < p \leqq p_{6000} = 59359$, there is only one other such example: $Q(2999)$ is divisible by $p_{431} = 3001$, and the latter and $p_{430} = 2999$ do form a twin prime pair.

The same question for $R(p)$ leads to the obvious examples for $p = 3$ and $p = 7$; there are no other examples for which p_{k+1} is a divisor of $R(p_k)$ in the abovementioned range. There are a few cases in which the second or third prime after p divides $Q(p)$ or $R(p)$—specifically, $7 \mid Q(3)$, $37 \mid R(23)$, $271 \mid Q(263)$, $307 \mid Q(283)$, and $673 \mid Q(659)$. There are no additional examples in the range $p \leqq 59359$.

3. Are there more cases in which the least prime factor of $Q(p)$ does not exceed $2p$?

This holds for $p = 2$ and for $p = 17$, as observed by Trigg. We found it to hold for exactly 32 values of p in the range $2 \leqq p \leqq 1987$, and the same holds true for $R(p)$ for 24 such values. These are shown in Table 2, together with the divisor or divisors less than $2p$.

4. What is the smallest value of p for which $Q(p)$ has four prime factors? Five prime factors?

$Q(53)$ is the least value of $Q(p)$ with four prime factors, and has exactly four. We found none with five prime factors, and $Q(97)$ is the least candidate for this property. $R(37)$ has exactly four prime factors, and is the least value of $R(p)$ with at least four; $R(79)$ might have as many as five.

In the course of these investigations some additional facts were noted. We mention three here:

TABLE 2

p	Prime divisors of $Q(p)$ not exceeding $2p$	p	Prime divisors of $R(p)$ not exceeding $2p$
2	3	3	5
17	19	7	11
41	61	23	37
53	73	83	139
89	131	167	331
107	149	239	349
239	313	241	389
263	271	397	599
283	307	421	761
443	463	463	631 and 647
499	827	499	569
587	1033	523	563
659	673	577	1093
677	809 and 877	641	881
739	1051	797	953
769	997 and 1297	877	911
811	1279	907	983
839	1109	919	1181
907	1259	941	1433
937	1031	1069	1327
1061	2029	1103	1283
1097	1381	1289	1811
1181	1667	1871	3467
1237	1663	1877	2531
1259	1867		
1423	2609		
1459	1471		
1481	1619		
1657	3203		
1663	2383		
1669	3041		
1987	3581		

First, some primes—for example, 13, 17, 23, and 41—divide none of the $Q(p)$ and none of the $R(p)$.

Second, several primes may divide two values of $Q(p)$, two values of $R(p)$, or one of each. All such between 2 and $p_{1001} = 7927$ are shown in Table 3, together with the $Q(p)$ and $R(p)$ they divide for $p \leqq 7919$.

TABLE 3

p	What p divides	p	What p divides
19	$Q(17)$ and $R(7)$	1051	$Q(211)$ and $Q(739)$
61	$Q(41)$ and $R(17)$	1069	$Q(523)$ and $R(359)$
131	$Q(89)$ and $R(23)$	1283	$Q(509)$ and $R(1103)$
139	$Q(53)$ and $R(83)$	1291	$Q(439)$ and $R(163)$
163	$R(67)$ and $R(79)$	1381	$Q(157)$ and $Q(1097)$
277	$Q(17)$ and $Q(59)$	1657	$Q(137)$ and $Q(557)$
313	$Q(239)$ and $R(73)$	1867	$Q(157)$ and $Q(1259)$
331	$Q(29)$ and $R(167)$	2609	$Q(1423)$ and $R(479)$
673	$Q(659)$, $R(149)$, and $R(193)$	3041	$Q(1277)$ and $Q(1669)$
881	$Q(137)$ and $R(641)$	3373	$Q(521)$ and $Q(1103)$
953	$Q(47)$ and $R(797)$	3467	$Q(59)$ and $R(1871)$
983	$Q(463)$ and $R(907)$	4871	$Q(613)$ and $R(139)$

Finally, we checked $Q(p)$ and $R(p)$ for prime factors less than 10^7 for $2 \leqq p \leqq 97$, for prime factors less than 10^5 for $101 \leqq p \leqq 541 = p_{100}$, and for prime factors less than 7930 for $547 \leqq p \leqq 1987$. As a result we know that $Q(p)$ is prime for six values of p, composite for 106 values of p, and unknown to us for the remaining 188 values of p. Similarly, $R(p)$ is a unit for $p = 2$, prime for six values of p, composite for 96 values of p, and unknown to us for the remaining 197 values of p. The largest number we actually computed was

$$Q(59359) = 62970292 \cdots 375361614691,$$

a number of 25706 digits.

Questions inevitably remain. What is the least value of $Q(p)$ having exactly (or at least) five prime factors? Or, six, or seven? Are any more of the $Q(p)$ prime? What are the answers to these questions for the $R(p)$? Note that $R(p)$ and $Q(p)$ form a twin prime pair for $p = 3, 5, 11$, and for no other prime $p \leqq 307$. Is there another such twin prime pair? Are there infinitely many primes dividing none of the $Q(p)$ and none of the $R(p)$? It is easy to show that none of the $Q(p)$ and none of the $R(p)$ can be perfect squares other than $R(2)$. We know all are square-free for $p \leqq 61$. Does this hold for all p?

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SYNTAX ERROR
1
1

20x=2*3*5*7*11*13*17*19*23*29*31*37*41*43*47*53*59*61-1
y=x/1193
y
98313815053987402333
x%1193
0
```

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May 3, 1975

Professor John L Selfridge
Northern Ill Univ

Dear John,

Would it be possible for you to factor this number for me? (I need the largest factor to complete a sequence for my book)

Product of primes

$$(2 . 3 . 5 . 7 . 11 . \cdots . 59) - 1$$

According to Math. Magazine, March 1975, page 93, it is a product of two large primes.

If you can factor this I would greatly appreciate it.

Best regards

Neil Sloane

(N. J. A. Sloane
Math Dept.)

P.S. Thank you for showing that $\frac{19^{19}-1}{18}$ is a prime (via John Brillhart)

-THE TRUTH OF THE MATTER IS ....
1922760350154212639069 EQUALS
69664915493 TIMES 27600124633

Thanks to Morrison
Brillhart
Wunderlich