BAMS 48 (1942) → 2545–2548

$$z = 1 + \frac{a_{2n-1}}{\bar{z}}, \qquad \bar{z} = 1 + \frac{a_{2n}}{z}, \qquad n \geqq 1.$$

This gives

$$a_{2n-1} = (z - 1)\bar{z},$$
$$a_{2n} = (\bar{z} - 1)z = \bar{a}_{2n-1},$$

and it is easily seen that all a_n lie on the boundary of the parabola. The theorem is now completely proved.

Bibliography

1. J. F. Paydon, *Convergence regions and value regions for continued fractions*, this Bulletin, abstract 47-11-473.
2. W. T. Scott and H. S. Wall, *A convergence theorem for continued fractions*, Transactions of this Society, vol. 47 (1940), pp. 155–172.
3. ———, *Value regions for continued fractions*, this Bulletin, vol. 47 (1941), pp. 580–585.

The Rice Institute

A TABLE OF COEFFICIENTS FOR NUMERICAL DIFFERENTIATION

ARNOLD N. LOWAN, HERBERT E. SALZER AND ABRAHAM HILLMAN[1]

The following table lists the coefficients $A_{m,s}$ for $m = 1, 2, \cdots, 20$ and $s = m, \cdots, 20$ in Markoff's formula for the mth derivative in terms of advancing differences, namely

$$\omega^m f^{(m)}(x) = \sum_{s=m}^{n-1} (-1)^{m+s} A_{m,s}\Delta^s f(x) + (-1)^{m+n}\omega^n A_{m,n} f^{(n)}(\xi).$$

In this formula ω is the tabular interval and

$$A_{m,s} = (-1)^{m+s} m B_{s-m}^{(s)}/s(s-m)!$$

and $B_{s-m}^{(s)}$ is the $(s-m)$th Bernoulli number of the sth order.

Presented to the Society, April 4, 1942 under the title *Coefficients of differences in the expansion of derivatives in terms of advancing differences*; received by the editors March 7, 1942.

[1] The results reported here were obtained in the course of the work done by the Mathematical Tables Project, Work Projects Administration, New York City.

If a function has been tabulated to su for some suitable interval of the argum accompanying table may be used to derivatives which in turn may be emplo the function in the complex plane within is analytic.

The coefficients were computed from t

$$sA_{m,s} = (s-1)A_{m,s-1} \ldots$$

and checked by independent calculation

$$x(x+1)(x+2)\cdots(x+s- \ldots$$

From the identity

$$(x + x^2/2 + x^3/3 + \cdots)^m \equiv A_{m,m}x \ldots$$

it was discovered that a prime p is not nominator of an $A_{m,s}$ for which $s < m$ prime factors in accordance with this the accuracy of the work.

The Markoff formula is used at the where advancing differences are the on discussion see L. M. Milne-Thomson, *ences*, chap. 7, pp. 157–159. According t simplicity of the remainder term is an difference formulae.

Comparison of the Markoff coefficien efficients shows the latter to be much sm venient for obtaining the derivatives o away from the ends of a table. Howev tions in applied mathematics such as B tions, use of the Markoff formula for a some fixed degree might yield a smaller lar form of its remainder term.

The first few coefficients of the vari H. T. Davis, *Table of the Higher Math* 73–77; Whittaker and Robinson, *Calc* and in an article by W. S. Bickley *N limits of a difference table*, Philosophica pp. 12–14. (This article lists coefficient to those of the 12th difference.)

$$1 + \frac{a_{2n-1}}{\bar{z}}, \qquad \bar{z} = 1 + \frac{a_{2n}}{z}, \qquad n \geqq 1.$$

$$a_{2n-1} = (z - 1)\bar{z},$$
$$a_{2n} = (\bar{z} - 1)z = \bar{a}_{2n-1},$$

at all a_n lie on the boundary of the parabola.
mpletely proved.

BIBLIOGRAPHY

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S. Wall, *A convergence theorem for continued fractions*, vol. 47 (1940), pp. 155–172.

for continued fractions, this Bulletin, vol. 47 (1941), pp.

OEFFICIENTS FOR NUMERICAL IFFERENTIATION

RBERT E. SALZER AND ABRAHAM HILLMAN[1]

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ences, namely

$$^{m+s}A_{m,s}\Delta^s f(x) + (-1)^{m+n}\omega^n A_{m,n} f^{(n)}(\xi).$$

tabular interval and

$$-1)^{m+s} m B^{(s)}_{s-m}/s(s-m)!$$

3ernoulli number of the sth order.

il 4, 1942 under the title *Coefficients of differences in ms of advancing differences*; received by the editors

re obtained in the course of the work done by the rk Projects Administration, New York City.

