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ON THE DIVISIBILITY OF FACTORIALS

By MAURICE KRAITCHIK

1. It is known that $n! = 1.2.3.4\ldots n$ is divisible by p^k, $p \leqslant n$,

$$k = E\left(\frac{n}{p}\right) + E\left(\frac{n}{p^2}\right) + E\left(\frac{n}{p^3}\right) + \ldots,$$

$E(x)$ designating the greatest integer contained in x.

For instance $40!$ is divisible by 5^k,

$$k = E\left(\frac{40}{5}\right) + E\left(\frac{40}{25}\right) = 8 + 1 = 9.$$

It may be shown that the expression of k may be written in a finite form, $k = (n - s)/(p - 1)$, where s is the sum of the digits of n written in the scale of notation with the base p.

Thus, in the case of $n = 40$, $p = 5$, since $40_{10} = 130_5$, $s = 1 + 3 = 4$, $k = (40 - 4)/(5 - 1) = 9$.

See my "*Récherches sur la Théorie des Nombres*, v. 1, p. 21."

It is easy to find the whole factorization of $n!$

2. More complicated is the question of factorization of the numbers $n! \pm 1$.

A general result is given by Wilson's theorem according to which $(p - 1)! + 1$ is divisible by p, p being a prime number.

Another general result follows from Wilson's theorem. It is: $(p - 2)! - 1$ is divisible by p, p being a prime number.

3. The number $n! + 1$ is a perfect square for $n = 4, 5, 7$. No other perfect squares may be found for $n < 1020$ (see my book, *loc. cit.*, p. 38–41). But $n! \pm 1$ may be divisible by a square. Thus

$$12! + 1 \text{ is divisible by } 13^2$$
$$9! - 1 \text{ is divisible by } 11^2$$
$$15! - 1 \text{ is divisible by } 31^2.$$

The following table gives the factorization of $n! \pm 1$, as well as that of $P_n \pm 1$, P_n being the product of all prime numbers not exceeding n.

FACTORIZATION OF $n! \pm 1$

2582 2583

$n! - 1$	n	$n! + 1$
0	1	2
1	2	3
5	3	7
23	4	5^2
7.17	5	11^2
719	6	7.103
5039	7	71^2
23.1753	8	61.661
11^2.2999	9	19.71.269
29.125131	10	11.329891
13.17.23.7853	11	39916801
479001599	12	13^2.2834329
1733.3593203	13	83.75024347
87178291199	14	23.3790360487
17.31^2.53.1510259	15	59.479.46271341
3041.6880233439	16	17.61.137.139.1059511
19.73.25644371677	17	661.537913.1000357
59.108514808571661	18	19.23.29.61.67.123610951
653.2383907.78143369	19	71.1713311273363831
2432902008176639999	20	20639383.117876683047
23.89.249589360877916l7	21	43.439429.270387581578 3
23.89.24958936087791617	22	23.521.93799610095769647

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INCOMPLETE RESULTS

$n! - 1$	n	$n! + 1$
Div. 109	22	
	23	Div. 47.79
	24	811
149.907	25	401
	26	1697
29	27	
239	28	29
31.59.311	29	
	30	31
787	31	257
	32	2281
	33	67
37.71	35	137.379
	36	37.83.739.1483
53.439	37	
41	39	79
	40	41.59.277

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2584

FACTORIZATION OF $P_n \pm 1$, $P_n = 2.3.5.7.....n$, n-prime

2585

$P_n - 1$	n	$P_n + 1$
1		1
1	2	3
5	3	7
29	5	31
11.~~19~~	7	211
2309	11	2311
30029	13	59.~~509~~
61.~~8369~~	17	19.~~97.277~~
53.~~197.929~~	19	347.~~27953~~
37.~~131.46027~~	23	317.~~703763~~
79.~~81894851~~	29	331.~~571.34231~~
228737.~~876817~~	31	200560490131
229.~~511.1519.38669~~	37	181.~~61011.676421~~
304250263527209	41	61.~~450451.11072701~~
141269.~~92608862041~~	43	167.~~78339888213593~~
191.~~32193182333447599~~	47	953.~~46727.13808181181~~
	53	73.~~139.173.18564761860301~~

INCOMPLETE RESULTS

$P_n - 1$	n	$P_n + 1$
	59	Div. 277
Div. 163	61	223
313	67	
163	73	
139	79	
	83	
	89	131

Editor's Note.—Professor Kraitchik's formula for k may be shown to lead to the following

Theorem. *If* $n = a_1a_2 \cdots a_{k-1}a_k$ *in the scale of notation with the base* p, *then* $k = a_1 + a_1a_2 + \cdots + a_1a_2\cdots a_{k-1}$ *in the same base.*

Thus, in the case of $n = 167$, $p = 7$, since $167 = 326_7$, $k = 3_7 + 32_7 = 35_7 = 26$.

$P_{37}+1$: 60611 vice 61011