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AN ASYMPTOTIC FORMULA FOR THE NUMBER
OF COMPLETE PROPOSITIONAL CONNECTIVES

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The present article is really a continuation of the author's earlier paper [1] on this subject. The line of investigation described previously is rounded off by deriving some further numerical results, which include, in particular, an asymptotic formula for the number of complete propositional connectives of n variables in 2-valued logic.

In conformity with nomenclature used by other writers, a function $f(p_1, \ldots, p_n)$ will be called a δ-*function* if $f(p, \ldots, p) = \neg\, p$. Also, the function $f^*(p_1, \ldots, p_n) = \neg\, f(\neg\, p_1, \ldots, \neg\, p_n)$ will be called the *dual* of f, and f will be called *self-dual* if $f^* = f$. With these definitions, Post's 2 necessary and sufficient conditions for $f(p_1, \ldots, p_n)$ to be a complete connective in 2-valued logic are

(1) f is a δ-function.   (2) f is not self-dual $(f^* \neq f)$.

We shall write

$$2^{2^n} = N = 4\,M^2.$$

Then N is the total number of propositional connectives of n variables (when all permutational variants are counted separately), and $2\,M\,(= \sqrt{N})$ is the total number of self-dual functions.

Similarly, $M^2\,(= \tfrac{1}{4}\,N)$ is the total number of δ-functions, and $M\,(= \tfrac{1}{2}\sqrt{N})$ is the total number of self-dual δ-functions.

We note, in particular, that this means that, as n increases, the *proportion* of self-dual functions becomes progressively less, or, putting it another way, Post's second condition becomes increasingly unimportant for purposes of enumeration. Expressing this informally, it can be said that, for large n, roughly $\tfrac{1}{4}$ of all functions, (virtually all the δ-functions), are complete connectives.

But our real interest is the number of essentially distinct connectives and it will be convenient to define the following 3 numerical functions:

$\left[\begin{array}{l} \end{array}\right.$ $t(n)$ the total number of distinct δ-functions of n variables, ~ Post fn

$s(n)$ the number of distinct self-dual δ-functions,

$c(n)$ the number of complete propositional connectives of n variables. — complete Post fn

Then

$$c(n) = t(n) - s(n).$$

These functions were evaluated in our previous paper for $n \leq 5$, and their values are incorporated in table 1 below, for comparison purposes.

Two other functions which will occur are

$$\tau(n) = \frac{M^2}{n!} \quad \text{and} \quad \sigma(n)$$

In fact, we shall prove that

$$c(n) \sim t(n) \sim \tau(n)$$

and

$$s(n) \sim \sigma(n)$$

We shall establish first that

$$s(n)/t(n) \to 0 \quad \text{as} \quad n \to \infty.$$

Since the frequency of every distribution is at least 1 and at most $n!$, it follows that

$$\frac{M^2}{n!} < t(n) < M^2$$

and

$$\frac{M}{n!} < s(n) < M;$$

therefore

$$\frac{s(n)}{t(n)} < M \cdot \frac{n!}{M^2},$$

which obviously tends very strongly to zero as $n \to \infty$. Hence,

$$c(n) \sim t(n)$$

The other necessary step in deriving the asymptotic formula is also prompted by the evidence accumulated in our previous paper. For general n, ω will be used to denote a partition $a_1|a_2|\ldots|a_n$ (frequency $n!$), of type $\{1^n\}$, in which all the n letters are distinguishable. [In the particular notations used before, this type was ε when $n = 4$ and η when $n = 5$.] The proportion of distributions of type ω increases strikingly as n increases. Thus, taking all the δ-functions,

when $n = 4$,   10752 out of   16384 (66%) are of type $\omega[\varepsilon]$;
when $n = 5$, 1030172160 out of 1073741824 (96%) are of type $\omega[\eta]$.

Indeed, it would be surprising if connectives which leave the a_i with any kind of symmetry did not become rapidly rarer, since each unit increase in n squares the total number of functions, and yet provides only one extra letter to help to 'accommodate' them all.

Any function not of type ω must have at least 2 symbols which are indistinguishable. For the sake of definiteness, let us enumerate first those functions in which a_1 and a_2 are indistinguishable. This means that every pair of products like $a_1 a_i a_j \ldots a_k$ and $a_2 a_i a_j \ldots a_k$ must have identical properties. Since an unrestricted selection $a_i a_j \ldots a_k$ is made from $a_3, a_4, \ldots, a_n$, the number of such products is 2^{n-2}. Hence, instead of the usual 2^n choices which have to be made in assigning the sequence of digits in the truth table valuation of the function, there are only $(2^n - 2^{n-2})$ choices when a_1 and a_2 are indistinguishable. Therefore, the total number of functions with this property is

$$2^{(2^n - 2^{n-2})} = N^{-\frac{1}{4}} \cdot N,$$

which we choose to leave in this form.