If a function has been tabulated to sufficiently great accuracy and for some suitable interval of the argument along the real axis, the accompanying table may be used to generate the values of the derivatives which in turn may be employed to generate the values of the function in the complex plane within a region where the function is analytic.

The coefficients were computed from the recurrence formula

$$sA_{m,s} = (s-1)A_{m,s-1} + mA_{m-1,s-1}$$

and checked by independent calculations using the identity

$$x(x+1)(x+2)\cdots(x+s-1) \equiv s!\sum_{j=1}^{s} A_{j,s}x^j/j!.$$

From the identity

$$(x + x^2/2 + x^3/3 + \cdots)^m \equiv A_{m,m}x^m + A_{m,m+1}x^{m+1} + \cdots$$

it was discovered that a prime p is not effectively present in the denominator of an $A_{m,s}$ for which $s<m+p-1$. The cancellation of prime factors in accordance with this rule was a further check on the accuracy of the work.

The Markoff formula is used at the beginning and end of a table where advancing differences are the only types available. For a full discussion see L. M. Milne-Thomson, *The Calculus of Finite Differences*, chap. 7, pp. 157–159. According to Milne-Thomson the relative simplicity of the remainder term is another advantage over central difference formulae.

Comparison of the Markoff coefficients with central difference coefficients shows the latter to be much smaller and obviously more convenient for obtaining the derivatives of a polynomial sufficiently far away from the ends of a table. However for many important functions in applied mathematics such as Bessel, error, and gamma functions, use of the Markoff formula for a polynomial approximation of some fixed degree might yield a smaller total error due to the particular form of its remainder term.

The first few coefficients of the various formulae may be found in H. T. Davis, *Table of the Higher Mathematical Functions*, vol. 1, pp. 73–77; Whittaker and Robinson, *Calculus of Observations*, pp. 62–65, and in an article by W. S. Bickley *Numerical differentiation near the limits of a difference table*, Philosophical Magazine, (7), vol. 33 (1942), pp. 12–14. (This article lists coefficients of the first 12 derivatives up to those of the 12th difference.)

Please enter 4

COEFFICIENTS $A_{m,s}$ IN MARKOFF'S EXPANSION

2547/2548
2545/2546

m \ s	1	2	3	4	5	6	7	8	9	10	11	12	13	14
1	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{6}$	$\frac{1}{7}$	$\frac{1}{8}$	$\frac{1}{9}$	$\frac{1}{10}$	$\frac{1}{11}$	$\frac{1}{12}$	$\frac{1}{13}$	$\frac{1}{14}$
2		1	1	$\frac{11}{12}$	$\frac{5}{6}$	$\frac{137}{180}$	$\frac{7}{10}$	$\frac{363}{560}$	$\frac{761}{1260}$	$\frac{7129}{12600}$	$\frac{671}{1260}$	$\frac{83711}{166320}$	$\frac{6617}{13860}$	$\frac{1145993}{2522520}$
3			1	$\frac{3}{2}$	$\frac{7}{4}$	$\frac{15}{8}$	$\frac{29}{15}$	$\frac{469}{240}$	$\frac{29531}{15120}$	$\frac{1303}{672}$	$\frac{16103}{8400}$	$\frac{190553}{100800}$	$\frac{128977}{69300}$	$\frac{9061}{4950}$
4				1	2	$\frac{17}{6}$	$\frac{7}{2}$	$\frac{967}{240}$	$\frac{89}{20}$	$\frac{4523}{945}$	$\frac{7645}{1512}$	$\frac{341747}{64800}$	$\frac{412009}{75600}$	$\frac{9301169}{1663200}$
5					1	$\frac{5}{2}$	$\frac{25}{6}$	$\frac{35}{6}$	$\frac{1069}{144}$	$\frac{285}{32}$	$\frac{31063}{3024}$	$\frac{139381}{12096}$	$\frac{1148963}{90720}$	$\frac{355277}{25920}$
6						1	3	$\frac{23}{4}$	9	$\frac{3013}{240}$	$\frac{781}{48}$	$\frac{242537}{12096}$	$\frac{48035}{2016}$	$\frac{1666393}{60480}$
7							1	$\frac{7}{2}$	$\frac{91}{12}$	$\frac{105}{8}$	$\frac{4781}{240}$	$\frac{13321}{480}$	$\frac{314617}{8640}$	$\frac{790153}{17280}$
8								1	4	$\frac{29}{3}$	$\frac{55}{3}$	$\frac{10831}{360}$	$\frac{897}{20}$	$\frac{944311}{15120}$
9									1	$\frac{9}{2}$	12	$\frac{99}{4}$	$\frac{1747}{40}$	$\frac{5551}{80}$
10										1	5	$\frac{175}{12}$	$\frac{65}{2}$	$\frac{491}{8}$
11											1	$\frac{11}{2}$	$\frac{209}{12}$	$\frac{1001}{24}$
12												1	6	$\frac{41}{2}$
13													1	$\frac{13}{2}$
14														1

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1, 11, 5, 137

1, 12, 6, ...

2547

See next page

$\omega^m D^m f(x) \sim \sum_{s=m}^{s=n} (-1)^{m+s} A_{m,s} \Delta^s f(x)$

m \ s	15	16	17
1	$\frac{1}{15}$	$\frac{1}{16}$	$\frac{1}{17}$
2	$\frac{1171733}{2702700}$	$\frac{1195757}{2882880}$	$\frac{143327}{360360}$
3	$\frac{30946717}{17199000}$	$\frac{39646461}{22422400}$	$\frac{58433327}{33633600}$
4	$\frac{406841}{71280}$	$\frac{35118025721}{6054048000}$	$\frac{4446371981}{756756000}$
5	$\frac{21939781}{1496880}$	$\frac{2065639}{133056}$	$\frac{2195261857}{134534400}$
6	$\frac{22463}{720}$	$\frac{277382447}{7983360}$	$\frac{38101097}{997920}$
7	$\frac{899683}{16200}$	$\frac{2271089}{34560}$	$\frac{86853967}{1140480}$
8	$\frac{35717}{432}$	$\frac{54576553}{518400}$	$\frac{8424673}{64800}$
9	$\frac{515261}{5040}$	$\frac{23915}{168}$	$\frac{76492463}{403200}$
10	$\frac{2485}{24}$	$\frac{324509}{2016}$	$\frac{59279}{252}$
11	$\frac{30217}{360}$	$\frac{1199}{8}$	$\frac{494351}{2016}$
12	$\frac{105}{2}$	$\frac{26921}{240}$	$\frac{6341}{30}$
13	$\frac{143}{6}$	65	$\frac{35269}{240}$
14	7	$\frac{329}{12}$	$\frac{238}{3}$

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COEFFICIENTS $A_{m,s}$ IN MARKOFF'S EXPANSION $\omega^m D^m f(x) \sim \sum_{s=m}^{s=n} (-1)^{m+s} A_{m,s} \Delta^s f(x)$