Now the frequency of a distribution with a_1 and a_2 indistinguishable is at most $\frac{1}{2}n!$ (and less, of course, if a_1 and a_2 belong to a group of 3 or more symmetrically occurring symbols, or, if there are other groups besides a_1, a_2). Hence, when the indistinguishable pair is allowed to range over all possible selections, instead of being confined to a_1, a_2, the total number of functions obtained will certainly be less than

$$(\tfrac{1}{2}n!\, N^{-\frac{1}{4}}) \cdot N,$$

and this total will include at least once every function which is not of type ω.

By an exactly similar argument, the total number of δ-functions which are not of type ω is less than

$$(\tfrac{1}{2}n!\, N^{-\frac{1}{4}}) \cdot M^2.$$

Now the multiplier $\frac{1}{2}n!\, N^{-\frac{1}{4}}$ tends (strongly) to zero as $n \to \infty$, and this proves that the proportion of δ-functions (and also of general functions) which are of type ω tends to 1.

But the frequency of type ω is $n!$, and the total number of all δ-functions is M^2, of which only $t(n)$ are really distinct. Hence,

$$t(n) \sim M^2/n! = \tau(n),$$

and so, using our earlier result,

$$c(n) \sim \tau(n).$$

In a similar way, it can be shown that the number of self-dual δ-functions which are not of type ω is less than

$$(\tfrac{1}{2}\, n!\, N^{-\frac{1}{5}}) \cdot M = o(M).$$

Hence

$$s(n) \sim M/n! = \sigma(n).$$

[Note that the evidence from small values of n tends to be misleading in this case. When $n = 3$ and $n = 4$, there are, in fact, no self-dual δ-functions of type ω, and when $n = 5$, only 44% are of this type. But, beyond this point, the proportion of type ω will increase rapidly. These accidential circumstances explain why, for small values of n, $s(n)$ and $\sigma(n)$ in table 1 do not agree as closely as $t(n)$ and $\tau(n)$.]

n	$\sigma(n)$	$\tau(n)$	$s(n)$	$t(n)$	$c(n)$
1	1	1	1	1	0
2	1	2	1	3	2
3	1	11	4	20	16
4	5	683	16	996	980
5	273	8947849	544	9333312	9332768
6	$2 \cdot 9826 \cdot 10^6$	$6 \cdot 4053 \cdot 10^{15}$			
7	$1 \cdot 8300 \cdot 10^{15}$	$1 \cdot 6879 \cdot 10^{34}$			
8	$4 \cdot 2198 \cdot 10^{33}$	$7 \cdot 1796 \cdot 10^{71}$			

Table 1

1*

In table 1, $\sigma(n)$ and $\tau(n)$ are given to the nearest integer for $n \leq 5$, and to 5 significant digits for $n \geq 6$. Note that

$$n \, \sigma(n) = 2\tau(n-1).$$

[It is the writer's guess that substantially correct values will be obtained by taking $\sigma(n)$ as an estimate for $s(n)$ for $n \geq 7$, and $\tau(n)$ as an estimate for $t(n)$ or $c(n)$ for $n \geq 6$.]

Once calculated, the functions we have introduced have various other applications, which are worth mentioning briefly, before leaving this subject. For example, the answer to the following query might be wanted. Of the total of N propositional functions of n variables, how many are essentially distinct? To take a trivial example, only 12 of the 16 connectives of 2 propositions p and q are really different, because although 8 of the connectives involve p and q symmetrically, the following 4 pairs are mere alphabetic variants of each other:

$$\begin{cases} p & \neg\, p & p \to q & p \,\&\, \neg\, q \\ q & \neg\, q & q \to p & q \,\&\, \neg\, p. \end{cases}$$

The answer to the question in the general case is $4t(n)$, and other similar results (which hold for $n \geq 1$) are summarized in table 2. For any of the 12 enumerative problems considered in this table, it is also a simple matter (using table 1 when necessary) to decide how many of the propositional functions which arise are genuinely dependent on all n variables, and how many are really gratuitous reappearances of functions of less than n variables. In addition, the asymptotic formulae which are obtainable for these and other numerical functions can often be expressed in terms of $\sigma(n)$ and $\tau(n)$, and so on.

Type of function to be enumerated	Total when all permutational variants are counted separately	Total when mere permutational variants are not counted separately
All possible connectives	$4M^2 = N$	$4t(n)$
Self-dual functions	$2M$	$2s(n)$
Non-self-dual functions	$2M(2M-1)$	$4t(n) - 2s(n)$
All δ-functions	M^2	$t(n)$
Self-dual δ-functions	M	$s(n)$
Complete connectives	$M(M-1)$	$t(n) - s(n) = c(n)$

Table 2

Reference

[1] ROGER F. WHEELER, Complete Propositional Connectives, this Zeitschr. 7 (1961), 185—198.