7	8	9	10	11	12	13	14	15	16	17	18	19	20	s / m
$\frac{1}{7}$	$\frac{1}{8}$	$\frac{1}{9}$	$\frac{1}{10}$	$\frac{1}{11}$	$\frac{1}{12}$	$\frac{1}{13}$	$\frac{1}{14}$	$\frac{1}{15}$	$\frac{1}{16}$	$\frac{1}{17}$	$\frac{1}{18}$	$\frac{1}{19}$	$\frac{1}{20}$	1
[illegible]	$\frac{363}{560}$	$\frac{761}{1260}$	$\frac{7129}{12600}$	$\frac{671}{1260}$	$\frac{83711}{166320}$	$\frac{6617}{13860}$	$\frac{1145993}{2522520}$	$\frac{1171733}{2702700}$	$\frac{1195757}{2882880}$	$\frac{143327}{360360}$	$\frac{42142223}{110270160}$	$\frac{751279}{2042040}$	$\frac{275295799}{775975200}$	2
[illegible]	$\frac{469}{240}$	$\frac{29531}{15120}$	$\frac{1303}{672}$	$\frac{16103}{8400}$	$\frac{190553}{100800}$	$\frac{128977}{69300}$	$\frac{9061}{4950}$	$\frac{30946717}{17199000}$	$\frac{39646461}{22422400}$	$\frac{58433327}{33633600}$	$\frac{344499373}{201801600}$	$\frac{784809203}{467812800}$	$\frac{169704792667}{102918816000}$	3
[illegible]	$\frac{967}{240}$	$\frac{89}{20}$	$\frac{4523}{945}$	$\frac{7645}{1512}$	$\frac{341747}{64800}$	$\frac{412009}{75600}$	$\frac{9301169}{1663200}$	$\frac{406841}{71280}$	$\frac{35118025721}{6054048000}$	$\frac{4446371981}{756756000}$	$\frac{80847323107}{13621608000}$	$\frac{2263547729}{378378000}$	$\frac{32262100943}{5360355000}$	4
[illegible]	$\frac{35}{6}$	$\frac{1069}{144}$	$\frac{285}{32}$	$\frac{31063}{3024}$	$\frac{139381}{12096}$	$\frac{1148963}{90720}$	$\frac{355277}{25920}$	$\frac{21939781}{1496880}$	$\frac{2065639}{133056}$	$\frac{2195261857}{134534400}$	$\frac{371446039969}{21794572800}$	$\frac{27566944753}{1556755200}$	$\frac{31938836201}{1743565824}$	5
	$\frac{23}{4}$	9	$\frac{3013}{240}$	$\frac{781}{48}$	$\frac{242537}{12096}$	$\frac{48035}{2016}$	$\frac{1666393}{60480}$	$\frac{22463}{720}$	$\frac{277382447}{7983360}$	$\frac{38101097}{997920}$	$\frac{1356664151597}{32691859200}$	$\frac{162356544377}{3632428800}$	$\frac{694142313941}{14529715200}$	6
	$\frac{7}{2}$	$\frac{91}{12}$	$\frac{105}{8}$	$\frac{4781}{240}$	$\frac{13321}{480}$	$\frac{314617}{8640}$	$\frac{790153}{17280}$	$\frac{899683}{16200}$	$\frac{2271089}{34560}$	$\frac{86853967}{1140480}$	$\frac{13195009}{152064}$	$\frac{227663026369}{2335132800}$	$\frac{2022480780283}{18681062400}$	7
	1	4	$\frac{29}{3}$	$\frac{55}{3}$	$\frac{10831}{360}$	$\frac{897}{20}$	$\frac{944311}{15120}$	$\frac{35717}{432}$	$\frac{54576553}{518400}$	$\frac{8424673}{64800}$	$\frac{334947281}{2138400}$	$\frac{9764119}{52800}$	$\frac{5013017410969}{23351328000}$	8
		1	$\frac{9}{2}$	12	$\frac{99}{4}$	$\frac{1747}{40}$	$\frac{5551}{80}$	$\frac{515261}{5040}$	$\frac{23915}{168}$	$\frac{76492463}{403200}$	$\frac{21878439}{89600}$	$\frac{4065163957}{13305600}$	$\frac{3975325483}{10644480}$	9
			1	5	$\frac{175}{12}$	$\frac{65}{2}$	$\frac{491}{8}$	$\frac{2485}{24}$	$\frac{324509}{2016}$	$\frac{59279}{252}$	$\frac{79243781}{241920}$	$\frac{11795941}{26880}$	$\frac{6063698587}{10644480}$	10
				1	$\frac{11}{2}$	$\frac{209}{12}$	$\frac{1001}{24}$	$\frac{30217}{360}$	$\frac{1199}{8}$	$\frac{494351}{2016}$	$\frac{1513391}{4032}$	$\frac{18843187}{34560}$	$\frac{367394203}{483840}$	11
					1	6	$\frac{41}{2}$	$\frac{105}{2}$	$\frac{26921}{240}$	$\frac{6341}{30}$	$\frac{5490071}{15120}$	$\frac{976163}{1680}$	$\frac{354467473}{403200}$	12
						1	$\frac{13}{2}$	$\frac{143}{6}$	65	$\frac{35269}{240}$	$\frac{46631}{160}$	$\frac{3965533}{7560}$	$\frac{10596053}{12096}$	13
							1	7	$\frac{329}{12}$	$\frac{238}{3}$	$\frac{136241}{720}$	$\frac{31521}{80}$	$\frac{6406481}{8640}$	14

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COEFFICIENTS $A_{m,s}$ IN MARKOFF'S EXPANSION
$\omega_m D^m f(x) \sim \sum_{s=m}^{s=n} (-1)^{m+s} A_{m,s} \Delta^s f(x)$

m \ s	15	16	17	18	19	20
15	1	$\frac{15}{2}$	$\frac{125}{4}$	$\frac{765}{8}$	$\frac{11519}{48}$	$\frac{50255}{96}$
16		1	8	$\frac{106}{3}$	114	$\frac{18017}{60}$
17			1	$\frac{17}{2}$	$\frac{119}{3}$	$\frac{1615}{12}$
18				1	9	$\frac{117}{4}$
19					1	$\frac{19}{2}$
20						1

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ON MAJORANTS OF SU
ANALYTIC FU

FRANTIŠEK

This paper represents a different
problems connected with majorants
same method has been used previous
tion of the Phragmén-Lindelöf th
approach is to prove first Lemma 4.
sults are easily deducible. Corollary
N. Levinson.[2] His theorem has ma
these results.

LEMMA 1. *If* (i) $0<f(x)\leqq 1$ *and* (ii

(1) $\int_a^b \log \left| \int_\xi^x \right.$

is a continuous function of ξ *in* $(a,$

We first suppose that $f(x)$ is non-
We get

$\int_0^x f(y)dy > \int_{x/2}^x f($

Hence

$\int_0^1 \log \left(\int_0^x f(y)dy \right) dx >$

(2) $>$

$=$

If $f(x)$ is replaced by $f(a+(b-a)$

Presented to the Society, Novemb
analytic functions; received by the edit
[1] Cf. the end of this paper and Jour
14 (1939), p. 208.
[2] *Gap and Density Theorems*, Amer
lications, vol. 26, 1940, p. 127, Theore